MATH 103 : CALCULUS I : SJMILLER

FIRST LECTURE: INTRO

① Personal

↳ who I am

↳ course details

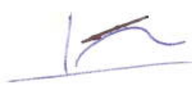
↳ study groups, read before class

② Goals of course

↳ learn Calculus + applications

↳ problem solving techniques: Warning: $\frac{16}{64}, \frac{19}{95}, \frac{26}{65}, \frac{49}{98}$

③ Fundamental Problems

↳ speed: derivative 

↳ distance: integral 

↳ Examples: Monte Carlo: Dartboard (start class with this!)

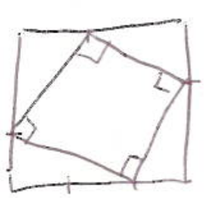
↳ Logarithm: IRS: Benford (Bring Al Capone)

↳ Computer science: efficient algs, disk storage

↳ Optimization problems (disk storage): generalizations: Airlines, MLB

↳ Drowning swimmers, dogs and calc, fractal geometry...

Solving these will require us to study different fns to model world: x^x in computer science, xe^{-x^2} baseball, ...

PROVE PYTHAG: 

CHAPTER 1: FUNCTIONS, GRAPHS AND MODELS

1.1. Functions and mathematical modeling

Common: πr^2, $\frac{4}{3}\pi r^3$, $s = \frac{1}{2}at^2$, $F = \frac{G m_1 m_2}{r^2}$, ...

Defn: Function

f: Domain $\to$ Range

for each x in domain associates unique output

Ex polynomials:

$f(x) = 2x^2 - x + 7$

$f(\text{input}) = 2(\text{input})^2 - \text{input} + 7$

so $f(-1) = 2(-1)^2 - (-1) + 7 = 2 + 1 + 7 = 10$

$f(x+1) = 2(x+1)^2 - (x+1) + 7$

$f(x+h) = 2(x+h) - (x+h) + 7$

Interval notation

$[a,b] = \{x : a \le x \le b\}$

$(a,b) = \{x : a < x < b\}$

$[a,b) =$

$(a,b) =$

Unbounded: $[a, \infty)$ or (a, ∞), never $[a, \infty]$

Domains of functions

$f(x) = 1/x$ $\quad x \neq 0$

$= \sqrt{2x+4}$ $\quad x \geq -2$

HW

DO: Page 10: 1, 14, 29, 34
42, 47

Suggest: T/F, Q + Discussion
36, 38

Determining fns: Compound interest

1.2 GRAPHS OF EQS AND FNS

Graph of Equation: all $(x,y) \in \mathbb{R}^2$ satisfy eq

ex: $x^2 + y^2 = r^2$ or $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = r^2$  

Vertical Line Test

↳ each vertical line thru point in domain of fn meets graph in exactly one point

$y = x^2$  yes

$y^2 = 1 - x^2$  no

Plotting:

↳ sample at many points, tabulate

↳ Problem: what is many? $f(x) = \sin(200\pi x)$, $f(\frac{k}{100}) = 1$ for any $k \in \mathbb{Z}$

LINES

$y = mx + b$, $m = \text{slope} = \frac{\text{rise}}{\text{run}}$, $b = y\text{-intercept}$

↳ This is slope-intercept form

Much of calculus is linearization, lines important!

↳ Point-slope: given m, $P_0 = (x_0, y_0)$

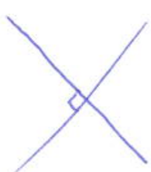
Then $\frac{y - y_0}{x - x_0} = m$ or $y - y_0 = m(x - x_0)$

↳ Point-Point: given $P_0 = (x_0, y_0)$, $P_1 = (x_1, y_1)$

find slope: $m = \frac{y_1 - y_0}{x_1 - x_0}$ $\left(= \frac{y_0 - y_1}{x_0 - x_1} \right)$

Lines: $Ax + By = C$

↳ Question: what is $0 \cdot \infty$? Answer: -1!

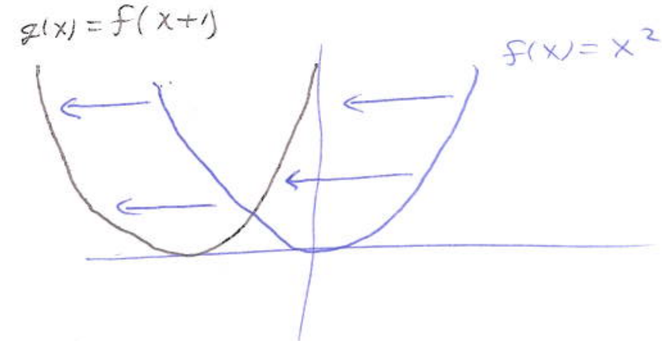
Sketch 

if perpendicular, prod slopes is -1 (assuming nonzero)

can check in some cases

EXTRA CREDIT: PROVE THIS! (soln on back)

