# EVALUATING GAUSS SUMS VIA POISSON SUMMATION

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ABSTRACT. Gauss famously discovered that the magnitude of the gauss sum (mod $p$) is precisely $\sqrt{p}$, and for quadratic gauss sums he went further and determined the value of the sum itself. We present Dirichlet's 1835 proof of Gauss' formula, with the language modernized.

Recall that the fourier transform of a function $f : \mathbb{Z}/q\mathbb{Z} \to \mathbb{C}$ is defined

$$\widehat{f}(a) := \frac{1}{\sqrt{q}} \sum_{n \leq q} f(n) e\left(-\frac{an}{q}\right),$$

where $e(\alpha) := e^{2\pi i \alpha}$ and the range of summation is over all positive integers $n$. It turns out that the fourier transform of a primitive Dirichlet character $\chi$ (mod $q$) is proportional to the complex conjugate of $\chi$:

$$\widehat{\chi}(a) = \frac{\chi(-1)\tau(\chi)}{\sqrt{q}} \overline{\chi}(a), \qquad \text{where} \qquad \tau(\chi) := \sum_{n \leq q} \chi(n) e\left(\frac{n}{q}\right).$$

When $(a, q) = 1$, this formula holds for imprimitive characters as well, and follows from a straightforward change of variables. By contrast, the case $(a, q) > 1$ requires primitivity:

**Exercise 1.** *Throughout, let $\chi$ (mod $q$) denote a primitive character, and let $d$ denote a proper divisor of $q$.*

*(a) Prove that* $\displaystyle\sum_{\substack{n \leq q \\ n \equiv r \ (\mathrm{mod}\ d)}} \chi(n) = 0$ *for every $r$.* [*Hint: If not, construct a character (mod $d$) inducing $\chi$.*]

*(b) Deduce that* $\sum_{n \leq q} \chi(n) e\left(\frac{an}{d}\right) = 0$. [*Hint: Rewrite as a double sum and use part (a).*]

*(d) Conclude that $\widehat{\chi}(a) = 0$ whenever $(a, q) > 1$.*

From the formula above we see that the behavior of $\widehat{\chi}$ is transparent, apart from the factor $\tau(\chi)$. This quantity is called the *gauss sum* and is ubiquitous in number theory, for example appearing in the functional equation of Dirichlet $L$-functions and in the study of quadratic forms. In applications it's desirable to have a simple way to evaluate $\tau(\chi)$, but this seems to be a difficult problem; the only cases for which we know a simple expression for $\tau(\chi)$ are for real characters $\chi$, a result going back to Gauss. What is known for arbitrary characters, however, is the magnitude of the gauss sum:

**Theorem 2.** *For any primitive $\chi$ (mod $q$) we have $|\tau(\chi)| = \sqrt{q}$ (or equivalently, $|\widehat{\chi}| = |\chi|$).*

*Proof.* Expanding the modulus squared, making the change of variable $h = mn^{-1}$, and applying Exercise 1 to extend the sum to all of $\mathbb{Z}/q\mathbb{Z}$ yields

$$\begin{aligned}
|\tau(\chi)|^2 &= \sum_{m,n \leq q} \chi(m)\overline{\chi(n)} e\left(\frac{m-n}{q}\right) = \sum_{h,n \in (\mathbb{Z}/q\mathbb{Z})^\times} \chi(h) e\left(\frac{hn-n}{q}\right) \\
&= \sum_{n \in (\mathbb{Z}/q\mathbb{Z})^\times} e\left(-\frac{n}{q}\right) \sum_{h \in (\mathbb{Z}/q\mathbb{Z})^\times} \chi(h) e\left(\frac{hn}{q}\right) = \sum_{n \leq q} e\left(-\frac{n}{q}\right) \sum_{h \leq q} \chi(h) e\left(\frac{hn}{q}\right) \\
&= \sum_{h \leq q} \chi(h) \sum_{n \leq q} e\left(\frac{(h-1)n}{q}\right).
\end{aligned}$$

The inner sum on the last line vanishes unless $h = 1$, and the claim follows. $\square$

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As Gauss discovered, computing the precise value of the gauss sum is more challenging. Since $\tau(\chi)$ is closely related to the fourier transform of $\chi$, it's natural to employ fourier analysis techniques to study it. Dirichlet was the first to explicitly do this, and his approach (which we describe below) gives the cleanest proof of Gauss' theorem.

