WIGDERSON & WIGDERSON'S APPROACH TO UNCERTAINTY PRINCIPLES

LEO GOLDMAKHER

ABSTRACT. Heisenberg's uncertainty principle, originally observed in the context of physics, was quickly recognized as a general mathematical phenomenon: a function and its fourier transform cannot simultaneously be too localized. Since the original uncertainty principle, many variants have been discovered. Most of these uncertainty principles have ad hoc proofs relying on special properties of the relevant fourier transform, but Avi and Yuval Wigderson recently discovered a simple unifying framework that produces proofs of many different versions. Moreover, their approach yields new uncertainty principles, and generalizes beyond the fourier transform to a much larger class of linear operators. My goal in this note is to demonstrate their approach in a very concrete case: Heisenberg's original uncertainty principle. Nothing in this essay is original, and my primary motivation to write it is the hope that the concrete example presented will inspire the reader to play more with the ideas and read Wigderson & Wigderson's manuscript, which not only covers far more material but is also beautifully written.

0. IMPLICIT ASSUMPTIONS

I will assume the reader has previously encountered the fourier transform, Plancherel's theorem, L^p norms, and the Cauchy-Schwarz inequality, but for ease of reference all these are defined in Appendix A. Throughout, I'll assume all functions are nice, without specifying precisely what that means; going through the arguments carefully and coming up with sufficiently strong hypotheses is probably a good character-building exercise.

1. HEISENBERG'S UNCERTAINTY PRINCIPLE(S)

In 1927, the 26-year-old physicist Werner Heisenberg (already famous for his foundational work on matrix mechanics two years prior) proposed that it's not possible to specify both the position and the velocity of an object at any given moment, or even to approximate both quantities to arbitrary precision. This was a shocking departure from classical mechanics, where one of the aims is to make predictions about a particle's motion based on some initial conditions; Heisenberg's uncertainty principle implies that one *can never know* the initial conditions, rendering the whole exercise moot!

Heisenberg originally formulated his result in terms of a certain measure of imprecision of position and momentum. Inspired by the fact that the probability densities of momentum and position are related via a fourier transform, Kennard and Weyl (independently) discovered the following general version of Heisenberg's principle: for any nice function ψ such that $|\psi|^2$ a probability density,

$$\left(\int_{-\infty}^{\infty} x^2|\psi(x)|^2\,dx\right)\left(\int_{-\infty}^{\infty} \xi^2|\widehat{\psi}(\xi)|^2\,d\xi\right) \geq C, \tag{1.1}$$

where C is a positive constant that's independent of ψ. Each factor on the left hand side of (1.1) is the variance of a probability distribution[1], measuring how spread out the distribution is along the real line. Thus we see that it's not possible for both ψ and $\widehat{\psi}$ to be localized. For an illustration of this, see Appendix B: figures 1, 3, and 5 give examples of localized probability distributions, and figures 2, 4, and 6 are their corresponding fourier transforms. In each case, we see that the fourier transforms aren't localized, as predicted by (1.1).

Remark. One can take $C = \frac{1}{16\pi^2}$ in (1.1), and this is known to be optimal. I'm suppressing the precise value of C in (1.1) to highlight that it's not the size of the constant, but merely its positivity, that makes this an uncertainty principle. Henceforth I'll use the following shorthand (a notation introduced by Vinogradov):

$$F(x) \gg G(x) \qquad \Longleftrightarrow \qquad \exists k>0 \text{ such that } F(x) \geq k\,G(x). \tag{$*$}$$

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[1] If $|\psi|^2$ is a probability density, then so is $\left|\widehat{\psi}\right|^2$ by Plancherel.

The hypothesis that $|\psi|^2$ is a probability density is an illusory constraint, since we can renormalize any nice ψ to satisfy this. This observation, combined with Plancherel, implies that (1.1) is equivalent to

Theorem 1.1 (Heisenberg Uncertainty Principle)**.** *Let* $L(f) := \int_{-\infty}^{\infty} x^2|f(x)|^2\,dx$*. Then* $L(f)L(\widehat{f}) \gg \|f\|_2^2\,\|\widehat{f}\|_2^2$.