Shifts of Functions

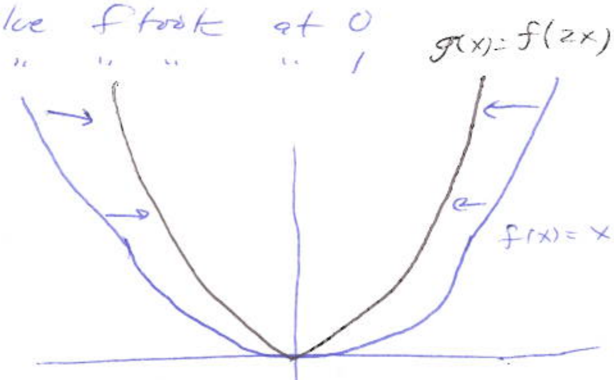


① Translations

$g(x) = f(x+a)$

↳ moves graph a units left

when $x = -a$, g takes on value f took at 0
$1-a$, " " " " " " 1

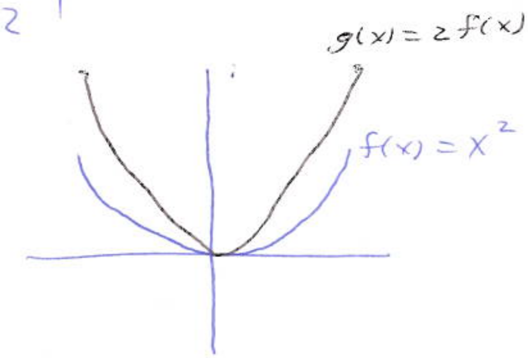


② Dilations

$g(x) = f(bx)$

↳ squishes x-axis by factor of b

when $x = 1/b$, g takes value f took at 1
$2/b$, " " " " " 2



③ Rescale

$g(x) = c\,f(x)$

↳ expands y-axis by factor of c

Most general: $g(x) = c\,f(bx+a)$

HW:
#5, #39, #65

Solving Quadratics

↳ Go from easy to hard

Suggested
#16, #44, #79

$$x^2 + c = 0 \to x^2 = -c \to x = \pm\sqrt{-c}$$

$$ax^2 + c = 0 \to ax^2 = -c \to x = \pm\sqrt{-c/a}$$

$$ax^2 + bx + c = 0 \to a\left(x^2 + \frac{b}{a}x\right) + c = 0$$

$$a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}\right) + c = 0$$

$$a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c = 0$$

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2-4ac}}{2a} \to x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

Notice easier study $a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2-4ac}{4a}$

1.3 POLYNOMIALS + ALGEBRAIC FNS

Plots of polys, ratios of polys, ...

Fund Thm of Alg

HW: 14, 22

$f+g$, $f \cdot g$, f/g ...

1.4 TRANSCENDENTAL FNS

trig fns: $A\sin(Bx+C)$ and $\sin x$

Exponential: 2^x : understand if x int, rational; if irrational?

Defn e^x: $\lim_{n\to\infty}(1+\frac{x}{n})^n = \sum_{n=0}^{\infty}\frac{x^n}{n!}$ Compound interest

↳ highly non-trivial show $e^x e^y = e^{x+y}$

De Moivre: $e^{i\theta} = \cos\theta + i\sin\theta$: all trig identities

Prove log laws: $\log_b x = y$ means $x = b^y$, Calculus: $\log x \ll x^\varepsilon$

Solving trans eqs hard: $x = \cos x$

HW: 14, 22, 49

Cow and pole problem: Extra Credit

Composition of fns:

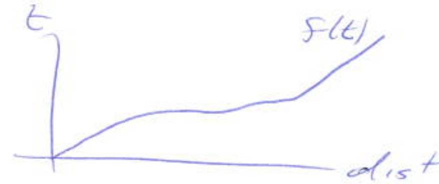
Suggested 1, 7, 8, 20

↳ $h(x) = f(g(x)) = (f \circ g)(x)$

$f(g(x)) \neq g(f(x))$ in general

CHAPTER 2: PRELUDE TO CALCULUS

2.1. Tangent Lines and Slope Predictors



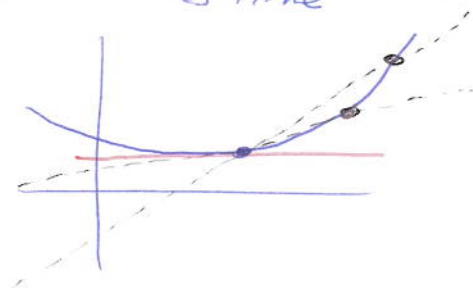
Estimating Speed

↳ global ave speed often poor locally

↳ ave in small window (count revs of wheels)

AveSpeed from time a to b is $\dfrac{f(b)-f(a)}{b-a} = \dfrac{\Delta \text{dist}}{\Delta \text{time}}$

↳ as $b \to a$ estimates inst speed



Derivative of f at a

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \quad \text{if lim exists}$$

will derive lim laws soon

Say $f(x) = ax^2+bx+c$, or $2x^2-x+3$ for definiteness

$$f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} \quad \underline{\text{do calc}}$$

$$= 3$$

When $x=1$, $f(1)=4$: Point $(1,4)$ slope 3

↳ eq tangent line is $y - 4 = 3(x-1)$ or $y = 3(x-1)+4$

Key Idea: near 1 replace complicated fn with line

↳ ex: $f(1.1) = 2(1.1)^2 - 1.1+3 = 2.42-1.1+3 = 4.32$

tangent line: $x=1.1$ gives $y = 3(1.1-1)+4 = 4.3$

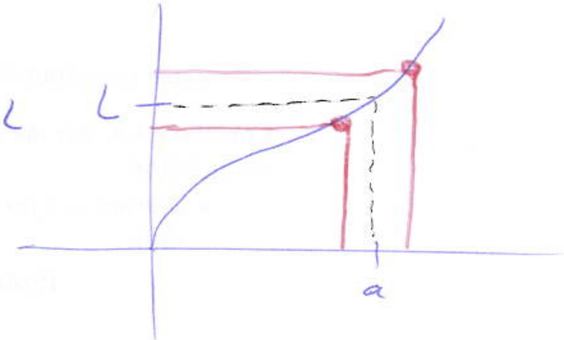
↳ not bad: within $2/100$!

HW: 5, 7, 15

Suggested: 33

2.2 THE LIMIT CONCEPT

Meaning of $\lim_{x \to a} f(x) = L$



↳ as x gets close to a, f(x) gets close to L

↳ do not need $f(a) = L$

↳ advanced: ε, δ

LIMIT LAWS (cor laws for derivs)

~~Examples~~

① Constant: If $f(x) \equiv C$ Then $\lim_{x \to a} f(x) = C$

If c constant $h(x) = c f(x)$ Then $\lim_{x \to a} h(x) = c \lim_{x \to a} f(x)$

② Sum Law: $f+g$, both limits exist

③ Prod Law: fg, both limits exist

④ Quot Law: f/g, both limits exist, $\lim_{x \to a} g(x) \neq 0$

⑤ Root Law: $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$

↳ MUST prove rigorously: ε, δ

↳ never have $\infty - \infty$ or $\infty \cdot 0$: $n^2 - n$, $n^2 - n^2$, $n^2 - n^3$, ...

⑥ Composition Law: Given: $\lim_{x \to a} g(x) = L$, $\lim_{x \to L} f(x) = f(L)$

Then $\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right) = f(L)$

↳ interchanging f and limit: Careful!

