FURSTENBERG'S TOPOLOGICAL PROOF OF THE INFINITUDE OF PRIMES

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Recall that if X is a set, a collection $\mathcal{B}$ of subsets of X is a *basis* for a topology on X if

(1) every element of X is contained in a basis element, and
(2) if $x \in B_1 \cap B_2$ for two basis elements B_i, there exists a basis element B_3 such that $x \in B_3 \subseteq B_1 \cap B_2$.

A set $S \subseteq X$ is called *open* in the topology generated by a basis $\mathcal{B}$ if for every $s \in S$, there is a basis element N such that $s \in N \subseteq S$. A set is *closed* iff its complement is open. Finally, closure is preserved under arbitrary intersections and finite unions.

Theorem 1. *There are infinitely many primes.*

Proof. Let $\mathcal{B}$ be the collection of all bi-infinite arithmetic progressions in $\mathbb{Z}$. This is easily checked to be a basis for a topology on $\mathbb{Z}$, in which $A \subseteq \mathbb{Z}$ is open iff it is a union of bi-infinite arithmetic progressions. In particular, for any prime p, the set $p\mathbb{Z}$ is both open and closed (since the complement of $p\mathbb{Z}$ is a union of $p-1$ arithmetic progressions).

Suppose there were only finitely many primes. Then the set

$$\bigcup_p p\mathbb{Z}$$

would be closed. But the complement of this set is $\{\pm 1\}$, which is clearly not open. This contradiction shows that there must be infinitely many primes. $\square$