A QUICK PROOF OF MERTENS' THEOREM

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We first prove a weak form of Stirling's formula:

$$\begin{aligned}\sum_{n\leqslant x}\log n &= \int_1^x \log t\, d[t] = [x]\log x - \int_1^x \frac{[t]}{t}\, dt \\ &= x\log x - \{x\}\log x - x + 1 + \int_1^x \frac{\{x\}}{t}\, dt \\ &= x\log x - x + O(\log x)\end{aligned}$$

We also know that

$$\sum_{d|n}\Lambda(d) = \sum_{p^j|n}\Lambda(p^j) = \sum_{p|n}\sum_{j\leqslant \mathrm{ord}_p(n)}\log p = \sum_{p|n}(\mathrm{ord}_p\, n)\log p = \log n$$

whence

$$\begin{aligned}x\log x - x + O(\log x) &= \sum_{n\leqslant x}\log n = \sum_{n\leqslant x}\sum_{d|n}\Lambda(d) = \sum_{d\leqslant x}\Lambda(d)\left[\frac{x}{d}\right] \\ &= x\sum_{d\leqslant x}\frac{\Lambda(d)}{d} + O\big(\psi(x)\big).\end{aligned}$$

Since $\psi(x) \ll x$, we deduce that

$$\sum_{d\leqslant x}\frac{\Lambda(d)}{d} = \log x + O(1).$$

But this sum is essentially the sum over the primes:

$$\begin{aligned}\sum_{d\leqslant x}\frac{\Lambda(d)}{d} &= \sum_{p\leqslant x}\frac{\log p}{p} + \sum_{p\leqslant x^{1/2}}\frac{\log p}{p^2} + \sum_{p\leqslant x^{1/3}}\frac{\log p}{p^3} + \cdots \\ &= \sum_{p\leqslant x}\frac{\log p}{p} + O\left(\sum_{p\leqslant x^{1/2}}\frac{\log p}{p^2}\left(\frac{1}{1-\frac{1}{p}}\right)\right) \\ &= \sum_{p\leqslant x}\frac{\log p}{p} + O(1).\end{aligned}$$

Thus, setting

$$R(x) = \sum_{p\leqslant x}\frac{\log p}{p} - \log x$$

we have shown that $R(x) \ll 1$. We are now in the position to prove Mertens' estimate:

Proposition. *There exists a constant $C > 0$ such that*

$$\sum_{p \leqslant x} \frac{1}{p} = \log\log x + C + O\left(\frac{1}{\log x}\right)$$

Proof.

$$\begin{aligned}
\sum_{p \leqslant x} \frac{1}{p} &= \int_{2^-}^{x} \frac{1}{\log t}\, d\left(\log t + R(t)\right) \\
&= \frac{\log x + R(x)}{\log x} + \int_2^x \frac{dt}{t\log t} + \int_2^x \frac{R(t)}{t\log^2 t}\, dt \\
&= 1 + O\left(\frac{1}{\log x}\right) + \log\log x - \log\log 2 + \int_2^\infty \frac{R(t)}{t\log^2 t}\, dt - \int_x^\infty \frac{R(t)}{t\log^2 t}\, dt \\
&= \log\log x + \left(\int_2^\infty \frac{R(t)}{t\log^2 t}\, dt + 1 - \log\log 2\right) + O\left(\frac{1}{\log x}\right)
\end{aligned}$$

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