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MATB43: INTRODUCTION TO ANALYSIS

Problem Set 3 – due Thursday, March 21, during the first 5 minutes of lecture.

INSTRUCTIONS: Please print and attach this page as the first page of your submitted problem set.

PROBLEM	MARK
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Please read the following statement and sign below:

I understand that I am not allowed to use the internet to assist (in any way) with this assignment. I also understand that I must write down the final version of my assignment in isolation from any other person.

SIGNATURE:________________________________

Problem Set 3

I recommend proceeding in order, as some problems are easier to solve using the results of prior problems.

3.1 Suppose (a_n) and (b_n) are sequences such that $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$.

(a) Prove that $\lim_{n\to\infty} (a_n + b_n) = A + B$.

(b) Prove that $\lim_{n\to\infty} a_n b_n = AB$.

(c) Prove that $\lim_{n\to\infty} |a_n| = |A|$.

3.2 Suppose (a_n) and (c_n) are convergent sequences.

(a) Prove that if $a_n \leq c_n$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} c_n$.

(b) Prove that if $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$, and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n$, then (b_n) is convergent. Moreover, show that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

3.3 (a) Given sequences (a_n) and (b_n) such that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} (a_n + b_n)$ converge. Prove that $\sum_{n=1}^{\infty} b_n$ must converge as well.

(b) Give a new proof (different from the one given in class) of the following statemnt: if $\sum_{n=1}^{\infty} |a_n|$ converges, then so does $\sum_{n=1}^{\infty} a_n$. [*Hint: consider* $\sum (a_n + |a_n|)$.]

3.4 Suppose (a_n) is a sequence such that $\lim_{n\to\infty} a_{2n} = A = \lim_{n\to\infty} a_{2n-1}$. Prove that $\lim_{n\to\infty} a_n = A$.

3.5 Determine (with proof!) $\lim_{n\to\infty} (\sqrt{n+3} - \sqrt{n})$.

3.6 Show (by example) that it's possible to have a bounded sequence (a_n) and a convergent sequence (b_n) such that both $(a_n + b_n)$ and $(a_n b_n)$ diverge.

3.7 Let $a_1 > 1$, and suppose $a_{n+1} = 2 - 1/a_n$ for all $n \geq 1$. Prove that (a_n) converges, and find its limit. [*Hint: the sequence is bounded and monotone.*]

3.8 Let $0 \leq \alpha < 1$, and let $f : \mathbb{R} \to \mathbb{R}$ be a function which satisfies $|f(x) - f(y)| \leq \alpha |x - y|$ for all $x, y \in \mathbb{R}$. Pick $a_1 \in \mathbb{R}$, and set $a_{n+1} := f(a_n)$ for all $n \in \mathbb{N}$. Prove that (a_n) converges. [*Hint: Cauchy criterion.*]

3.9 Suppose (a_n) is bounded, and that every convergent subsequence of (a_n) has limit A. Prove that

$$\lim_{n\to\infty} a_n = A.$$

3.10 Suppose (a_n) is a monotonically decreasing sequence of positive numbers, and that $\sum_{n=1}^{\infty} a_n$ converges. Prove that $\lim_{n\to\infty} na_n = 0$.

3.11 Prove that if $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n^2$ converges.

3.12 For which values of x do the following series converge?

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{x^{2n-1}}{(2n-1)!}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n(x-1)^n}{2^n}$

3.13 Let (a_n) be a bounded sequence of real numbers. Does there exist a subsequence of (a_{n_k}) which is convergent, and such that n_k is a perfect square for all k?

3.14 Let (a_n) be a sequence of non-negative real numbers, and suppose $\sum_{n=1}^{\infty} a_n$ diverges. Prove that $\sum_{n=1}^{\infty} \sqrt{a_n}$ also diverges.

3.15 For each of the following, determine (with proof!) whether $\sum_{n=1}^{\infty} a_n$ converges or diverges.

(a) $a_n = \dfrac{n}{2^n}$

(b) $a_n = \dfrac{n!}{n^n}$

(c) $a_n = (\log n)^{-n}$

(d) $a_n = \dfrac{(n!)^2}{(2n)!}$

(e) $a_n = \begin{cases} \frac{1}{n} & \text{if } n \text{ is not a multiple of } 3 \\ \frac{-1}{n} & \text{if } n \text{ is a multiple of } 3 \end{cases}$