GROUPS AND SYMMETRY: LECTURE 3

LEO GOLDMAKHER

Recall that we're studying the plane isometries, in particular trying to prove the following classification theorem:

Theorem 1. *Every isometry of the plane is either a translation, a rotation, or a glide reflection.*

We saw last time that drawing pictures is not helpful: we constructed a complicated composition of isometries which, by the theorem, must be a glide reflection, but which would be impossible to guess from a picture. Instead, we approach the problem algebraically. In other words, we first describe translations, rotations, and glide reflections in terms of variables; then we write down all relationships among these; then we manipulate the relationships to simplify them; then we translate back into the language of isometries. In this lecture, we focused on the translation into variables.

TRANSLATIONS: Of the three types of isometries, these are the easiest to describe algebraically. If we want to translate every point over 1 unit and down 2 units, we could just apply the function

$$\begin{aligned} T : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (x+1, y-2) \end{aligned}$$

Of course, if we're going to deal with different translations, calling this particular one just 'T' isn't great; what would we call the translation which moves everything over 4 and up 1? Instead, we incorporate the instructions into the name of the function. For example, the above translation we'll call $T_{(1,-2)}$. More generally, given $A \in \mathbb{R}^2$ we can define translation by A:

$$\begin{aligned} T_A : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ X &\longmapsto X + A \end{aligned}$$

ROTATIONS: These are not so easy to define. Let's start with the simplest type of rotation: we'll denote by R_α the counterclockwise rotation by α radians around the origin. How can we describe this algebraically? Kaidi suggested looking at the plane in polar coordinates, rather than in rectangular. Recall that this means that rather than labeling a point P as (x, y), we instead label it (r, θ), where r is the distance from P to the origin and θ is the angle formed between the x-axis and the segment connecting P to the origin. The advantage of polar coordinates is that it's easy to rotate around the origin: $R_\alpha(r, \theta) = (r, \theta + \alpha)$. The disadvantage is that usually we think about points in terms of rectangular coordinates, not polar. But this isn't a big problem: we can always translate between polar and rectangular when we need to, by using the relations

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta.$$

For example, what is $R_{\pi/3}(-1, \sqrt{3})$? First, we convert to polar coordinates:

$$\begin{aligned} \text{rectangular} &\rightsquigarrow \text{polar} \\ (-1, \sqrt{3}) &\rightsquigarrow (2, 2\pi/3) \end{aligned}$$

Date: September 13, 2013.

Rotating by $\pi/3$ gives the point $(2, \pi)$, in polar coordinates. Translating back into rectangular yields

$$R_{\pi/3}(-1, \sqrt{3}) = (-2, 0).$$

Having warmed up on this example, let's turn to the general case. What is $R_\alpha(x, y)$ in rectangular coordinates? Let (r, θ) denote the name of (x, y) in polar. Then

$$\begin{aligned}
R_\alpha(x, y) &= R_\alpha(r, \theta) \\
&= (r, \theta + \alpha) \qquad \textit{(in polar)} \\
&= \big(r\cos(\theta+\alpha), r\sin(\theta+\alpha)\big) \qquad \textit{(in rectangular)} \\
&= \big(r(\cos\theta)(\cos\alpha) - r(\sin\theta)(\sin\alpha), r(\cos\theta)(\sin\alpha) + r(\sin\theta)(\cos\alpha)\big) \\
&= (x\cos\alpha - y\sin\alpha, x\sin\alpha + y\cos\alpha)
\end{aligned}$$

Thus, we've solved the problem of expressing rotations around the origin in algebraic terms. However, this formula is ugly. It turns out we can make it look a lot nicer by using the language of linear algebra. Given a point (x, y) in the plane, denote it by $\begin{pmatrix} x \\ y \end{pmatrix}$. Then we see that

$$R_\alpha \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

In other words, the isometry R_α acts like multiplication by $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$.

To get a better feel for this, we tried an example. What is $R_{\pi/2}(x, y)$? Well, the matrix corresponding to $R_{\pi/2}$ is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. We have $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$, whence we deduce that $R_{\pi/2}(x, y) = (-y, x)$. In other words, when rotating a point around the origin by $\pi/2$, you can just flip the two coordinates and put a minus sign in front of the first one. Thus, we see that

$$R^2_{\pi/2}(x, y) = R_{\pi/2}(-y, x) = (-x, -y) = -(x, y),$$

or in other words,

$$R^2_{\pi/2} = -\mathbb{1}.$$

This looks a lot like the definition of the imaginary number i:

$$i^2 = -1.$$

As it turns out, this isn't a coincidence. We spent some time discussing the complex numbers

$$\mathbb{C} := \{a + bi : a, b \in \mathbb{R}\}.$$

Originally they arose from the cubic formula, but eventually were seen to be useful for all sorts of applications. Among other things, we can think of the plane $\mathbb{R}^2$ as $\mathbb{C}$, by relabeling the point (x, y) by the complex number $x + yi$. What is the analogue of polar coordinates in $\mathbb{C}$? Well, we know that $x = r\cos\theta$ and $y = r\sin\theta$, so in terms of r and θ we can write

$$x + yi = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta).$$

Euler noticed a funny thing about this way of writing complex numbers. Plugging in the Taylor series for $\cos\theta$ and $\sin\theta$, we see that

$$\cos\theta + i\sin\theta = 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots$$

