LECTURE 24: SUMMARY

Last time, we discussed the notion of a limit point: in a metric space (X, d), a point $\ell \in X$ is a limit point of $S \subseteq X$ iff ℓ is the limit of some sequence (of distinct elements) in S. Further, we showed that this holds iff every neighbourhood of ℓ contains some point of S (other then ℓ itself).

What are the limit points of the open interval $(0, 1)$ (in $\mathbb{R}$, under the usual metric)? First, every single point in $(0, 1)$ is a limit point of $(0, 1)$ (can you prove this?). But there are also two points which aren't in $(0, 1)$ which are limit points: 0 and 1. Thus, the set of limit points of the open interval $(0, 1)$ is the closed interval $[0, 1]$. The set of limit points of the closed interval $[0, 1]$ is simply itself; no sequence of points ever converges to something outside the set itself. Inspired by this, we say that a set is closed if no sequence of points in the set converges to something outside the set. More precisely:

Definition. *Given (X, d) a metric space. We say $\mathcal{C} \subseteq X$ is* closed *iff $\mathcal{C}$ contains all of its limit points.*

Thus, for example, the closed interval $[a, b]$ is closed in $\mathbb{R}$ (under the usual metric). The unit disk $\{z \in \mathbb{R}^2 : |z| \leq 1\}$ is closed in $\mathbb{R}^2$ (under the Euclidean metric). And in any metric space, the set consisting of a single point is closed, since there are no limit points of such a set!

We now arrive at a fundamental result connecting open and closed sets.

Theorem 1. *Given a metric space (X, d) and a set $\mathcal{A} \subseteq X$, let $\mathcal{A}^c := X - \mathcal{A}$. Then $\mathcal{A}$ is open iff $\mathcal{A}^c$ is closed.*

Proof. As usual, we do the two directions separately.

$(\Longrightarrow)$

> Given $\mathcal{A}$ open. We want to show $\mathcal{A}^c$ is closed, i.e. that $\mathcal{A}^c$ contains all of its limit points. We do this by showing that no point of $\mathcal{A}$ is a limit point of $\mathcal{A}^c$.
>
> Take any $\alpha \in \mathcal{A}$. Since $\mathcal{A}$ is open, there exists a neighbourhood $\mathcal{N}$ of α which is entirely contained in $\mathcal{A}$. It follows that $\mathcal{N} \cap \mathcal{A}^c = \emptyset$, so α cannot be a limit point of $\mathcal{A}^c$.

$(\Longleftarrow)$

> Suppose $\mathcal{A}^c$ is closed. We want to show that $\mathcal{A}$ is open, i.e. that every point of $\mathcal{A}$ lives in a neighbourhood entirely contained in $\mathcal{A}$.

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Pick any $\alpha \in \mathcal{A}$. Since $\mathcal{A}^c$ is closed, it contains all its limit points, so α can't be a limit point of $\mathcal{A}^c$. It follows that there exists a neighbourhood $\mathcal{N}$ of α such that $\mathcal{N} \cap \mathcal{A}^c = \emptyset$. But this implies that $\mathcal{N} \subseteq \mathcal{A}$.

□

Note that this theorem does *not* say that every subset of X is either open or closed. Some sets (such as $[0, 1)$ in $\mathbb{R}$) are neither; other sets (such as $\emptyset$, or the entire space X) are both. Still, the duality in the theorem above is useful. For example:

Corollary 2. *Arbitrary intersections of closed sets are closed.*

Proof. Given a collection $\{\mathcal{C}_\alpha\}$ of closed sets. Then

$$\bigcap_\alpha \mathcal{C}_\alpha = \bigcap_\alpha (X - \mathcal{C}_\alpha^c) = X - \bigcup_\alpha \mathcal{C}_\alpha^c = \left(\bigcup_\alpha \mathcal{C}_\alpha^c\right)^c.$$

Since $\mathcal{C}_\alpha^c$ is open for each α, and arbitrary unions of opens are open, we conclude that $\bigcup_\alpha \mathcal{C}_\alpha^c$ is open. But this implies that $\left(\bigcup_\alpha \mathcal{C}_\alpha^c\right)^c = \bigcap_\alpha \mathcal{C}_\alpha$ is closed. □

Similarly, we can easily prove

Corollary 3. *Finite unions of closed sets are closed.*

Note that the assumption of finiteness is necessary here. For example,

$$\bigcup_{n \in \mathbb{N}} \left[\frac{1}{n}, 1 - \frac{1}{n}\right] = (0, 1).$$

We finished lecture by discussing the Cantor set. This is a subset of $[0, 1]$ which is formed by removing countably many disjoint intervals from $[0, 1]$. Despite the fact that the sum of all the lengths of the removed intervals equals 1, the Cantor set contains uncountably many points. It is also closed (in $\mathbb{R}$).