Note that (again by Plancherel) we could have written the lower bound as $\|f\|_2^4$. The form above is more complicated, but emphasizes the duality between f and $\widehat{f}$ and points the way towards generalizations.

2. The prototypical uncertainty principle

When first encountering fourier transforms, one is usually presented with a bevy of relationships (*What happens to the fourier transform when you differentiate the function?*, etc.) but a natural question that gets short shrift is: how large does $\widehat{f}$ get? Right away by triangle inequality we have

$$|\widehat{f}(\xi)| \le \int_{-\infty}^{\infty} |f|.$$

The right hand side is simply $\|f\|_1$, the L^1 norm of f, and is independent of ξ. Thus we deduce

$$\sup_{\xi\in\mathbb{R}} |\widehat{f}(\xi)| \le \|f\|_1.$$

We recognize the quantity on the left hand side as the L^∞ norm, so we've proved

$$\|\widehat{f}\|_\infty \le \|f\|_1.$$

Similarly,

$$\|f\|_\infty \le \|\widehat{f}\|_1.$$

Multiplying these two inequalities produces

$$\frac{\|f\|_1}{\|f\|_\infty} \cdot \frac{\|\widehat{f}\|_1}{\|\widehat{f}\|_\infty} \ge 1. \tag{2.1}$$

Now, we've barely done anything to arrive at (2.1), but Wigderson & Wigderson realized this inequality is a crude uncertainty principle. Indeed, $\frac{\|f\|_1}{\|f\|_\infty}$ tells you something about how spread out f is over the real line: it is large only if the support of f is. Thus the inequality (2.1) asserts that at least one of the two functions f and $\widehat{f}$ must be somewhat spread out—they cannot both be very localized.

What's lovely about this observation is that it inspires other uncertainty principles. For example, is there a version of (2.1) that uses norms other than L^1 and L^∞? By just a bit of playing around, one is led to:

Proposition 2.1. *For any* $q \ge 1$ *we have* $\dfrac{\|f\|_1}{\|f\|_q} \cdot \dfrac{\|\widehat{f}\|_1}{\|\widehat{f}\|_q} \ge 1.$

Proof. Note that $\|f\|_q^q = \int_{-\infty}^{\infty} |f|^q \le \|f\|_\infty^{q-1}\|f\|_1$ whence $\dfrac{\|f\|_1}{\|f\|_q} \ge \left(\dfrac{\|f\|_1}{\|f\|_\infty}\right)^{1-\frac{1}{q}}$. The claim now follows immediately from (2.1). □

More important than the details of this argument is the strategy: to prove an uncertainty principle for the localization measure $\frac{\|f\|_1}{\|f\|_q}$, we bounded it below by a function of the localization measure $\frac{\|f\|_1}{\|f\|_\infty}$, for which we already knew an uncertainty principle. Wigderson & Wigderson use this type of bootstrapping to great effect, deriving many uncertainty principles from Proposition 2.1. Here we'll focus on a single example: Heisenberg's uncertainty principle.

3. Proof of the Heisenberg Uncertainty Principle

Inspired by the strategy from the proof of Proposition 2.1, we wish to deduce

$$L(f)L(\widehat{f}) \gg \|f\|_2^2 \|\widehat{f}\|_2^2 \tag{3.1}$$

from uncertainty principles we've already proved. (Recall that the measure of the localization of f that we're using is $L(f) := \int_{-\infty}^{\infty} x^2 |f(x)|^2 \, dx$.) Given that the L^2 norm is involved, we try to derive (3.1) from the $q = 2$ case of Proposition 2.1:

$$\frac{\|f\|_1}{\|f\|_2} \cdot \frac{\|\widehat{f}\|_1}{\|\widehat{f}\|_2} \geq 1.$$

