Assignment 2 Solutions

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(1) Note that $n^2 - 1 = (n-1)(n+1)$. Hence $n^2 - 1$ is prime if and only if $n - 1 = 1$.

(2) (a) If $d \mid n$ and $d \mid n+1$, then d divides their difference: $d \mid (n+1) - n = 1$. Hence $d = 1$.
(b) If $d \mid n(n+1)+1$ and $d \mid n+1$, then $d \mid (n(n+1)+1) - (n+1)(n) = 1$, hence $d = 1$.
(c) Observing the pattern, we see that $(n(n+1)+1)(n+1)(n)+1$ is relatively prime to n, $n+1$ and $n(n+1)+1$.
(d) We see that we can create infinitely many numbers that are relatively prime to each other by multiplying all the existing ones and adding 1 to them. By Fundamental Thm of Arithmetic (FTA), the k^{th} term of this sequence is divisible by some prime p_k. All these primes must be distinct (else those two terms wouldn't be relatively prime).

(3) (a) Let $1 \leqslant a \leqslant b \leqslant n$, $x = 1 + a(n!)$ and $y = 1 + b(n!)$. So x and y both belong to A_n. We want to show that they are relatively prime. Towards a contradiction, assume they are not. Then there must be a prime number p such that $p|x$ and $p|y$. Then p divides their difference as well, that is, $p|y - x = (b-a)n!$. Since p is a prime number, we either have $p|n!$ or $p|(b-a)$. If $p|n!$, however, then $p|x - a(n!) = 1$, which is not possible. Since p does not divide $n!$, we get that $p > n$. Now we should have $p|(b-a)$, however, $b - a$ is less than n, hence this is not possible either. We get a contradiction.
(b) We can keep contructing larger and larger sets A_n, which consist of relatively prime numbers. As in question 2(d), we will get infinitely prime numbers.

(4) Note that $\frac{(\log x)^k}{x} \to 0$ if and only if $(\frac{(\log x)^k}{x})^{1/k} = \frac{\log x}{x^{1/k}} \to 0$ as x goes to infinity, so we will show the latter. Both $\log x$ and $x^{1/k}$ tend to ∞ with x, so we may apply L'Hôpital's rule:

$$\frac{\frac{1}{x}}{\frac{x^{(1-k)/k}}{k}} = \frac{k}{x^{1/k}},$$

which clearly goes to 0 as $x \to \infty$.

(5) We again use L'Hôpital's rule. First, we need to check that both functions go to infinity. We have $\log t \leqslant t$ for all $t \geqslant 2$, whence

$$\int_2^x \frac{dt}{\log t} \geqslant \int_2^x \frac{dt}{t} = \log x - \log 2$$

for every $x \geqslant 2$; it immediately follows that $\displaystyle\int_2^x \frac{dt}{\log t}$ tends to infinity with x. Similarly, $\log x \leqslant \sqrt{x}$ for all $x > 0$, whence

$$\frac{x}{\log x} \geqslant \sqrt{x}.$$

It follows that $\frac{x}{\log x}$ also tends to infinity with x.

Now we can apply L'Hôpital's rule:

$$\lim_{x\to\infty} \frac{\int_2^x \frac{dt}{\log t}}{\frac{x}{\log x}} = \lim_{x\to\infty} \frac{\frac{1}{\log x}}{\frac{\log x - 1}{(\log x)^2}} = \lim_{x\to\infty} \frac{1}{1 - \frac{1}{\log x}} = 1.$$

Hence $\int_2^x \frac{dt}{\log t} \sim \frac{x}{\log x}$.

(6) (a) We solve for $\log x$:

$$\frac{\log x}{\log 2} + 1 \leqslant 2\log x$$
$$\log 2 \leqslant \log x(2\log 2 - 1)$$
$$\log x \geqslant \frac{\log 2}{2\log 2 - 1}$$
$$x \geqslant e^{\frac{\log 2}{2\log 2 - 1}}.$$

(b) We solve for $\log x$:

$$\frac{\log x}{\log 3} + 1 \leqslant \log x$$
$$\log 3 \leqslant \log x(\log 3 - 1)$$
$$\log x \geqslant \frac{\log 3}{\log 3 - 1}$$
$$x \geqslant e^{\frac{\log 3}{\log 3 - 1}}.$$

(c) We solve for $\log n$:

$$\frac{(1+\log 2)n}{\log \frac{n}{2}} \leqslant \frac{2n}{\log n}$$
$$\frac{(1+\log 2)}{\log n - \log 2} \leqslant \frac{2}{\log n}$$
$$(1+\log 2)n\log n \leqslant 2n(\log n - \log 2)$$
$$(1+\log 2)\log n \leqslant 2\log n - 2\log 2$$
$$\log n(2 - 1 - \log 2) \geqslant 2\log 2$$
$$\log n \geqslant \frac{2\log 2}{1 - \log 2}$$
$$n \geqslant e^{\frac{\log 4}{1-\log 2}}.$$

(d) This is a bit more complicated than the previous parts, but the idea is straightforward. We are trying to show that

$$\frac{2n\log 2}{\log(2n+1)} - 1 \geqslant \frac{\log 2}{2} \cdot \frac{2n}{\log(2n)}$$

for all large n. We will prove this by showing that

$$\text{LHS} \geqslant \frac{\text{n}\log 2}{\log \text{n}} \geqslant \text{RHS}$$

whenever n is large enough. The second inequality is easily seen to hold for all n, so we focus on the first inequality. This is equivalent to showing

$$\frac{1}{2n\log 2} + \frac{1}{2\log n} \leqslant \frac{1}{\log(2n+1)}.$$

To prove this, it suffices to show

$$\frac{1}{n} + \frac{1}{2\log n} \leqslant \frac{1}{1+\log n}, \qquad (*)$$

since $\frac{1}{2n\log 2} \leqslant \frac{1}{n}$ for all n and $\frac{1}{1+\log n} = \frac{1}{\log(en)} \leqslant \frac{1}{\log(2n+1)}$ for all $n \geqslant 2$. Now, to prove (*) it suffices to prove

$$\frac{1}{n} + \frac{1}{2\log n} \leqslant \frac{1}{\frac{6}{5}\log n} \qquad (**)$$

because $\frac{1}{\frac{6}{5}\log n} \leqslant \frac{1}{\log(2n+1)}$ whenever $n \geqslant e^5$. The bound (**) is equivalent to

$$3\log n \leqslant n,$$

which holds for all $n \geqslant e^2$ (a good calculus exercise; compare the initial values / derivatives). This proves the claim.

(7) It is obvious that $x^n(1-x)^n$ is positive for $0 \leqslant x \leqslant 1$. Hence $0 \leqslant I_n$. For the other bound, it is an easy calculus exercise to find that the maximum of $x(1-x)$ on the interval $[0,1]$ occurs at $x = 1/2$; it follows that

$$I_n = \int_0^1 x^n(1-x)^n\,dx \leqslant \int_0^1 \frac{1}{2^n}\Big(1-\frac{1}{2}\Big)^n\,dx = \int_0^1 \frac{1}{4^n}\,dx = \frac{1}{4^n}$$

as claimed.