Note $\delta\varepsilon \neq \varepsilon\delta$

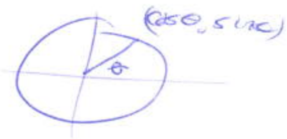
HW: 1, 4, 6, 21, 37

Examples

$\lim_{x \to 2} \frac{x-1}{x+2}$, $\lim_{x \to 2} \frac{x^2-4}{x^2+x-6}$, $\lim_{x \to 0} \frac{x}{|x|}$

Suggested: 27, 38

2.3 More About Limits



Trig: $\lim_{\theta\to 0} \cos\theta = 1$, $\lim_{\theta\to 0} \sin\theta = 0$, $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$

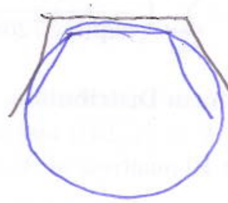
↳ don't like book proof, as need to know area circle is πr^2

↳ proof of that often use $\lim \frac{\sin\theta}{\theta} = 1$!

↳ only $=1$ in radians!

Squeeze Law

$f(x) \le g(x) \le h(x)$ for x near a, if $\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$ then $\lim_{x\to a} g(x) = L$



Trig: Archimedes close to calc

↳ inscribed and circum $\odot$.

↳ ~~Perim~~ Area: $n \cdot \sin\frac{A}{2n} \cos\frac{A}{2n}$ where have A units of angle

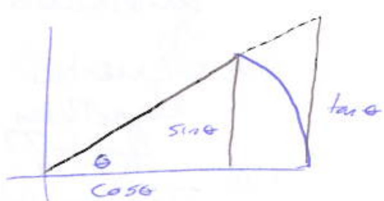
inscribed $= \frac{n}{2} \sin\left(\frac{A}{n}\right) = \frac{A}{2} \cdot \frac{\sin(A/n)}{A/n}$

circumscribed ~~Area~~: $n \cdot 1 \cdot \tan\left(\frac{A}{2n}\right) = \frac{A}{2} \frac{\tan(A/2n)}{A/2n} = \frac{A}{2} \frac{\sin(A/2n)}{A/2n} \cdot \frac{1}{\cos(A/2n)}$

Choose units A st $\lim_{n\to\infty} \frac{\sin(A/n)}{A/n} = 1$

↳ Perim ~~Area~~: $2n \sin\left(\frac{A}{2n}\right) = A \cdot \frac{\sin(A/2n)}{A/2n}$

Choose units st length of arc of unit circle is its angle



$$\frac{1}{2} \sin\theta \cos\theta \le \frac{\theta}{2\pi} \pi \le \frac{\sin\theta}{2\cos\theta}$$

$$\frac{\sin\theta}{\theta} \cos\theta \le 1 \le \frac{\sin\theta}{\theta} \frac{1}{\cos\theta}$$

left hand limit: $\lim_{x\to a^-} f(x)$ $\xrightarrow{x}$ a right hand $\lim_{x\to a^+} f(x)$ a $\xleftarrow{x}$

HW: 1, 2, 7, 25, 32

two sided: left = right

infinite limits: $\lim_{x\to 0^-} \frac{1}{x} = -\infty$ $\lim_{x\to 0^+} \frac{1}{x} = +\infty$

Suggested: 70

2.4 THE CONCEPT OF CONTINUITY

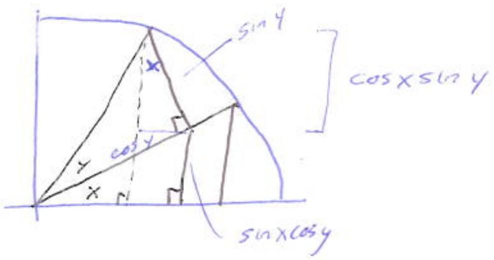
Defn: Cont at a point : f cont at a if $\lim_{x \to a} f(x) = f(a)$

need (1) f defined at a
(2) left, right limits exist
(3) (1) and (2) same

Sum, diff, prod cont fns are cont

↳ knowing c, x^n cont → $\sum_{i=0}^{n} a_i x^i$ cont

Trig laws : $\sin(x+y)$, $\cos(x+y)$



Thus $\sin(x+y) = \sin x \cos y + \cos x \sin y$

comment: why these aux lines

Cont of $\sin x$, $\cos x$

$\lim_{x \to a} \sin x = \lim_{h \to 0} \sin(a+h) = \lim_{h \to 0} (\sin a \cos h + \cos a \sin h) = \sum \lim = \sin a$

↳ of course, this needs as input sin/cos cont at 0!

Cont of Composition

g cont at a, f cont at g(a), $h = f \circ g$ cont at a

Cont in intervals

on (a,b) : cont from right at a, left at b

IVT: f cont [a,b]. If K b/w f(a) and f(b) then $\exists c \in [a,b]$ st $f(c) = K$

↳ false if f not cont

↳ Proof: Divide and Conquer
wlog $f(a) < K < f(b)$
$[a_k, b_k]$ st $f(a_k) \le K \le f(b_k)$
limits / squeeze thm

↳ App: finding zeros: sign changes
↳ 10 iters give ≈ 3 decimals: Newton's Method better (more input)
↳ dangers: double zeros
↳ App: $\mathcal{Y}(s)$! MY RESEARCH

HW: 1, 6, 15, 20, 49

Suggested: 47, 52, 61
63, 66

CHAPTER 3: THE DERIVATIVE

3.1 THE DERIV AND RATES OF CHANGE

Defn: $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$

↳ gives tangent line approx $y = f'(a)(x-a) + f(a)$

↳ algebra can be tedious, not always easy simplify

↳ examples: ① $f(x) = c$ ② $f(x) = x$ ③ $f(x) = x^2$ ④ $f(x) = 2x^2 - 3x$

Notation: $y = f(x)$ then $\Delta y = f(x+\Delta x) - f(x)$

$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{\Delta x}$ take $\lim_{\Delta x \to 0}$, get $f'(x)$

use $\frac{dy}{dx}$ to denote $f'(x)$ ← better notation when more vars

App: Sign of Deriv

↳ if pos: fn is ↑; if neg, fn is ↓

↳ app: Laffer Curve: deriv won Cold War (Simple Model)