To get from here to (3.1), it suffices to prove

$$L(f)L(\widehat{f}) \gg \frac{\|f\|_1^4}{\|f\|_2^2} \cdot \frac{\|\widehat{f}\|_1^4}{\|\widehat{f}\|_2^2}.$$

This inspires the following guess:

Claim. *We have $L(h) \gg \frac{\|h\|_1^4}{\|h\|_2^2}$ for any nice function h.*

Proof. We're trying to bound $L(h)$ from below in terms of $\|h\|_1$. Staring at the definitions of these two quantities, we're led to try to relate them using Cauchy-Schwarz:

$$\int_{-\infty}^{\infty} |h| \leq \left(\int_{-\infty}^{\infty} \frac{1}{x^2} \, dx \right)^{1/2} \left(\int_{-\infty}^{\infty} x^2 |h(x)|^2 \, dx \right)^{1/2}$$

This immediately fails, since the first factor on the right hand side diverges. We can fix this by restricting our domain of integration: for any $T > 0$ we have

$$\int_{|x| \geq T} |h(x)| \, dx \leq \left(\int_{|x| \geq T} \frac{1}{x^2} \, dx \right)^{1/2} \left(\int_{|x| \geq T} x^2 |h(x)|^2 \, dx \right)^{1/2} \ll \frac{1}{\sqrt{T}} \sqrt{L(h)}. \tag{3.2}$$

What we would like is to bound the left hand side below by something involving $\|h\|_1$, since that would give us an inequality that looks like our claim. Note that the smaller T is, the more of the mass of h the above integral contains. In particular, we'll choose T small enough to guarantee that at least half the mass of h is outside of $[-T, T]$, or equivalently, that at most half the mass of h is inside the interval $[-T, T]$. We thus seek to bound $\int_{-T}^{T} |h|$ above by $\frac{1}{2}\|h\|_1$. Cauchy-Schwarz implies

$$\int_{-T}^{T} |h| \leq \|\mathbb{1}_{[-T,T]}\|_2 \, \|h\|_2 = \sqrt{2T} \, \|h\|_2.$$

This isn't what we wanted, but now we can cheat: simply choose T to satisfy $\sqrt{2T}\|h\|_2 = \frac{1}{2}\|h\|_1$. Plugging this back into (3.2) we deduce

$$\frac{1}{2}\|h\|_1 \ll \frac{\|h\|_2}{\|h\|_1} \sqrt{L(h)},$$

and the claim immediately follows. □

In fact, the above method yields more general results. For example,

Exercise 3.1. *Prove that $L(f)L(\widehat{f}) \gg_q \|f\|_q^2 \|\widehat{f}\|_q^2$ for any $q \geq 1$. (Here the subscript $\gg_q$ indicates that the implicit constant is allowed to depend on q, but on nothing else.)*

Moreover, because very few properties of the fourier transform were required, it applies to a large class of linear operators. I leave such explorations to the interested reader.

Appendix A. Useful notions

Here are various notions that were used:

- **Fourier transform.** The fourier transform $\widehat{f}$ of a function f is defined

$$\widehat{f}(t) := \int_{-\infty}^{\infty} f(x)e(-tx)\,dx,$$

where $e(\alpha) := e^{2\pi i\alpha}$. If f is nice enough, we can express f in terms of its fourier transform:

$$f(x) := \int_{-\infty}^{\infty} \widehat{f}(t)e(tx)\,dt.$$

- L^q **norms.** Given $q \geq 1$, the L^q norm of nice function f is defined

$$\|f\|_q := \left(\int_{-\infty}^{\infty} |f|^q\right)^{1/q}.$$

The L^∞ norm is defined

$$\|f\|_\infty := \sup_{x\in\mathbb{R}} |f(x)|.$$

- **Plancherel's theorem.** For any nice function, $\|f\|_2 = \|\widehat{f}\|_2$.
- **Support of a function.** The support of a function is the set of all inputs that don't get mapped to 0. More formally, $\mathrm{supp}(f) := \{x \in \mathbb{R} : f(x) \neq 0\}$.
- **Cauchy-Schwarz and Hölder inequalities.** The Cauchy-Schwarz inequality gives a way of bounding the L^1 norm of a product in terms of L^2 norms of the factors:

