THE CHANGE OF VARIABLE THEOREM

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1. Statement

Theorem 1.1 (Change of Variables Formula in the Plane)**.** *Let S be an elementary region in the xy-plane (such as a disk or parallelogram for example). Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be an invertible and differentiable mapping, and let $T(S)$ be the image of S under T. Then*

$$\int\int_S 1 \cdot dxdy \ = \ \int\int_{T(S)} 1 \cdot \left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right| dudv,$$

or more generally

$$\int\int_S f(x,y) \cdot dxdy \ = \ \int\int_{T(S)} f\left(T^{-1}(u,v)\right) \cdot \left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right| dudv.$$

Some notes on the above:

(1) We assume T has an inverse function, denoted T^{-1}. Thus $T(x,y) = (u,v)$ and $T^{-1}(u,v) = (x,y)$.

(2) We assume for each $(x,y) \in S$ there is one and only one (u,v) that it is mapped to, and conversely each (u,v) is mapped to one and only one (x,y).

(3) The derivative of $T^{-1}(u,v) \ = \ (x(u,v), y(u,v))$ is

$$(DT^{-1})(u,v) \ = \ \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix},$$

and the absolute value of the determinant of the derivative is

$$\left|\det\left(DT^{-1})(u,v)\right)\right| \ = \ \left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right|,$$

which implies the area element transforms as

$$dxdy \ = \ \left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right| dudv.$$

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(4) Note that f takes as input x and y, but when we change variables our new inputs are u and v. The map T^{-1} takes u and v and gives x and y, and thus we need to evaluate f at $T^{-1}(u, v)$. Remember that we are now integrating over u and v, and thus the integrand must be a function of u and v.

(5) Note that the formula requires an absolute value of the determinant. The reason is that the determinant can be negative, and we want to see how a small area element transforms. Area is supposed to be positively counted. Note in one-variable calculus that $\int_a^b f(x)dx = -\int_b^a f(x)dx$; we need the absolute value to take care of issues such as this.

(6) While we stated T is a differentiable mapping, our assumptions imply T^{-1} is differentiable as well.

2. Sketch of Proof

The Change of Variable Theorem (or Formula) is one of the most important results of multivariable calculus. The reason is that numerous problems have a natural coordinate system where, if we look at it from the right perspective, the analysis greatly simplifies. It is thus very important to be able to convert from one coordinate system to another and be able to exploit the advantages of each.

Our first example was mapping the unit square to a rectangle (see Figure 1). Note the original square, S, has area 1 and the region it maps to, $T(S)$, has area 6. Thus $dxdy$ corresponds to $\frac{1}{6}dudv$. If we compute the derivative matrix associated to T^{-1}, since

$$x(u,v) \ = \ u/2 \quad \text{and} \quad y(u,v) \ = \ v/3,$$

we find

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \ = \ \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{pmatrix},$$

which has a determinant of $1/6$. This verifies the formula's prediction, namely that the exchange rate from xy-space to uv-space is $1/6$. In other words,

$$\int\int_S 1 \cdot dxdy \ = \ \int\int_{T(S)} \frac{1}{6} \cdot dudv;$$

clearly we don't expect $\int\int_S 1 \cdot dxdy$ to equal $\int\int_{T(S)} \cdot dudv$; the absolute value of the determinant of the derivative matrix gives the exchange rate.

We now sketch the proof. It will involve several of the major concepts we've discussed throughout the semester, from the cross product to determinants and areas to the definition of the derivative being that the tangent