Skipping much of section for now

HW: ~~Pb~~ Find the deriv of $f(x) = 3x^2 - 4x + 1$

3.2 BASIC DIFF RULES

① Constant Rule ③ Prod Rule ⑤ Quotient Rule

② Sum/Diff Rule ④ Reciprocal Rule ⑥ Power Rule: x^n

All proofs start ⓦ defn Deriv

Power Rule: $f(x) = x^n$ $n \geq 0$ int, $f'(x) = n x^{n-1}$

↳ Input: Binomial Thm: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

↳ proof: combinatorics, expand

↳ for applications, just need $x^n + n x^{n-1} y + O(y^2)$

↳ Proof of power rule

App: $f(x) = \sum_{k=0}^{n} a_k x^k$ then $f'(x) = \sum_{k=1}^{n} k a_k x^k$

Neg exponents: use quotient rule, extend $(x^n)' = n x^{n-1}$ to $n \in \mathbb{Z}$

↳ What about $f(x) = x^{p/q}$ or x^r?

HW: 3, 8, 16, 25, 42

Suggested: 55, 61, 66, 73

THOREAU: Simplify, Simplify

↳ ex: $f(x) = (2x^2 - 3)(x^2 + 5x)$

↳ product rule or expand and diff

Statement: g diff at x, f diff at $g(x)$, $h = f \circ g$ diff at x
and $h'(x) = f'(g(x)) \cdot g'(x)$

"Proof": $h'(x) = \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h}$

$= \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \frac{g(x+h) - g(x)}{h}$

(okay if $g(x+h) \neq g(x)$)
lim quotient = quotient limits
↳ first is $\lim_{\omega \to g(x)} \frac{f(\omega) - f(g(x))}{\omega - g(x)} = f'(g(x))$

Examples: $h(x) = (3x^2+17)^{2008}$
↳ must write $h(x) = f(g(x))$

Ask Class for other problems

Generalized Power Rule: $h(x) = (f(x))^n$ Then $h'(x) = n f(x)^{n-1}$ $n \in \mathbb{Z}$
↳ Corr: Deriv of $x^{p/q} = \frac{p}{q} x^{\frac{p}{q}-1}$
↳ proof: $f(x) = x^{p/q}$ $h(x) = f(x)^q = x^p$. Differentiate
↳ Corr: Deriv $(f(x))^n$ is $n(f(x))^{n-1} * f'(x)$ for $n \in \mathbb{Q}$. What of $n \in \mathbb{R}$?

Alternate Notation

$g'(x) = \frac{dg}{dx}$ Write $y = f(u)$ and $u = g(x)$

Then $\frac{dy}{du} = f'(u)$ but $\frac{dy}{dx} = f'(u) g'(x) \Big|_{u=g(x)} = \frac{dy}{du} \frac{du}{dx}$

warning! $\lim_{n \to \infty} \frac{\sin x}{n} \neq 6$

↳ informally cancel du's : remember 19/95 ...

Example: refinery makes u liters of gas from x barrels oil ~~refinery makes u barrels oil from x g~~
then makes y grams petroproduct from u liters of gas
↳ ~~if increase~~ say $u = 75x$, $y = 3u$ Then $y = 225u$
↳ increase u by 1 unit, increase y by 3
increase x by 1 unit, increase u by 75 ⇒ ↑x by 1, ↑y by 225

Ints: Not $f(g(x))\, g'(x)$, ... $\boxed{\text{Pendulum } 2\pi\sqrt{\frac{L}{g}}}$

HW: 1, 7, 39, 42, 54

App: Related Rates
↳ will do more later, say sphere where $\frac{dr}{dt} = 2$ m/sec

Suggested: 33, 49, 52, 63

3.4 Derivs of Algebraic Fns

↳ did most of this already

↳ Notation: $D_x f(x) = \frac{d}{dx} f(x) = f'(x)$

↳ Can be cont and not diff: $f(x) = |x| = \sqrt{x^2}$

↳ "almost all" cont fns are diff nowhere!

↳ weird: Russel's Paradox: intuition can fail

HW: 1, 17, 31, 47, 57, 59, 62, 63, 71

Suggested: 43, 58, 60, 61, 65

Diff ⇒ Cont $\lim_{x \to a} \left(f(x) - f(a)\right] = \lim_{x \to a} \left[(x-a) \cdot \frac{f(x)-f(a)}{x-a}\right]$ lim Quot = Quot lim

Vertical line tangent at $x=a$ if f cont and $|f'(x)| \to +\infty$ as $x \to a$

↳ Extra Credit: Could $f'(x) = +\infty$ from both sides?

Skipping Ahead

3.7 Derivs of Trig Fns

Thm (In Radians) $\frac{d}{dx} \cos x = -\sin x$ (minus sign mnemonic), $\frac{d}{dx} \sin x = \cos x$

↳ Proof: $\frac{d}{dx} \sin x = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} = \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$

$= \lim_{h \to 0} \left[\sin x \cdot \frac{\cos h - 1}{h} + \cos h \frac{\sin h - 0}{h}\right]$ note $1 = \cos 0$

↳ thus just need deriv of sin and cos at 0

$\frac{d}{dx} \cos x = \lim_{h \to 0} \left[\cos x \cdot \frac{\cos h - 1}{h} - \sin x \frac{\sin h - 0}{h}\right]$

↳ Know $\lim_{h \to 0} \frac{\sin h - 0}{h} = \sin' 0 = 1$

Assume fns diff: $f(x) = \sin^2 x + \cos^2 x = 1$

$\Longrightarrow f'(x) = 2 \sin x \cdot \sin' x + 2 \cos x \cdot \cos' x = 0$

↳ take $x = 0$ to see $\cos' 0 = 0$

HW: 1, 7, 8 (Thoreau), 13, 20, 41, 45

Suggested: 21, 23, 38, 50, 61, 71, 77

Ex: $\tan x = \frac{\sin x}{\cos x}$

Ex: $h(x) = \sin^2\left((2x-1)^{3/2}\right)$

3.8 Exponential and Log Fns

$f(x) = a^x \quad f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{a^{x+h}-a^x}{h} = a^x \lim_{h\to 0}\frac{a^h-1}{h}$

