GROUPS AND SYMMETRY: LECTURE 5

LEO GOLDMAKHER

Last time, we proved the following:

Proposition 1. *An isometry ϕ is a nontrivial rotation of the plane* if and only if $\exists h \in \mathbb{C}$ *and a real* $\alpha \neq 0$ *such that* $\phi = T_h \circ R_\alpha$.

(Actually, we only proved the forward implication; the converse is an exercise on this week's problem set.) Here is one immediate consequence.

Corollary 2. *Every isometry of the form $T_h \circ R_\alpha$ is either a translation or a rotation.*

Proof. If $\alpha = 0$, then it's a translation. If $\alpha \neq 0$, the Proposition implies that the isometry is a rotation. □

Recall that the two key ideas in the proof of the Proposition were

(1) (Eric) We can describe a rotation around C by first changing our point of view so that C is the new origin, then rotating around this origin, then changing our point of view so that C is back where it belongs. In other words, the rotation around C is $T_C \circ R_\alpha \circ T_{-C}$.
(2) (Dan) We can switch the order of translations and rotations, as long as we do it carefully. More precisely, we figured out that $T_h \circ R_\alpha = R_\alpha \circ T_{R_{-\alpha}(h)}$. This allows us to simplify $T_C \circ R_\alpha \circ T_{-C}$ to something of the form $T_h \circ R_\alpha$.

We tested our understanding of the second idea by trying to perform the analogous "flipping" operation on $R_\alpha \circ \rho$. This wasn't as straightforward as the previous example, and we had a couple of false starts. Once we realized that the starting and ending isometries both have to take the *same input* the problem became clearer, and Jerry demonstrated that $R_\alpha \circ \rho = \rho \circ R_{-\alpha}$.

We next observed that Dan's idea can be applied to simplify an arbitrary (finite) string of translations and rotations to something of the form $T_h \circ R_\alpha$. Dan then suggested that if we understood how translations and reflections interact, we would be able to simplify any string composed of translations, rotations, and reflections to one of the two forms $T_h \circ R_\alpha$ or $T_h \circ R_\alpha \circ \rho$. We discussed this for a bit. Among other things, we figured out that $\rho^2 := \rho \circ \rho = \mathbb{1}$, an easy but useful tool in simplifying complicated compositions of isometries.

Proposition 1 shows that any rotation can be expressed in terms of our primitive isometries T_h, R_α, and ρ. What about reflections across an arbitrary line $\mathcal{L}$? Call such a reflection $\sigma_\mathcal{L}$. Kishan and Steph suggested the following beautiful idea. First, there exists some isometry ϕ such that $\phi(\mathcal{L})$ is

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the x-axis; moreover, ϕ can be written in the form $T_h \circ R_\alpha$. Next, we have

$$\begin{aligned}
\sigma_{\mathcal{L}} &= \phi^{-1} \circ \rho \circ \phi \\
&= (T_h \circ R_\alpha)^{-1} \circ \rho \circ T_h \circ R_\alpha \\
&= R_\alpha^{-1} \circ T_h^{-1} \circ \rho \circ T_h \circ R_\alpha \\
&= T_{-\alpha} \circ T_{-h} \circ \rho \circ T_h \circ R_\alpha \\
&= \cdots \\
&= T_a \circ R_\theta \circ \rho.
\end{aligned}$$

Thus, our alphabet of T_h's, R_α's, and ρ is robust enough to describe arbitrary reflections and rotations. Moreover, we immediately deduce that any glide reflection can also be described in the form $T_a \circ R_\theta \circ \rho$, since a glide reflection is simply a reflection composed with an additional translation. This led us to make the following conjecture: every isometry can be written in the form $T_h \circ R_\theta \circ \rho^j$. We will explore this next lecture.

WWW.MATH.TORONTO.EDU/LGOLDMAK/C01F13/
E-mail address: lgoldmak@math.toronto.edu