LECTURE 8: SUMMARY

Last time we discussed the Prime Number Theorem, which asserts that

$$\pi(x) \sim \frac{x}{\log x}.$$

(We also discussed the Riemann Hypothesis, which is a more precise form of this.) We won't be able to prove this theorem in this course, since the proof requires a background in complex analysis. Instead, we devote the next couple of lectures to proving a somewhat weaker version of this theorem, due to Chebyshev in the 1850s:

Theorem 1. *There exist positive constants a and b such that*

$$\frac{ax}{\log x} < \pi(x) < \frac{bx}{\log x}$$

for all $x \geq 2$.

We started by reviewing the symbol $\binom{n}{k}$, read 'n choose k'. It represents the number of ways of choosing k objects out of a set of n objects. For example, $\binom{4}{2} = 6$. There's an explicit formula for this quantity:

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}.$$

Perhaps the most famous appearance of this is in the binomial theorem:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

Having review all this, we are now ready to begin proving Chebyshev's theorem. We begin by proving an upper bound on $\pi(x)$. Our approach is due to Ramanujan, who significantly simplified Chebyshev's original method.

Consider the binomial coefficient $\binom{2n}{n}$. We will bound it from above and below; comparing the two bounds will give us Chebyshev's upper bound. We have

$$\binom{2n}{n} < \sum_{k=0}^{2n} \binom{2n}{k} = (1+1)^{2n} = 2^{2n}. \tag{†}$$

On the other hand, observe that if a prime $p \leq 2n$, then $p \mid (2n)!$. If moreover $p > n$, the $p \nmid n!$. This (combined with problem 1.9 from your homework) shows that

$$\Big(\prod_{p \in (n, 2n]} p\Big) \,\Big|\, \binom{2n}{n}.$$

Date: January 31st, 2013.

We deduce that

$$\binom{2n}{n} \geq \prod_{p \in (n, 2n]} p > \prod_{p \in (n, 2n]} n = n^{\pi(2n) - \pi(n)}. \tag{‡}$$

Combining equations (†) and (‡), taking logs, and simplifying, we find

$$\pi(2n) - \pi(n) < \frac{(2 \log 2)n}{\log n}.$$

This already looks quite promising; next lecture we'll conclude the proof of Chebyshev's upper bound with relative ease, and move on the lower bound.