↳ enough to find deriv at 0

↳ numerics: lim < 1 if $a=2$, > 1 if $a=3$, so somewhere b/w say is 1

↳ call that number e

↳ other defns: $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$

Extra Credit: Show two defns equal

↳ law of exponents must be PROVED: $e^x e^y = e^{x+y}$ combinatorics

↳ make this Extra Credit as well

__Deriv of e^x__: If $\frac{d}{dx}\sum = \sum\frac{d}{dx}$, $\frac{d}{dx}e^x = e^x$

↳ study deriv of a^x. Let $\log_b x = y$ mean $x = b^y$

So $\ln x = \log_e x$ is the inverse of $\exp x$: $\exp(\ln x) = \ln e^x = x$

$a = e^{\omega} \rightarrow \omega = \ln a$

So $\lim_{h\to 0}\frac{a^h-1}{h} = \lim_{h\to 0}\frac{(e^{\omega})^h-1}{h} = \lim_{h\to 0}\omega\cdot\frac{e^{\omega h}-1}{\omega h} = \ln a\cdot\lim_{k\to 0}\frac{e^k-1}{k}$

So $\frac{d}{dx}a^x = (\ln a)a^x$

↳ __Question__: $10 = e^{\ln 10} = \sum_{n=0}^{\infty}\frac{(\ln 10)^n}{n!}$, $10^2 = \sum_{n=0}^{\infty}\frac{(2\ln 10)^n}{n!}$ obvious?

↳ does this reduce to 100?

__Power Rule__: $f(x) = x^r = e^{r\ln x} = \exp(r\ln x)$

↳ if know deriv of $\ln x$, know $f'(x)$

↳ $r=1$: $x = \exp(\ln x)$: Chain Rule

$1 = \exp'(\ln x)\cdot\ln' x = \exp(\ln x)\cdot\ln' x = x\ln' x$

$\Longrightarrow \ln' x = 1/x$

$\Longrightarrow (x^r)' = \exp'(r\ln x)\cdot r\ln' x = rx^{r-1}$

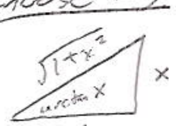
__Log Diff__: $y = f(x)$
$\ln y = \ln f(x)$
$\frac{1}{y}\frac{dy}{dx} = \frac{1}{f(x)}f'(x)$

↳ Calculate deriv of $\log_b x$

__HW__: 1, 7, 18, 33, 39, 59

__suggested__: 63, 64, 69, 71, 72

__Inverse Trig__ $x = \tan(\arctan x)$



$1 = \tan'(\arctan x)\cdot\arctan' x$

$\arctan' x = \cos^2(\arctan x) = \frac{1}{1+x^2}$

3.5 Maxima and Minima on Closed Intervals

↳ Define absolute max/min
↳ Define local max/min

Thm: f cont on finite closed [a, b] then f attains max and min

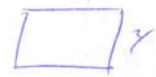
↳ Proof: Bolzano-Weierstrass: seq

<u>Searching for max/min for diff fn</u>

↳ check value at endpoints
↳ check where $f'(x) = 0$ (called critical points)

<u>Example: Pen</u>

↳ 40m build rectangular pen



↳ must make 1-var as only know 1 var calculus — Thoreau!

Area $= xy$ but $2x + 2y = 40$ so $A(x) = x(20 - x) = 20x - x^2$

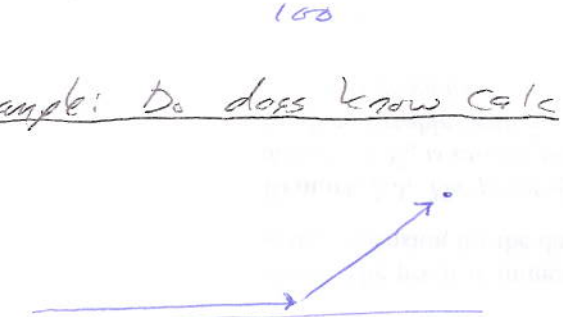
↳ 40m but one side "free": water

↳ Symmetry argument! Imagine 80m fence!

↳ math "lazy": reduce to previous problem: base on fire

<u>Example: Drowning Swimmer</u>

HW: 1, 9, 18, 28, 47, 48, 52



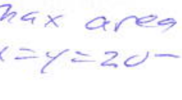
(1) quartic: phantom roots

Suggested: 41, 42, 45, 49, 50, 51

(2) as $V_L > V_w$ should be to right of straight line

<u>Other:</u>

(1) closest point?



<u>Example: Do dogs know calc?</u>

↳ arith/geo mean

(2)

Perim = 20
max area
$x = y = 20 - 10\sqrt{2}$

3.6 Applied Optimization Problems

5 Steps: ① Find quantity to max/min; draw good picture

② Make everything depend on 1 variable

③ Find critical points

④ Evaluate fn at C.P. and endpoints

⑤ interpret results / answer Question

HW: 1, 7, 13, 20, 47

Suggested: 15, 27, 33, 42, 48, 49, 62

For 3.7
HW: 1, 7, 8, 13
20, 41, 45, ~~[illegible]~~
Suggested: 21, 23, 38
50, 71, 71, 77

3.7 Implicit Diff and Related Rates

$x^2+y^2=1$ vs $y=\sqrt{1-x^2}$ implicit vs explicit

if (x_0,y_0) satisfies eq, consider $y=y(x)$, diff w.r.t. x, solve i/o above

ex: $\sin(x+2y)=2x\cos y$: find tangent line at $(0,0)$ → Check to make sure pt. on curve. Ex: $1+2+4+8+\dots=-1$

Related Rates

① Sphere: $dr/dt = r'(t) = 2$ m/sec. How fast volume ↑ when $r=3$ m?