This is highly reminiscent of the Taylor series for e^t:

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots$$

A little thought shows that the two series agree if we plug in $t = i\theta$. In other words, we have

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Thus, we have the following dictionary between polar coordinates in $\mathbb{R}^2$ and complex polar coordinates:

$$\begin{gathered}\text{polar in } \mathbb{R}^2 \rightsquigarrow \text{polar in } \mathbb{C} \\ (r,\theta) \rightsquigarrow re^{i\theta}\end{gathered}$$

This may look like just more notation, but it's actually a marvelous computational tool. For example, suppose we want to figure out $R_{\pi/3}(\sqrt{3},-1)$. First, we convert $(\sqrt{3},-1)$ to complex polar coordinates:

$$(\sqrt{3},-1) \rightsquigarrow 2e^{-i\pi/6}.$$

Next, we apply the rotation, which simply adds to the angle:

$$\begin{aligned} R_{\pi/3}(2e^{-i\pi/6}) &= 2e^{i(-\pi/6+\pi/3)} \\ &= 2e^{i\pi/6}. \end{aligned}$$

Finally, we convert back to rectangular coordinates:

$$2e^{i\pi/6} = 2(\cos\pi/6 + i\sin\pi/6) = \sqrt{3} + i.$$

Thus, $R_{\pi/3}(\sqrt{3},-1) = (\sqrt{3},1)$.

Having explored this example, we reinterpreted R_α in terms of $\mathbb{C}$. Given any point $z \in \mathbb{C}$, we can write in complex polar coordinates as $re^{i\theta}$. Rotating by α simply adds to the angle, whence we have

$$\begin{aligned} R_\alpha(z) &= re^{i(\theta+\alpha)} \\ &= re^{i\theta}e^{i\alpha} \\ &= e^{i\alpha}z. \end{aligned}$$

In other words, we now have an extremely short algebraic description of rotation, as long as we interpret the plane as $\mathbb{C}$ rather than as $\mathbb{R}^2$:

$$\begin{aligned} R_\alpha : \mathbb{C} &\longrightarrow \mathbb{C} \\ z &\longmapsto e^{i\alpha}z; \end{aligned}$$

in other words, rotation by α acts on the complex plane like multiplication by $e^{i\alpha}$.

We concluded our discussion of rotations by reconsidering our earlier example $R_{\pi/2}$. In the complex interpretation, we have $R_{\pi/2}(z) = e^{i\pi/2}z = iz$. Thus,

$$R_{\pi/2}(x + yi) = i(x + yi) = ix + yi^2 = -y + xi$$

exactly as before. Moreover, $R^2_{\pi/2}(z) = i^2 z = -z$. So, $R^2_{\pi/2}$ behaves like multiplication by -1.

<u>REFLECTIONS:</u> We finished class by discussing reflections. As a warm-up, we started by considering the reflection across the x-axis, which we called ρ (this is the Greek letter *rho*):

$$\begin{aligned} \rho : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (x, -y) \end{aligned}$$

We can also state this in terms of the complex interpretation:

$$\begin{aligned} \rho : \mathbb{C} &\longrightarrow \mathbb{C} \\ z &\longmapsto \bar{z} \end{aligned}$$

where $\bar{z}$, called the *conjugate of* z, is defined by

$$\overline{x + yi} := x - yi$$

for all $x, y \in \mathbb{R}$. We ended by stating (without proof) a few nice properties of conjugation:

(1) For all $z \in \mathbb{C}$, we have $|z|^2 = z\bar{z}$, where $|z|$ denotes the distance from z to the origin.

(2) For all $z, w \in \mathbb{C}$, we have $\overline{z + w} = \bar{z} + \bar{w}$ and $\overline{zw} = \bar{z}\bar{w}$.

Next lecture, we'll turn to more general rotations and reflections (i.e. not just those around the origin / across the x-axis).

WWW.MATH.TORONTO.EDU/LGOLDMAK/C01F13/
E-mail address: lgoldmak@math.toronto.edu