Instructor: Leo Goldmakher

NAME: ____________________________

**University of Toronto Scarborough**
**Department of Computer and Mathematical Sciences**

# MATC01: GROUPS AND SYMMETRY

**Problem Set 6 – due Monday, October 28st**

**INSTRUCTIONS:**
To receive credit, you must turn this in during the first 5 minutes of lecture on the due date. Please print and attach this page as the first page of your submitted problem set.

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Please read the following statement and sign below:

*I understand that I am not allowed to use the internet to assist (in any way) with this assignment. I also understand that I must write down the final version of my assignment in isolation from any other person.*

**SIGNATURE:**____________________________

# Problem Set 6

**6.1** Let $GL_2(\mathbb{Q})$ denote the set of all $2 \times 2$ matrices with rational entries and nonzero determinent. Prove that $GL_2(\mathbb{Q})$ is a group under matrix multiplication.

**6.2** Prove that $\Big(\{0,1\}, \oplus\Big)$ is a group, where $\oplus$ is defined by the following table:

| $\oplus$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 1 | 0 |

**6.3** Make the set $\{0,1,2\}$ into a group, by defining a binary operation on it. Present your operation in the form of a table as above.

**6.4** (*Courtesy of V. Blomer*) Let $c$ be the speed of light. According to Einstein, two velocities $v_1, v_2$ pointing in the same direction can be added by the rule

$$v_1 @ v_2 := \frac{v_1 + v_2}{1 + v_1 v_2 / c^2}.$$

(Imagine a person walking inside a moving train from the back to the front of the train; his own speed and the speed of the train are "added" by the rule above.) Prove that the interval $I = (-c, c)$ under this operation is a group, i.e. show that $I$ is closed under the operation, and verify that the group axioms are satisfied. Note that according to this model, nothing can move faster than the speed of light. [*Hint: to prove closure, consider the product* $(c+v_1)(c+v_2)$. *Why must it be positive? Deduce from this that* $v_1 @ v_2 > -c$. *Now use a similar trick to prove the upper bound* $v_1 @ v_2 < c$.]

**6.5** Let $\mathcal{G}_{\{\pm 1 \pm i\}}$ denote the symmetries of $\{\pm 1 \pm i\}$. Prove that this is a group.

**6.6** For each of the following, list all the ways in which it fails to be a group. Whenever a group axiom fails to be satisfied, give an example illustrating the failure.

(a) $\big(\mathbb{Z}^*, \times\big)$ where $\mathbb{Z}^*$ is the set of all non-zero integers and $\times$ denotes ordinary multiplication.

(b) The set of all subsets of $\{1,2,3,4,5,6,7,8,9,10\}$ under the operation $\cap$. (Recall that given any two sets $A$ and $B$, their intersection (written $A \cap B$) is the set consisting of all elements belonging to *both* $A$ and $B$.)

(c) The set of all positive integers, under the operation @ defined by

$$a @ b := \gcd(a, b).$$

(Recall that given two positive integers $a$ and $b$, the *greatest common divisor* of $a$ and $b$, denoted $\gcd(a,b)$, is the largest positive integer dividing both $a$ and $b$.)

(d) The set of all positive integers, under the operation $\oplus$ defined by

$$a \oplus b := \operatorname{lcm}(a, b).$$

(Recall that given two positive integers $a$ and $b$, the *least common multiple* of $a$ and $b$, denoted $\operatorname{lcm}(a,b)$, is the smallest positive integer which is a multiple of both $a$ and $b$.)

(e) The set of all non-negative integers (i.e. $\{0,1,2,\ldots\}$), under the operation $\odot$ defined by

$$a \odot b := |a - b|.$$

(In other words, $a \odot b$ is the distance between $a$ and $b$.)