Steps: ① Draw picture

② Record values of vars and their rates of change

③ Find eq relating vars

④ Differentiate (implicitly or explicitly)

⑤ Substitute

Try to figure out if should be pos or neg

② Cars on x, y-axis: $\left\{\begin{array}{l} x\text{-axis: speed } 20 \text{ mph} \\ y\text{-axis: speed } 10 \text{ mph}\end{array}\right\}$ How fast dist changing when 30 miles on x-axis, 40 miles on y-axis

③ Ostrich 5 ft walks towards light ~~[illegible]~~ 12 ft at 4 ft/sec. When 5 ft away, how fast is shadow changing (not: will be neg)

HW: ~~[illegible]~~ 5, 13, 15, 37, 44 Suggested: ~~[illegible]~~ 23, 38, 50

3.10 Newton's Method

Compare to Divide and Conquer: better, but more input

Get $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

Square root $= \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$

Data for $\sqrt{3}$, first guess 2

$x_0 = 2$

$x_1 =$

$x_2 =$

$x_3 =$

$x_4 =$

$\sqrt{3} =$

HW: 1, ~~[illegible]~~

Suggested: 21

mention Chaos Color - cubic

Hill + Coin Benford

CHAPTER 4: ADDITIONAL APPLICATIONS OF THE DERIVATIVE

4.2 Increments, Differentials and Linear Approx

Linear approx: $L(x) = f(a) + f'(a)(x-a)$

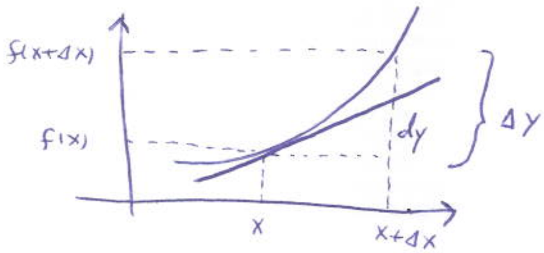
↳ how accurate? will see later

Example: approx $(122)^{2/3}$ $(= 24.5984)$

↳ Step 1: find a good fn st know $f(a)$

try $f(x) = x^{2/3}$

Step 2: $L(122) = f(125) + f'(125)\cdot(125-122) = 24.6$



Notation: $f(x) - L(x) = \Delta y - dy$

$\Delta x = dx$

write $dy = f'(x)\,dx$ $\quad$ (ie, $\frac{dy}{dx} = f'(x)$)

HW: 2, 11, 28, 42

4.3 Increasing and Decreasing Fns and the MVT

$f'(x) > 0$: $f(x) \uparrow$ on interval
$\quad\;\; < 0$: $f(x) \downarrow$ " "

MVT: f cont $[a,b]$, diff on (a,b). Then $\exists c \in (a,b)$ st $f'(c) = \frac{f(b)-f(a)}{b-a}$

↳ or $f(b) - f(a) = f'(c)\cdot(b-a)$

↳ follows from Rolle's Thm

Rolle's Thm: Same conds, $f(a) = f(b) = 0$ Then $\exists c \in (a,b)$ st $f'(c) = 0$

↳ Proof: Assume $f'(a)$, $f'(b)$ exist
Get opposite signs somewhere and use IVT

↳ MVT: Apply Rolle's Thm to $g(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a)$

Corr: ① $f'(x) = 0$ on $(a,b) \rightarrow f$ constant on $[a,b]$
② $f'(x) = g'(x)$ on $(a,b) \rightarrow \exists K$ st $f(x) = g(x) + K$
③ $f'(x) > 0$ on $(a,b) \rightarrow f(x)$ is $\uparrow$

④ Assume f'' exists
$f(x)$, $L(x) = f(a) + f'(a)(x-a)$, $g(x) = f(x) - L(x)$ and $g(a) = 0$
$g(b) = g(b) - g(a) = g'(c)(b-a)$
↳ now $g'(x) = f'(x) - f'(a)$ and $g'(a) = 0$

so $g'(c) = g'(c) - g'(a) = g''(\tilde{c})(c-a)$

$\Rightarrow |f(b) - L(b)| \leq \max_{\tilde{c}\in[a,b]} g''(\tilde{c}) \cdot (b-a)^2$

HW: 15, 17, 25, 26, 33, 45

Suggested: 1, 4, 5, 16, 18, 27, 48, 61

Where is $f \uparrow$ or $\downarrow$?

Say $f(x) = \frac{1}{3}x^3 - 4x^2 + 12x + 1701$
$f'(x) = x^2 - 8x + 12$
$f'(x) = (x-6)(x-2)$

$+ + + + 0 - - - - - - - - 0 + + + +$
$\quad\quad\quad\;\; 2 \quad\quad\quad\quad\quad\quad\; 6$

4.4 THE FIRST DERIVATIVE TEST AND APPLICATIONS

Critical Point $f'(x)=0$

1st Deriv Test Local Extrema

↳ f diff on I except possibly at c

If $f'<0$ for $x<c$ and $f'>0$ for $x>c$, local min at c

↳ also local extrema when $f'(x)=0$

↳ to find absolute max/min: Find critical points + test endpoints

↳ if on infinite interval, see limits

HW: 1, 11, 27, 29

Suggested: 30, 31, 34, 42

4.5 Simple Curve Sketching and 4.6 Higher Derivs and Concavity

$$f''(x) = (f'(x))' = \frac{d^2 f}{dx^2} = D_x^2 f(x) = D_x(D_x f(x))$$

↳ do example: $f(x) = \frac{1}{3}x^3 - 4x^2 + 12x + 1701$

Sign of f''

① $f'' > 0$: first deriv is increasing ; concave up / bends up

ex:  in both f' ~~ass~~ increasing; first $f' < 0$, then $f' > 0$

② $f'' < 0$: first deriv is decreasing: concave down / bends up

ex: 

2nd Deriv Test

f twice diff on $[a,b]$, $f'(c) = 0$, then

↳ if $f'' > 0$ on $[a,b]$, $f(c)$ is min of f on $[a,b]$

if $f'' < 0$ on $[a,b]$, " " max " " "

↳ if instead $f''(c) > 0$ local min, $f''(c) < 0$ local max

Proof: ① $f'' > 0$ means looks like 

② Let $g(x) = f(x) - [f(c) + f'(c)(x - c)]$ $g(c) = 0$

Then $g(x) - g(c) = g(x) = g'(\tilde{c})(x - c)$

$= [f'(\tilde{c}) - f'(c)](x - c)$

$= f''(\tilde{\tilde{c}})(\tilde{c} - c)(x - c)$

if $f'' > 0$ and $x > c$ then $g(x) > 0$ which means $f(x) > f(c)$

$f'' > 0$ and $x < c$ then $g(x) > 0$ " " " "