$$\|fg\|_1 \leq \|f\|_2\|g\|_2.$$

This is a special case of Hölder's inequality: for any real numbers $p, q \geq 1$ with $\frac{1}{p} + \frac{1}{q} = 1$,

$$\|fg\|_1 \leq \|f\|_p\|g\|_q.$$

Appendix B. A few localized probability distributions and their transforms

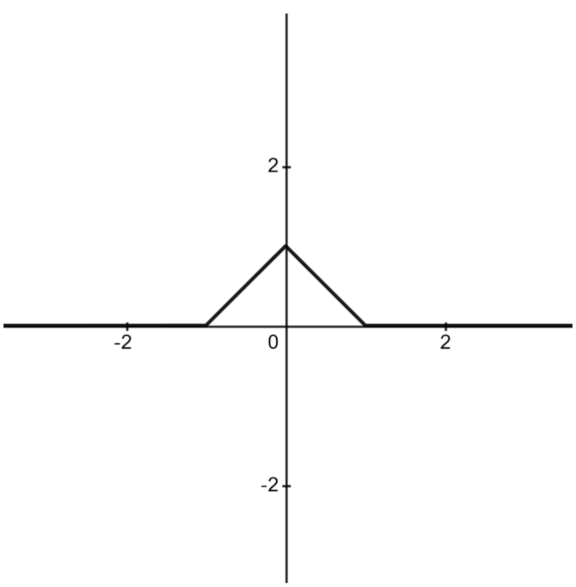

Figure 1. Continuous bump f

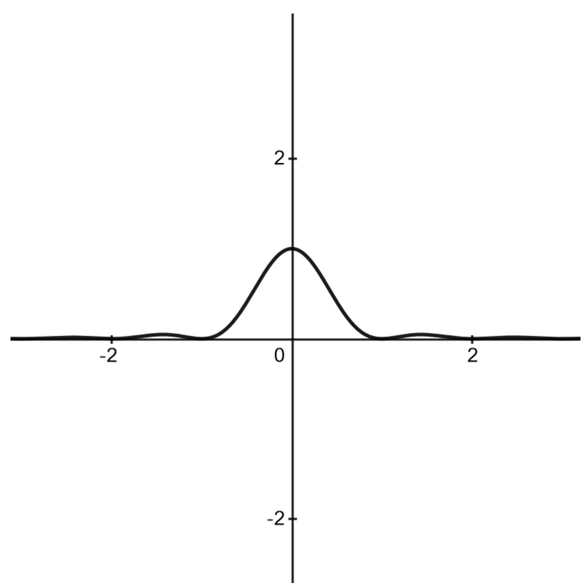

Figure 2. Plot of $\widehat{f}$