**Theorem 3** (Gauss)**.** *If $\chi \pmod p$ is the nontrivial real character, then*

$$\tau(\chi) = \begin{cases} \sqrt{p} & \text{if } p \equiv 1 \pmod 4 \\ i\sqrt{p} & \text{if } p \equiv -1 \pmod 4. \end{cases}$$

*Remark.* We can restate this in terms of fourier transforms: for quadratic $\chi \pmod p$ we have

$$\widehat{\chi} = \begin{cases} \chi & \text{if } p \equiv 1 \pmod 4 \\ -i\chi & \text{if } p \equiv -1 \pmod 4. \end{cases}$$

In other words, quadratic characters are eigenfunctions of the fourier transform, with eigenvalue either $1$ or $-i$.

One of the primary challenges in studying gauss sums is that they mix a multiplicative character with an additive one; indeed, the key idea in the proof of Theorem 2 was to disentangle the two. Although we cannot emulate this when evaluating $\tau(\chi)$, it turns out that when $\chi$ is quadratic we can rewrite the sum in a way that doesn't involve multiplicative characters at all.

**Exercise 4.** *Prove that $\tau(\chi) = \sum_{n \in \mathbb{F}_p} e\left(\frac{n^2}{p}\right)$ for the quadratic character $\chi \pmod p$.*

We can thus interpret a quadratic gauss sum as a discrete analogue of the gaussian integral. Indeed, in the proof below we will see that this analogy is made rigorous.

*Dirichlet's proof of Theorem 3.* Recall Poisson summation: for any $f$ that's piecewise smooth and of compact support,

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{k \in \mathbb{Z}} \widehat{f}(k),$$

where $\widehat{f}$ denotes the fourier *transform* of $f$. As usual in fourier analysis, for this to hold we need to interpret $f$ as being slightly fuzzy: $f(n)$ doesn't literally mean the value of $f$ at $n$, but rather, the typical behavior in a neighborhood of $n$. If $f$ is continuous at $n$ then this really is the value at $n$, but if, for example, $f$ has a jump discontinuity at $n$, then we interpret $f(n)$ as the average of the limits from the left and from the right.

Dirichlet's key idea is to use Poisson summation by choosing $f$ to make $\sum_{n \in \mathbb{Z}} f(n) = \tau(\chi)$. The easiest way to do this is to set $f(x) := e\left(\frac{x^2}{p}\right)$ on $[0, p]$ and $f(x) := 0$ otherwise, so that Poisson summation yields

$$\sum_{n \leq p} e\left(\frac{n^2}{p}\right) = \sum_{k \in \mathbb{Z}} \widehat{f}(k). \tag{1}$$

(As usual, the range $n \leq p$ on the left hand side denotes all *positive* integers $\leq p$.) Writing out the definition of the fourier transform and applying standard tricks—completing the square and changing variables—we find

$$\widehat{f}(k) = e\left(-\frac{pk^2}{4}\right) \sqrt{p} \int_{-\frac{k}{2}\sqrt{p}}^{\left(1-\frac{k}{2}\right)\sqrt{p}} e\left(x^2\right)\, dx.$$

In particular, we deduce that

$$\sum_{k \text{ even}} \widehat{f}(k) = \sqrt{p} \int_{-\infty}^{\infty} e\left(x^2\right)\, dx \qquad \text{and} \qquad \sum_{k \text{ odd}} \widehat{f}(k) = e\left(-\frac{p}{4}\right) \sqrt{p} \int_{-\infty}^{\infty} e\left(x^2\right)\, dx.$$

Equation (1) therefore implies

$$\sum_{n \le p} e\left(\frac{n^2}{p}\right) = \left(1 + e\left(-\frac{p}{4}\right)\right) \sqrt{p} \int_{-\infty}^{\infty} e\left(x^2\right)\, dx. \tag{2}$$

It remains only to compute the value of the integral on the right hand side. With some elbow grease this can be done directly, but there's a slicker approach that eschews calculus. Observe that nowhere in the above argument did we require that $p$ be prime. We may thus take $p = 1$ in equation (2), which implies

$$\int_{-\infty}^{\infty} e\left(x^2\right)\, dx = \frac{1}{1-i}.$$

Plugging this back into (2) and applying Exercise 4 yields the claim. □

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