Inflection Point

↳ $f''(x) = 0$

Curve Sketching

ex: $\frac{1}{3}x^3 - 4x^2 + 12x + 170$

① Find critical, inflection points ________ x

② evaluate f there and endpoints (or see $\lim x \to \pm\infty$) ________ f'

________ f''

③ Find signs 1st, 2nd deriv

④ Sketch

HW: sketch $f(x) = -\frac{1}{3}x^3 + 5x^2 - 16x + 2004$

4, 13, 33, 35

Suggested: 77, 80, 82, 83, 89

(Skip 4.7)

4.8 Indeterminate Forms and L'Hôpital's Rule

L'Hôpital's Rule: f, g diff, $g'(x) \neq 0$ in nbhood a (except maybe at a)

suppose $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = 0$

Then $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = \lim\limits_{x\to a} \frac{f'(x)}{g'(x)}$

↳ other variants ① replace a by $\pm\infty$

② instead of $0/0$ have ∞/∞ or $-\infty/-\infty$

③ if $f'(a)/g'(a) = 0/0$, apply again!

Proof (Simple Case): $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = \lim\limits_{x\to a} \frac{f(x)-f(a)}{g(x)-g(a)}$

$= \lim\limits_{x\to a} \frac{f(x)-f(a)}{x-a} \Big/ \frac{g(x)-g(a)}{x-a}$

$= \lim(\;)/\lim(\;) = f'(a)/g'(a)$

Ex: $\lim\limits_{x\to 0^+} \frac{\ln 2x}{\ln x}$: Simplify (Thoreau and avoid!)

Ex: $\lim\limits_{x\to 0} \frac{\cos x - \left(1 - \frac{x^2}{2}\right)}{\sin x - x}$

Application: Order of Magnitude

As $x\to\infty$: $\ln x \ll x^r \ll e^x$ for any $r > 0$

↳ Proof: $\lim\limits_{x\to\infty} x^r/e^x = 0$ (diff enough times)

↳ alternate: $e^x \geq \frac{x^{n+1}}{(n+1)!} = x^n \cdot \frac{x}{(n+1)!} \gg x^n$ x large

$e^x \geq x$, take $x = \ln(y^{\frac{r}{2}}) \Rightarrow y^{\frac{r}{2}} \geq \frac{r}{2} \ln y$

so y^r will be $> \ln y$ by a lot, as $y^{\frac{r}{2}} \gg \frac{r}{2}$

Good Question: Why can't we use L'Hôpital to calculate $\lim\limits_{x\to 0} \frac{\sin x}{x}$? Or can we?

HW: 1, 2, 9, 42

Suggested 62, 68, 70, 73

CHAPTER 5: THE INTEGRAL

5.2 ANTIDERIVS AND INITIAL VALUE PROBLEMS

Differential Eqs

Newton's law of cooling: rate of Δ at temp T is prop to diff b/w temp and surrounding medium

$$dT/dt = -k(T-A)$$ law and order

Antiderivative

F is an antideriv of f if $F'(x) = f(x)$

↳ not unique: $f(x) = x^2$, $F(x) = \frac{x^3}{3}$, $\frac{x^3}{3} + \pi$ ...

↳ if F, G both antiderivs then $F(x) - G(x) = \text{const}$ (MVT)

↳ Notation: $\int f(x)\,dx = F(x) + C$ iff $F'(x) = f(x)$

↳ call this indefinite integral

↳ diff mech, integration hard: Extra Credit: $f(X) = \ln x$, $(\ln x)^2$

Examples

$f(x) = 2\sin x\cos x$: $F(x) = \sin^2 x$ or $-\frac{\cos 2x}{2}$

$$\int x^n\,dx = \begin{cases} \frac{x^{n+1}}{n+1} & \text{if } n \neq -1 \\ \ln x & \text{if } n = 1 \end{cases}$$

$$\int \cos x\,dx = \sin x$$

$$\int \sin x\,dx = -\cos x$$

Rules

① Constant: $\int cf = c\int f$

② Sum/diff: $\int f \pm g = \int f \pm \int g$

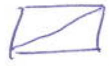
IVP: Diff Eq, Starting Conds

$$\frac{dv}{dt} = g, \quad v(0) = 5 \rightarrow v(t) = gt + 5$$

HW 1, 3, 16, 37

Suggested 58, 68

5.3 ELEMENTARY AREA COMPUTATIONS

Know area of rectangle, right triangle , general triangle 

↳ area under curve: approx with rectangles

Inscribe + circumscribe: squeeze $\lim$: true value between

$\underline{\text{Ex: } f(x) = x, \quad 0 \le x \le 1}$

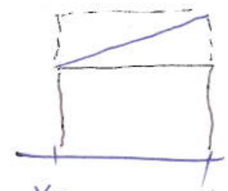
↳ clear area just $1/2$ 

Divide $[0,1]$ into n equal pieces, $0 = x_0 \le x_1 \le \cdots \le x_n = 1$

so $x_i = i/n$ and $x_{i+1} - x_i = \frac{1}{n}$

let $I_i = [x_i, x_{i+1}]$

on I_i, $f(x)$ satisfies $f(x_i) \le f(x) \le f(x_{i+1})$



so $\displaystyle\sum_{i=0}^{n-1} f(x_i) \cdot \frac{1}{n} \le \text{Area} \le \sum_{i=0}^{n-1} f(x_{i+1}) \frac{1}{n}$

$\displaystyle\sum_{i=0}^{n-1} \frac{i}{n} \frac{1}{n} \le \text{Area} \le \sum_{i=0}^{n-1} \frac{i+1}{n} \frac{1}{n} = \sum_{i=0}^{n-1} \frac{i}{n^2} + \frac{n}{n^2}$