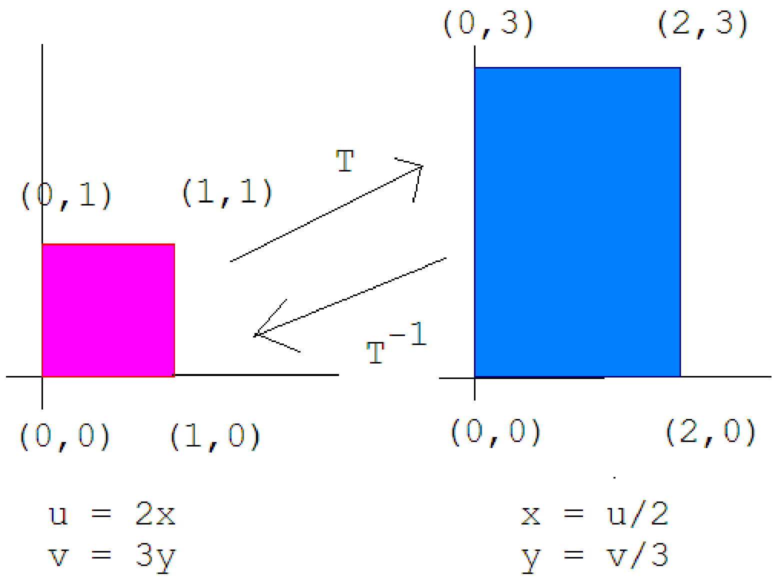

FIGURE 1. Mapping the unit square via u = 2x and v = 3y

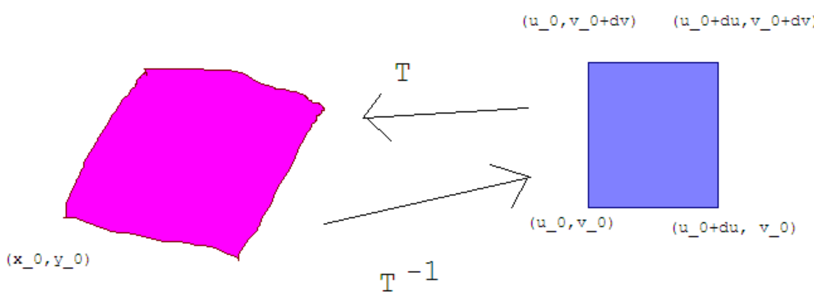

FIGURE 2. Mapping of the general case: $x(u,v)$ and $y(u,v)$. Note: to save time I've written $u_0 + du$ for $u_0 + \Delta u$, and similarly for the v's, above.

plane is a great approximation. Recall we have

$$T^{-1}(u,v) \;=\; (x(u,v), y(u,v)).$$

We want to see what a small rectangle in uv-space corresponds to in xy-space; see Figure 2. We want to see where the four corners of the rectangle in the uv-plane are mapped. Recall that if we have a function $f(u,v)$, then

$$f(u,v) \;=\; f(u_0,v_0) + (\nabla f)(u_0,v_0)\cdot(u-u_0, v-v_0) + \text{small}$$

if (u, v) is close to (u_0, v_0). There are many ways to look at this. It is a Taylor expansion, it is a definition of the derivative, it is taking a directional derivative.

As T^{-1} is differentiable, we can write

$$\begin{aligned} x(u,v) &= x(u_0,v_0) + (\nabla x)(u_0,v_0)\cdot(u-u_0,v-v_0) + \text{small} \\ y(u,v) &= y(u_0,v_0) + (\nabla y)(u_0,v_0)\cdot(u-u_0,v-v_0) + \text{small}, \ , \end{aligned}$$

so long as (u, v) is close to (u_0, v_0). This is simply the definition of the derivative in several variables, namely the statement that we can approximate a complicated function locally by a plane. What makes it a little confusing is that x is now our function name, not a coordinate (this is why we considered the example with a function f above). Thus, while the square with lengths Δu and Δv in uv-space doesn't map to exactly a square, rectangle or parallelogram in xy-space, it maps to almost a parallelogram. Let's see where the four corners of the rectangle map to; expanding the gradient we find

$$\begin{aligned} x(u,v) &= x_0 + \frac{\partial x}{\partial u}(u-u_0) + \frac{\partial x}{\partial v}(v-v_0) + \text{small} \\ y(u,v) &= y_0 + \frac{\partial y}{\partial u}(u-u_0) + \frac{\partial y}{\partial v}(v-v_0) + \text{small}. \end{aligned}$$

Note that $x(u_0, v_0)$ is what we're calling x_0 and $y(u_0, v_0)$ is what we're calling y_0, the base point of the square. For definiteness we are assuming the four corner's orientation is preserved under the mapping (we had to choose how to draw / discuss things). We have

$$\begin{aligned} (x(u_0+\Delta u,v_0), y(u_0+\Delta u,v_0)) &= \left(x_0 + \frac{\partial x}{\partial u}\Delta u, y_0 + \frac{\partial y}{\partial u}\Delta u\right) \\ (x(u_0,v_0+\Delta v), y(u_0,v_0+\Delta v)) &= \left(x_0 + \frac{\partial x}{\partial v}\Delta v, y_0 + \frac{\partial y}{\partial v}\Delta v\right). \end{aligned}$$

The original rectangle in uv-space had sides given by the vectors $(u_0 + \Delta u, v_0) - (u_0, v_0)$ and $(u_0, v_0 + \Delta v) - (u_0, v_0)$. Thus the area is equivalent to that of a rectangle given by the vectors $(\Delta u, 0)$ and $(0, \Delta v)$, for an area of $\Delta u \Delta v$.