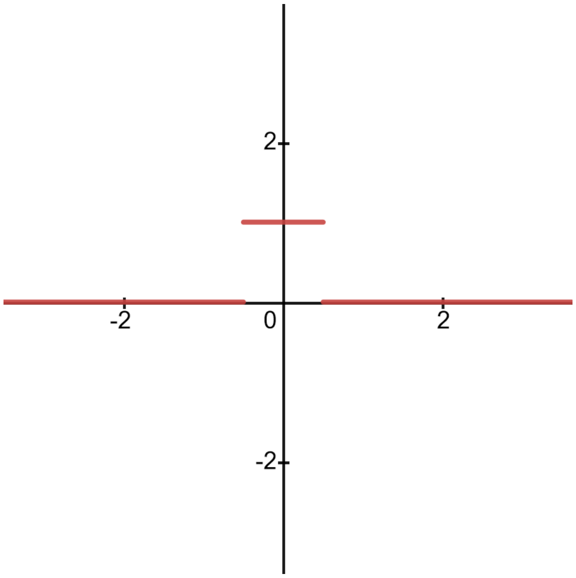

FIGURE 3. Discontinuous bump g

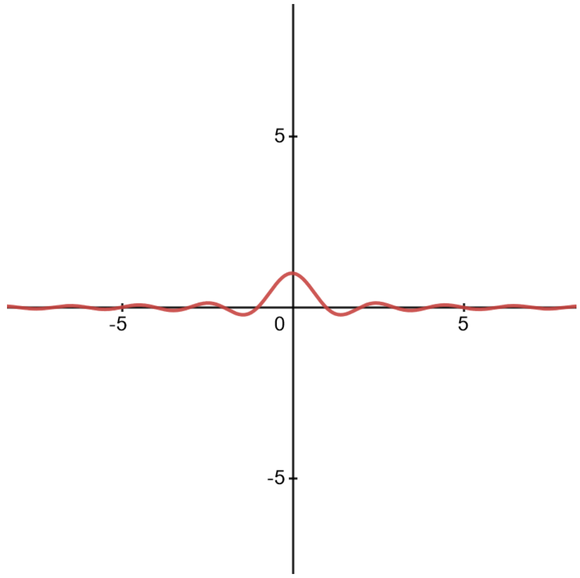

FIGURE 4. Plot of $\widehat{g}$

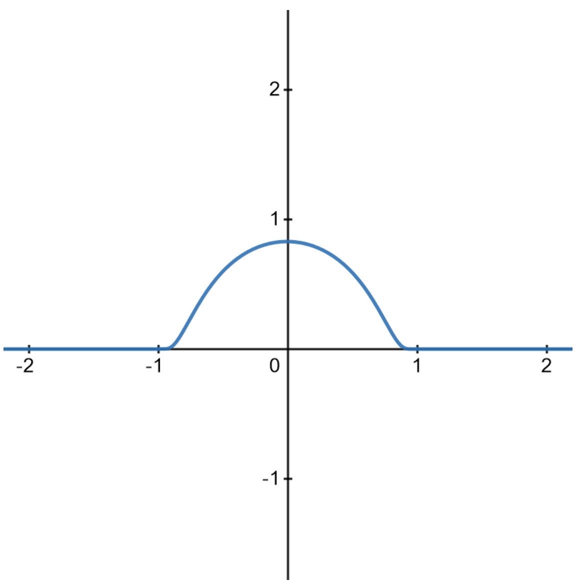

FIGURE 5. Smooth bump h

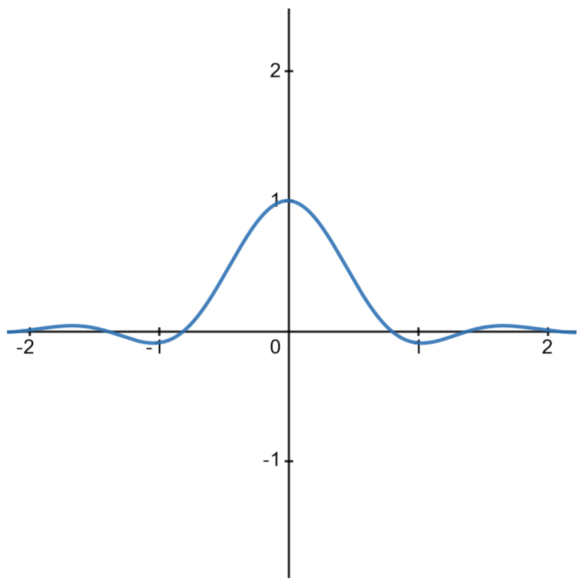

FIGURE 6. Plot of $\widehat{h}$

DEPT OF MATHEMATICS AND STATISTICS, WILLIAMS COLLEGE, WILLIAMSTOWN, MA, USA 01267
Email address: `Leo.Goldmakher@williams.edu`