LECTURE 9: SUMMARY

Last time we proved the relation

$$\pi(2n) - \pi(n) < \frac{2n \log 2}{\log n}. \tag{*}$$

The first half of today's lecture was devoted to deducing from this a more explicit result:

Theorem 1. *For all sufficiently large integers n, we have*

$$\pi(n) \leq \frac{2n}{\log n}.$$

Note that you will figure out the precise meaning of 'sufficiently large' in your next problem set.

Proof sketch. We proceed by induction. Assume n is large, and suppose that the theorem has been proved for all (sufficiently large) $k < n$. Our aim is to show that the claimed bound also holds for n. There are two cases.

- n is even.

  By (*) and induction, we have

  $$\begin{aligned} \pi(n) &< \pi(n/2) + \frac{n \log 2}{\log \frac{n}{2}} \\ &\leq \frac{n}{\log \frac{n}{2}} + \frac{n \log 2}{\log \frac{n}{2}} \\ &= \frac{(1 + \log 2)n}{\log \frac{n}{2}} \end{aligned}$$

  It is an exercise to show that

  $$\frac{(1 + \log 2)n}{\log \frac{n}{2}} \leq \frac{2n}{\log n}$$

  for all sufficiently large n. This concludes the proof for large, even n.
- n is odd.

  First, observe that $\pi(n) \leq \pi(n-1) + 1$. It follows, by induction and (*), that

  $$\begin{aligned} \pi(n) &\leq \pi(n-1) + 1 \\ &< \pi\Big(\frac{n-1}{2}\Big) + \frac{(n-1)\log 2}{\log \frac{n-1}{2}} + 1 \\ &\leq \frac{n-1}{\log \frac{n-1}{2}} + \frac{(n-1)\log 2}{\log \frac{n-1}{2}} + 1. \end{aligned}$$

  As before, it is an exercise to prove that this is bounded above by $\frac{2n}{\log n}$. □

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Aside from the constant 2, this upper bound cannot be improved:

Theorem 2. *For all sufficiently large n we have*

$$\pi(n) \geq \frac{\log 2}{2} \cdot \frac{n}{\log n}.$$

Proof (due to M. Nair, 1982). Let

$$I_n := \int_0^1 x^n (1-x)^n \, dx.$$

Note that I_n is positive. Expanding the integrand by the binomial theorem, exchanging the order of integration and summation, and simplifying, yields an expression of the form

$$I_n = \frac{a_1}{n+1} + \frac{a_2}{n+2} + \cdots + \frac{a_{n+1}}{2n+1}$$

where $a_i \in \mathbb{Z}$ for all i. It follows that $I_n \cdot [n+1, n+2, \ldots, 2n+1]$ is a positive integer. (Here $[n_1, n_2, \ldots]$ denotes the least common multiple of the n_i.) In particular, we deduce that

$$I_n \cdot [n+1, n+2, \ldots, 2n+1] \geq 1.$$

Since $I_n \leq \frac{1}{4^n}$ (as you will prove on your assignment), it follows that

$$[n+1, n+2, \ldots, 2n+1] \geq 4^n. \tag{†}$$

By the Fundamental Theorem of Arithmetic, we can write

$$[n+1, n+2, \ldots, 2n+1] = \prod_p p^{a_p}$$

where the product runs over all primes p (and the a_p are uniquely determined non-negative integers). From the definition of the LCM, there must exist some $k \in \{n+1, n+2, \ldots, 2n+1\}$ such that $p^{a_p} \mid k$; it follows that $p^{a_p} \leq k \leq 2n+1$ for all p. Moreover, it is clear that $a_p = 0$ for all $p > 2n+1$. Thus, we have

$$[n+1, n+2, \ldots, 2n+1] = \prod_p p^{a_p} \leq \prod_{p \leq 2n+1} (2n+1) = (2n+1)^{\pi(2n+1)}.$$

Combining this with (†) yields

$$(2n+1)^{\pi(2n+1)} \geq 4^n.$$

Taking logs and simplifying gives

$$\pi(2n+1) \geq \frac{2n \log 2}{\log(2n+1)}.$$

The latter quantity is always larger than the lower bound claimed in the theorem, so it suffices to prove the claim for even inputs. But in this case,

$$\pi(2n) \geq \pi(2n+1) - 1 \geq \frac{2n \log 2}{\log(2n+1)} - 1$$

which can be shown to exceed the claimed bound for all sufficiently large n. □

Combining our two theorems, we conclude that there exist positive constants a, b such that

$$\frac{an}{\log n} < \pi(n) < \frac{bn}{\log n}$$

for all $n \geq 2$.