GROUPS AND SYMMETRY: LECTURE 6

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Today we proved (most of) our conjecture from last time: that every isometry is built out of the primitive isometries T_h, R_α, and ρ. Recall that $\mathcal{G}$ denotes the set of all plane isometries. We prove:

Lemma 1. *Given $\phi \in \mathcal{G}$, there exist $h \in \mathbb{C}$, $\alpha \in [0, 2\pi)$, and $j \in \{0, 1\}$ such that*

$$\phi = T_h \circ R_\alpha \circ \rho^j.$$

Moreover, h, α, and j are uniquely determined by ϕ.

The key insight is this: if $\{e_1, e_2\}$ is an orthonormal basis of $\mathbb{R}^2$, then so is $\{\phi(e_1), \phi(e_2)\}$. Thus, writing an arbitrary point P of $\mathbb{R}^2$ in terms of the basis $\{e_1, e_2\}$, we will be able to describe $\phi(P)$ in terms of the new basis $\{\phi(e_1), \phi(e_2)\}$. The picture is complicated by the fact that the origin is mapped to some random point which is probably not the origin. Fortunately, this is simple to resolve, and once we do so, the rest of the proof falls into place.

Proof. The proof is somewhat long, so I'll break it into steps.
<u>STEP 1:</u> Renormalize ϕ so that it fixes the origin.

More precisely, define $f := T_{-\phi(0)} \circ \phi$. It is easy to verify that $f(0) = 0$. Moreover, $f \in \mathcal{G}$, since it is a composition of isometries. It will turn out that f is much easier to deal with than ϕ. //

<u>STEP 2:</u> f preserves dot products (and hence, angles)

More precisely, we will prove that for any $X, Y \in \mathbb{R}^2$, we have $f(X) \cdot f(Y) = X \cdot Y$, where $\cdot$ denotes the vector dot product. To see this, first observe that since $f \in \mathcal{G}$, we have

$$|f(X) - f(Y)|^2 = |X - Y|^2.$$

Recall that $|A|^2 = A \cdot A$. Using this to expand both sides, we find

$$|f(X)|^2 + |f(Y)|^2 - 2f(X) \cdot f(Y) = |X|^2 + |Y|^2 - 2X \cdot Y.$$

Next, Jay pointed out that for any $A \in \mathbb{R}^2$,

$$|f(A)| = |f(A) - f(0)| = |A - 0| = |A|.$$

We deduce that $f(X) \cdot f(Y) = X \cdot Y$ as claimed. //

<u>STEP 3:</u> f is linear.

In other words, we'll prove that for any $\alpha, \beta \in \mathbb{R}$ and any $X, Y \in \mathbb{R}^2$, we have

$$f(\alpha X + \beta Y) = \alpha f(X) + \beta f(Y).$$

We do this in two steps. First, we show that for all $\alpha \in \mathbb{R}$ and all $X \in \mathbb{R}^2$,

$$f(\alpha X) = \alpha f(X). \tag{1}$$

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Why is this? Let $v_1 = f(\mathbf{1})$ and $v_2 = f(\mathbf{i})$, where $\mathbf{1}$ represents the vector $(1, 0)$ and $\mathbf{i}$ the vector $(0, 1)$. Then $|v_1| = |v_2| = 1$, and Step 2 implies that $v_1 \cdot v_2 = 0$, whence $v_1 \perp v_2$. Now observe that

$$\begin{aligned}\big(f(\alpha X) - \alpha f(X)\big) \cdot v_1 &= f(\alpha X) \cdot v_1 - \alpha f(X) \cdot v_1 \\ &= f(\alpha X) \cdot f(\mathbf{1}) - \alpha f(X) \cdot f(\mathbf{1}) \\ &= \alpha X \cdot \mathbf{1} - \alpha(X \cdot \mathbf{1}) \\ &= 0.\end{aligned}$$

Similarly, we see that $\big(f(\alpha X) - \alpha f(X)\big) \cdot v_2 = 0$. This implies that the vector $f(\alpha X) - \alpha f(X)$ is perpendicular to both v_1 and v_2. But this is impossible unless

$$f(\alpha X) - \alpha f(X) = 0,$$

which proves (1). A similar proof (on your homework) shows that $f(X + Y) = f(X) + f(Y)$ for all $X, Y \in \mathbb{R}^2$. Combining these shows that

$$f(\alpha X + \beta Y) = f(\alpha X) + f(\beta Y) = \alpha f(X) + \beta f(Y)$$

as claimed. //

<u>STEP 4:</u> $f = R_\alpha \circ \rho^j$ for some α and j.

Recall our notation $v_1 = f(\mathbf{1})$ and $v_2 = f(\mathbf{i})$. Since $|v_1| = 1$, we can write it in the form

$$v_1 = e^{i\alpha}.$$

Since $v_1 \perp v_2$, we have

$$v_2 = e^{i(\alpha \pm \pi/2)} = e^{i\alpha} e^{\pm i\pi/2} = \pm i e^{i\alpha}.$$

Finally, by linearity, we see that for an arbitrary point $\alpha + \beta i$ we have

$$\begin{aligned}f(\alpha + \beta i) &= f(\alpha\,\mathbf{1} + \beta\,\mathbf{i}) \\ &= \alpha v_1 + \beta v_2 \\ &= \alpha e^{i\alpha} \pm \beta i e^{i\alpha} \\ &= e^{i\alpha}(\alpha \pm \beta i) \\ &= R_\alpha(\alpha + \beta i) \quad \text{or} \quad R_\alpha \circ \rho(\alpha + \beta i)\end{aligned}$$

as claimed. //

We are now striking distance from finishing the proof of the Lemma. We shall do this on Friday, as well as mount an attack on the classification theorem. □

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