

A SHORT(ER) PROOF OF THE DIVERGENCE OF THE HARMONIC SERIES

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It is a classical fact that the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

diverges. The standard proof involves grouping larger and larger numbers of consecutive terms, and showing that each grouping exceeds $1/2$. This proof is elegant, but has always struck me as slightly beyond the reach of students – how would one come up with the idea of grouping more and more terms together?

It turns out that one can remove this step without losing the essence of the proof:

Theorem 1. *The harmonic series diverges.*

Proof. Suppose the series converges to H , i.e.

$$H = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots$$

Then

$$\begin{aligned} H &\geq 1 + \frac{1}{2} + \underbrace{\frac{1}{4} + \frac{1}{4}} + \underbrace{\frac{1}{6} + \frac{1}{6}} + \underbrace{\frac{1}{8} + \frac{1}{8}} + \cdots \\ &= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \\ &= \frac{1}{2} + H. \end{aligned}$$

This contradiction concludes the proof. □

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