GROUPS AND SYMMETRY: LECTURE 8

LEO GOLDMAKHER

We started with an abstract question: what does it mean for a shape to be symmetric? For example, what makes a square symmetric? Building on ideas of Dan and Eric, we eventually arrived at the following notion:

Definition. *Given $S \subseteq \mathbb{R}^2$, a* symmetry *of S is any $\gamma \in \mathcal{G}$ such that $\gamma(S) = S$. (Recall that $\mathcal{G}$ denotes the set of all isometries.) Further, we denoted the set of* all *isometries of S by $\mathcal{G}_S$. In symbols:*

$$\mathcal{G}_S := \{\gamma \in \mathcal{G} : \gamma(S) = S\}.$$

Note that $\mathbb{1} \in \mathcal{G}_S$ automatically, for any S (even the empty set!).

We next discussed a couple of examples. First, what is $\mathcal{G}_{\{0\}}$? Dinu pointed out that R_α and ρ both live in $\mathcal{G}_{\{0\}}$. We then noted that compositions of these must also live in $\mathcal{G}_{\{0\}}$. What else? Building on an idea of Nakita, we proved the following:

Proposition 1. $\mathcal{G}_{\{0\}} = \big\{R_\alpha \rho^j : \alpha \in [0, 2\pi), j \in \{0, 1\}\big\}$.

Proof. Given $\phi \in \mathcal{G}_{\{0\}}$. By definition, this means $\phi \in \mathcal{G}$ and $\phi(0) = 0$. By our Lemma, we know that ϕ can be written (uniquely) in the form

$$\phi = T_h R_\alpha \rho^j.$$

Thus $0 = \phi(0) = T_h R_\alpha \rho^j(0) = h$. This shows that $\mathcal{G}_{\{0\}} \subseteq \{R_\alpha \rho^j\}$. On the other hand, we've already seen that $\mathcal{G}_{\{0\}} \supseteq \{R_\alpha \rho^j\}$. Thus the two sets are equal, as claimed. □

Next, we turned to symmetries of the square. To make the discussion more precise, consider the four points $\{\pm 1 \pm i\}$; these are the four vertices of a square centered at the origin. What is $\mathcal{G}_{\{\pm 1 \pm i\}}$? Xiang came up with eight isometries which live in this space: $\mathbb{1}$, $R_{\pi/2}$, R_π, $R_{3\pi/2}$, ρ, $\rho R_{\pi/2}$, ρR_π, and $\rho R_{3\pi/2}$. Are there others? Dan suggested the reflection across the line $y = x$, but Theepi observed that this is just $R_{\pi/2}$ in disguise. A couple of other suggestions were seen to be already on our list, albeit written in a different form. Are these all eight isometries all the symmetries of the square? We'll resume the discussion on Friday.

WWW.MATH.TORONTO.EDU/LGOLDMAK/C01F13/
E-mail address: lgoldmak@math.toronto.edu

Date: September 30, 2013.