LECTURE 18: SUMMARY

We began this lecture by proving the following result, which was the final missing step in our proof of the Cauchy Criterion:

Theorem (Monotone Subsequence Theorem). *Every sequence has a monotone subsequence.*

Proof. Let (a_n) denote a sequence. We call a term a_k a *peak* iff $a_k \geq a_m$ for all $m \geq k$. There are two cases:

(1) There are infinitely many peaks.
(2) There are finitely many peaks.

In the first case, the subsequence consisting of the peaks forms a monotonically decreasing sequence. In the second case, there exists some largest M such that a_M is a peak. Let $m_1 = M + 1$. Since a_{m_1} isn't a peak, there exists $m_2 > m_1$ such that $a_{m_1} < a_{m_2}$. Since a_{m_2} isn't a peak, there exists $m_3 > m_2$ such that $a_{m_2} < a_{m_3}$. We can continue this process indefinitely, thus creating a monotonically increasing subsequence (a_{m_k}). □

Before leaving the topic of sequences and series, we gave one final convergence test for series. Recall that most of our convergence tests – in particular, the root, ratio, and integral tests – all depend on the terms of the series being all non-negative. What about series whose terms are sometimes positive and sometimes negative? One classic example of this is the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

Let S_N denote the Nth partial sum of this series. By playing around with this a bit, we noticed that the subsequence (S_{2n}) is monotonically increasing. Moreover, it is bounded between 0 and 1. It follows (by the MCT) that the subsequence converges, say to some number $x \in \mathbb{R}$. We then proved that S_n also converges to x. This gives us an example of a series which converges, but which does NOT converge absolutely. I mentioned (but didn't prove) a cool theorem due to Riemann, which asserts that given any real number α, there exists a rearrangement of the terms of the series above which converges to α. This is not the case for any series which converges absolutely.

We then applied the intuition gained in the last series to prove a more general result:

Theorem 1 (Alternating Series Test). *Given a monotonically decreasing sequence (a_n) of non-negative real numbers such that $a_n \to 0$ as $n \to \infty$. Then the series*

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$$

converges.

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Proof. Let S_n denote the nth partial sum of the series. First, note that the subsequence (S_{2n}) is monotonically increasing; this is because for every $n \in \mathbb{N}$,

$$S_{2n} = S_{2(n-1)} + (a_{2n-1} - a_{2n}) \geq S_{2(n-1)}.$$

Moreover, I claim that this subsequence is bounded. On one hand, we have

$$\begin{aligned} S_{2n} &= (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2n-1} - a_{2n}) \\ &\geq 0. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} S_{2n} &= a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2n-2} - a_{2n-1}) - a_{2n} \\ &\leq a_1 \end{aligned}$$

Since (S_{2n}) is monotone and bounded, we conclude (by the MCT) that it converges. Let

$$x := \lim_{n\to\infty} S_{2n}.$$

I claim that the whole sequence (S_n) converges to x. Given $\epsilon > 0$. We already know that there exists K such that if $2k > K$, then $|S_{2k} - x| < \epsilon/2$. But also, since $a_n \to 0$ as $n \to \infty$, there exists M such that $|a_m| < \epsilon/2$ for all $m > M$. Pick any $n > \max\{M, K\} + 1$. If n is even, we have already established that

$$|S_n - x| < \epsilon/2 < \epsilon.$$

If n is odd, then

$$\begin{aligned} |S_n - x| &= |S_n - S_{n-1} + S_{n-1} - x| \\ &\leq |S_n - S_{n-1}| + |S_{n-1} - x| \\ &= |a_n| + |S_{n-1} - x| \\ &< \epsilon \end{aligned}$$

Thus, we have shown that the partial sums (S_n), and hence the infinite series, converge. □

This concludes our discussion of sequences and series in $\mathbb{R}$. We now move on to our final topic of the term: metric spaces. Before a formal definition, here's the idea. On $\mathbb{R}$, and more generally in $\mathbb{R}^n$, we have a good idea of how to measure the distance between two points: draw the straight line between them, and measure its length. But what if we're not on $\mathbb{R}^n$? What if, for example, we're living on the surface of a sphere? What's the distance between two points? What if we're living on the surface of a saddle? These are 2-dimensional surfaces, curved in some funny way in 3 dimensions. What if we live in some very bizarre 3-dimensional space, curved in 4 dimensions? Or, going even further into abstraction, is there a notion of distance for spaces of functions? Or for even more abstract spaces?

All of this is meant to motivate the following overarching question: given a set X, is there a reasonable notion of how to measure distance between two points of X? The following definition is an attempt to answer this.

Definition (Metric)**.** *Given a nonempty set X, we say that $d : X \times X \to \mathbb{R}$ is a* distance *(or* metric*) on X iff it satisfies the following three properties:*

(1) *$d(x, y) = 0$ iff $x = y$;*

(2) $d(x, y) = d(y, x)$ *for all* $x, y \in X$*; and*
(3) $d(x, y) + d(y, z) \geq d(x, z)$ *for all* $x, y, z \in X$*. (Triangle Inequality)*

The first property says that the distance from a point to itself is 0, and that the distance between two distinct points is always nonzero. The second says that distance is measured between two points, not from one point to another. The third says that the direct route from x to z is shorter than if you take any detour (through y) along the way. We will discuss this further next lecture.