__Claim:__ $\displaystyle\sum_{i=0}^{m} i = \frac{m(m+1)}{2}$ (induction, match largest + smallest, ...)

Thus $\displaystyle\frac{1}{n^2} \frac{(n-1)n}{2} \le \text{Area} \le \frac{1}{n^2} \frac{(n-1)n}{2} + \frac{n}{n^2}$

take $\lim_{n\to\infty}$, get $\frac{1}{2} \le \text{Area} \le \frac{1}{2}$ so Area is $1/2$

$\underline{\text{Ex: } f(x) = x^2, \quad 0 \le x \le 1}$

↳ Before $f(x_i) = i/n$, now $f(x_i) = (i/n)^2$

Get $\displaystyle\sum_{i=0}^{n-1} \frac{i^2}{n^2} \frac{1}{n} \le \text{Area} \le \sum_{i=0}^{n-1} \frac{i^2}{n^2} \frac{1}{n} + \frac{n^2}{n^3}$

__Claim:__ $\displaystyle\sum_{i=0}^{m} i^2 = \frac{m(m+1)(2m+1)}{6}$

↳ yields $\frac{1}{3} \le \text{Area} \le \frac{1}{3}$ after taking limits HW: 3, 9, 19, 22, 39

Generalizations

↳ $f(x) = x^3$: need $\displaystyle\sum_{i=0}^{m} i^3 = \frac{m^2(m+1)^2}{4}$ Suggested: 52, 53

↳ have to prove "hard" sums each time

↳ what of more general f, say $f(x) = x^2 \sin x$?

↳ $[a,b]$ set $x_i = a + \frac{b-a}{n}$

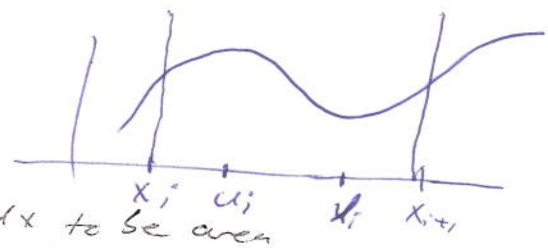
5.4 RIEMANN SUMS AND THE INTEGRAL 5.6 THE FTC

Will not prove in greatest generality

↳ each n partition $[a,b]$ into n pieces, find upper + lower sums, limit

↳ will assume all pieces equal

↳ will assume $[a,b]=[0,1]$



$\underline{FTC}$: Let f be cont on $[a,b]$. Set $\int_a^b f(x)\,dx$ to be area under f from a to b. If F is any anti-deriv of f

Then $\int_a^b f(x)\,dx = F(b) - F(a)$

↳ Note: G any other anti-deriv: $F(x) = G(x) + C$, so $F(b) - F(a) = G(b) - G(a)$

↳ will prove assuming f' ~~cont~~ bounded: simplifies proof Talk about bounded

$\underline{\text{Proof}}$ (The One With The MVT)

↳ $I_i = [x_i, x_{i+1}]$: let max be at u_i, min at l_i

so $\forall x \in I_i$, $f(l_i) \le f(x) \le f(u_i)$

so $L(n) = \sum_{i=0}^{n-1} f(l_i)\frac{1}{n} \le \text{Area} \le \sum_{i=0}^{n-1} f(u_i)\frac{1}{n} = U(n)$

$\underline{\text{Claim}}$: $\lim_{n\to\infty} U(n) - L(n) = 0$

$\underline{\text{Proof}}$: Equals $U(n) - L(n) = \frac{1}{n}\sum_{i=0}^{n-1}\left[f(u_i) - f(l_i)\right]$

By MVT, $= \frac{1}{n}\sum_{i=0}^{n-1} |f'(c_i)|\,|u_i - l_i|$

$\le \frac{1}{n}\sum_{i=0} B\cdot\frac{1}{n} = \frac{B}{n} \to 0$

Let $x_i^* \in I_i$ be any seq of points

↳ $L(n) \le \sum_{i=0}^{n-1} f(x_i^*)\frac{1}{n} \le U(n)$: choose clever seq

Apply MVT to F on I_i: choose x_i^* st $F(x_{i+1}) - F(x_i) = \underset{f(x_i^*)}{\underset{\shortparallel}{F'(x_i^*)}}(x_{i+1} - x_i)$

so $L(n) \le \sum_{i=0}^{n-1} f(x_i^*)\frac{1}{n} = \sum_{i=0}^{n-1} F(x_{i+1}) - F(x_i) \le U(n)$

└ telescoping: is $F(1) - F(0)$ or $F(b) - F(a)$

so $L(n) \le F(1) - F(0) \le U(n)$

↓ Area   Thus ↗ goes to area   ↓ Area

5.4 AND 5.6 CONTINUED

Call f Riemann Integrable if above method works

Not all f are RI: $f(x) = 0$ if $x \in \mathbb{Q}$, 1 if $x \notin \mathbb{Q}$

↳ Lebesgue Integral: agrees when RI exists, generalizes

↳ Throw down coins: RI: sum in order, LI: group pennies, nickels...

Average Values

If f integrable on $[a,b]$, ave value is $\frac{1}{b-a}\int_a^b f(x)\,dx$

↳ if f cont, $\exists\, \bar{x} \in [a,b]$ st $f(\bar{x}) =$ Ave Value

HW: Sec 5.6: 1, 13, 27, 45

Suggested: Sec 5.6: 41, 43, 66

*FTC: Define $F(x) = \int_a^x f(x)\,dx$ Then $F'(x) = f(x)$

EXTRA CREDIT: Say $G(x) = \int_a^{x^3} f(x)\,dx$. Express $G'(x)$ in terms of nice fns related to this problem.

5.5 EVALUATION OF INTEGRALS

① Constant: $\int_a^b c\,dx = c(b-a)$

② Multiple: $\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$

③ Sum/Diff: $\int_a^b (f(x) \pm g(x))\,dx =$

④ Interval Union: $a < c < b$: $\int_a^b f = \int_a^c f + \int_c^b f$

⑤ Comparison: $f \leq g$ on $[a,b]$ Then $\int_a^b f \leq \int_a^b g$

↳ special case: $m \leq f(x) \leq M \rightarrow m(b-a) \leq \int_a^b f \leq M(b-a)$

DO MANY EXAMPLES!

HW: 1, 13, 33, 54

Suggested: 59, 60