What about the region it is mapped to? It is essentially a parallelogram; this is the content of the function T^{-1} being differentiable. The side $(u_0 + \Delta u, v_0) - (u_0, v_0)$ which was equivalent to the vector $(\Delta u, 0)$ corresponds to

$$(x(u_0+\Delta u, v_0) - (x_0, y_0)),$$

which is just

$$\left(\frac{\partial x}{\partial u}\Delta u, \frac{\partial y}{\partial u}\Delta u\right);$$

similarly the other side corresponds to

$$\left(\frac{\partial x}{\partial v}\Delta v, \frac{\partial y}{\partial v}\Delta v\right).$$

To find the area of a parallelogram with sides $\overrightarrow{w}_1$ and $\overrightarrow{w}_2$ we need only take the cross product. We must be careful, though. The cross product takes as input two vectors with three components and outputs a vector with three components. We can consider our vectors as living in three-dimensional space by appending a zero as the third component, and then the area of the parallelogram is the length of the cross product. We must compute

$$\left(\frac{\partial x}{\partial u}\Delta u, \frac{\partial y}{\partial u}\Delta u, 0\right) \times \left(\frac{\partial x}{\partial v}\Delta v, \frac{\partial y}{\partial v}\Delta v, 0\right).$$

Recall this is given by

$$\left|\begin{array}{ccc} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial x}{\partial u}\Delta u & \frac{\partial y}{\partial u}\Delta u & 0 \\ \frac{\partial x}{\partial v}\Delta v & \frac{\partial y}{\partial v}\Delta v & 0 \end{array}\right| = \left(0, 0, \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)\Delta u \Delta v\right),$$

and the length is clearly just

$$\left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right| \Delta u \Delta v,$$

or equivalently

$$dxdy \sim \left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right| \Delta u \Delta v.$$

The outline above highlights the key ideas in the proof. One needs to perform a careful analysis of the error terms, but the main points are above. The idea is that locally any differentiable map is linear (and takes rectangles to parallelograms), and then we piece the contributions over the entire region together. The absolute value of the determinant of the derivative map gives us the exchange rate between the two different areas.

3. Special Cases

Theorem 3.1 (Change of Variables Theorem: Polar Coordinates)**.** *Let*

$$x = r\cos\theta, \quad y = r\sin\theta$$

with $r \geq 0$ *and* $\theta \in [0, 2\pi)$*; note the inverse functions are*

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan(y/x).$$

Let D be an elementary region in the xy-plane, and let D^* be the corresponding region in the $r\theta$-plane. Then

$$\int\int_D f(x,y)dxdy \ = \ \int\int_{D^*} f(r\cos\theta, r\sin\theta)rdrd\theta.$$

For example, if D is the region $x^2+y^2 \le 1$ in the xy-plane then D^* is the rectangle $[0,1]\times[0,2\pi]$ in the $r\theta$-plane.

Theorem 3.2 (Change of Variables Theorem: Cylindrical Coordinates)**.** *Let*

$$x \ = \ r\cos\theta, \quad y \ = \ r\sin\theta, \quad z \ = \ z$$

with $r \ge 0$, $\theta \in [0,2\pi)$ and z arbitrary; note the inverse functions are

$$r \ = \ \sqrt{x^2+y^2}, \quad \theta \ = \ \arctan(y/x), \quad z \ = \ z.$$

Let D be an elementary region in the xyz-plane, and let D^* be the corresponding region in the $r\theta z$-plane. Then

$$\int\int\int_D f(x,y,z)dxdydz \ = \ \int\int\int_{D^*} f(r\cos\theta, r\sin\theta, z)rdrd\theta dz.$$

Theorem 3.3 (Change of Variables Theorem: Spherical Coordinates)**.** *Let*

$$x \ = \ \rho\sin\phi\cos\theta, \quad y \ = \ \rho\sin\phi\sin\theta, \quad z \ = \ \rho\cos\phi$$

with $\rho \ge 0$, $\theta \in [0,2\pi]$ and $\phi \in [0,\pi)$. Note that the angle ϕ is the angle made with the z-axis; many books (such as physics texts) interchange the role of ϕ and θ. Let D be an elementary region in the xyz-plane, and let D^* be the corresponding region in the $\rho\theta\phi$-plane. Then

$$\int\int\int_D f(x,y,z)dxdydz \ =$$
$$\int\int\int_{D^*} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin(\phi)d\rho d\theta d\phi.$$

Note that the most common mistake is to have incorrect bounds of integration.