# GROUPS AND SYMMETRY: LECTURE 1

LEO GOLDMAKHER

All information about the course, including the syllabus, can be found on the course webpage:

`www.math.toronto.edu/lgoldmak/C01F13/`

Lecture summaries and assignments will all be posted to the website. In fact, the 0th assignment has already been posted.

## 1. Introduction

Despite the cryptic title, this course is about algebra. In school you've seen that algebra is useful for solving word problems. How does this work? First, we transform the concepts into symbols (aka variables); then we write down all the relationships we can think of between these symbols (ie identities or inequalities); then we manipulate these relationships and try to simplify them; and finally, we translate the symbols back into words, thus solving the original question.

There are several reasons for this approach. First, symbols are easier to write. More importantly, dealing purely with symbols forces us to ignore external associations – biases, emotions, etc. – which might make the concepts harder to think about objectively. Finally, although there are many different ways a word problem could be stated, algebraic relations are very precise. Two equivalent word problems might appear completely different on the surface, but it's usually obvious when two algebraic relations are equivalent.

Algebra is useful in other contexts as well, such as geometry or calculus. For example, the Pythagorean theorem is usually presented in an algebraic form ($a^2 + b^2 = c^2$) even though the content is purely geometric. Actually, we're so used to thinking in terms of algebra that it's difficult to conceive of a time before we knew about it. And yet, algebra is very abstract and far from obvious.[1] The ultimate goal of this course is to study algebra, not as a method for solving problems, but as its own subject.

We will work our way up to this by exploring increasingly abstract applications of algebra. Our first application, to classical geometry, will use algebra in a way you're probably not so familiar with – most of our variables will be functions, rather than numbers or vectors. As you will see, the advantage of the algebraic approach is that we won't get distracted by complicated pictures. In fact, we won't have to draw pictures at any time, a particularly important advantage to artistically-challenged people like me. The downside, of course, is that algebraic statements are more abstract than pictures. But pictures, while more concrete, are much more complicated to draw and manipulate.

Enough talk. Let's do some math.

---

*Date*: September 6, 2013.

[1]Even Archimedes, one of the most creative humans in history, had no inkling of algebra. Remarkably, this didn't prevent him from inventing integration, in particular discovering a formula for the area between a line and a parabola. His method, which we discussed in lecture, involves filling the region with infinitely many triangles, and summing the areas of all these. Both steps require remarkable ingenuity, even with the convenience of algebra; but Archimedes was working about 500 years before algebra was available.

## 2. CONGRUENCES

In school, you studied Euclidean geometry in the plane. A fundamental concept in this area is that of *congruence*, for example of triangles. What does it mean for two figures to be congruent? The first proposed definition – that they have the same size and shape – was a good start, but not suitable as a precise definition. What is shape, exactly? And how do we measure size? If we're just talking about triangles these questions can be answered, but for general shapes the situation is much more difficult. A second proposed definition was that two shapes are congruent if it's possible to pick up one of them and move it (without changing its shape) so that it lines up precisely on top of the second shape. This is better, but is still somewhat vague – what does it mean to move a shape, or for two shapes to line up precisely? We quickly realized two things:

(1) a shape is just a set of points in $\mathbb{R}^2$; and
(2) moving a set of points in the plane just means applying a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ to that set.

But for this movement to be rigid – for it to preserve the shape of the set of points – $f$ can't be just any function. After extensive discussion and lots of ideas, Mac proposed the following as a precise version of what it means to move a set of points without changing the shape of the set.

**Definition.** *A* rigid motion *(of the plane) is a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ which preserves distances, i.e. such that for all $X, Y \in \mathbb{R}^2$ we have*

$$d\Big(f(X), f(Y)\Big) = d\Big(X, Y\Big),$$

*where $d(A, B)$ denotes the distance $|A - B|$ between $A$ and $B$.*

With this notion in hand, congruence becomes easy to define precisely: two sets $A, B \subseteq \mathbb{R}^2$ are congruent iff there exists a rigid motion $\phi$ such that $\phi(A) = B$. For brevity, we write $A \cong B$.

Note that in general, two congruent triangles are *not* the same triangle; they're located in two different places. However, we use congruence as a notion of 'sameness'. Even the notation is reminscent of equality! This is no accident, as congruence is an *equivalence relation*. Recall that this means congruence satisfies three properties:

(i) *Reflexive.* For all $A \subseteq \mathbb{R}^2$, we have $A \cong A$.
(ii) *Symmetry.* If $A \cong B$, then $B \cong A$.
(iii) *Transitive.* If $A \cong B$ and $B \cong C$, then $A \cong C$.

These properties capture what it means for a comparison to be a reasonable notion of 'sameness'. In class, we discussed other examples of equivalence relations (e.g. hair color). Soon, we will run across a geometric example of an equivalence relation which is quite different from congruence.

The fact that $\cong$ is an equivalence relation is secretly expressing something about the set of all rigid motions. Specifically, property (i) says that the identity function (i.e. the function which sends every point to itself) is a rigid motion; property (ii) says that for every rigid motion $\phi$, the inverse function $\phi^{-1}$ is also a rigid motion; and property (iii) says that given any two rigid motions $\phi$ and $\psi$, the composition $\phi \circ \psi$ is also a rigid motion.[2]

WWW.MATH.TORONTO.EDU/LGOLDMAK/C01F13/
*E-mail address*: lgoldmak@math.toronto.edu

[2] Recall that the composition $f \circ g$ is the function defined by $\big(f \circ g\big)(x) := f\big(g(x)\big)$.