Math 105: Solutions to Practice Problems

Steven Miller

May 13, 2010

Abstract

Below are detailed solutions to some problems similar to some assigned homework problems.

Contents

1 The Geometry of Euclidean Space

2 Differentiation

2.1 The geometry of real-valued functions

2.2 Limits and continuity

2.3 Differentiation

2.4 Introduction to paths and curves

2.5 Properties of the derivative

2.6 Gradients and directional derivatives

The assignment is: Section 2.6: #2ab, #4a, #6a, #16 (the (in)famous Captain Ralph problem), #18.

Question: #2c: Compute the directional derivative of $f(x,y) = e^x \cos(\pi y)$***,*** $(x_0, y_0) = (0, -1)$ ***and*** $\overrightarrow{v} = \left(-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$***.***

Solution: We have

$$(\nabla f)(x,y) \;=\; \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \;=\; \left(e^x \cos(\pi y), -\pi e^x \sin(\pi y)\right),$$

and thus if we evaluate at $(0,-1)$ we find

$$(\nabla f)(0,-1) \;=\; (-1, 0)\,.$$

The directional derivative in general is

$$(\nabla f)(x_0, y_0) \cdot \overrightarrow{v},$$

so for this problem the answer is

$$(-1, 0) \cdot \left(-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) \;=\; \frac{1}{\sqrt{5}}.$$

Note that $\overrightarrow{v}$ is a unit length vector.

Question: #2d: Compute the directional derivative of $f(x,y) = xy^2 + x^3 y$***,*** $(x_0, y_0) = (4, -2)$ ***and*** $\overrightarrow{v} = \left(\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right)$***.***

Solution: We have

$$(\nabla f)(x,y) \;=\; \left(y^2 + 3x^2 y, 2xy + x^3\right),$$

and thus if we evaluate at $(4,-2)$ we find

$$(\nabla f)(4,-2) \;=\; (-92, 48)\,.$$

The directional derivative in general is

$$(\nabla f)(x_0, y_0) \cdot \overrightarrow{v},$$

so for this problem the answer is

$$(-92, 48) \cdot \left(\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right) \;=\; \frac{52}{\sqrt{10}}.$$

Note that $\overrightarrow{v}$ is a unit length vector.

Question: #3a: Compute Df ***for*** $f(x,y) = (x^2 y, e^{-xy})$**.**

Solution: Note that $f : \mathbb{R}^2 \to \mathbb{R}^2$, and thus Df will be a 2×2 matrix. Writing $f(x,y)$ as $(f_1(x,y), f_2(x,y))$, we have the first row of Df is ∇f_1, while the second row is ∇f_2. Explicitly,

$$(Df)(x,y) \;=\; \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} \;=\; \begin{pmatrix} 2xy & x^2 \\ -ye^{-xy} & -xe^{-xy} \end{pmatrix}.$$

Question: #4c: Find the plane tangent to $xyz = 1$ ***at the point*** $(1,1,1)$**.**

Solution: We use equation (1) on the bottom of page 167, which says that if $f(x,y,z) = k$ (for some constant k) then the tangent plane at (x_0, y_0, z_0) is given by

$$(\nabla f)(x_0,y_0,z_0) \cdot (x - x_0, y - y_0, z - z_0) \;=\; 0.$$

For our problem, $f(x,y,z) = xyz$ and $k = 1$. We have

$$(\nabla f)(x,y,z) \;=\; (yz, xz, xy),$$

which yields

$$(\nabla f)(1,1,1) \;=\; (1,1,1).$$

Thus the tangent plane is all (x,y,z) satisfying

$$(1,1,1) \cdot (x-1, y-1, z-1) \;=\; 0,$$

or equivalently it is

$$x + y + z - 3 \;=\; 0.$$

Question: #6b: Compute the gradient of $f(x,y,z) = xy + yz + xz$**.**

Solution: The gradient is defined as the vector of partial derivatives:

$$\mathrm{grad}(f) \;=\; \nabla f \;=\; \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right).$$

A straightforward computation shows

$$\nabla f \;=\; (y+z, x+z, x+y).$$

Question: #6c: Compute the gradient of $f(x,y,z) = 1/(x^2+y^2+z^2)$**.**

Solution: By symmetry, it suffices to compute $\frac{\partial f}{\partial x}$, as $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$ are obtained through analogous computations. To compute $\frac{\partial f}{\partial x}$, we use the one-variable chain (or power) rule:

$$\begin{aligned}
\frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1} &= -(x^2+y^2+z^2)^{-2}\frac{\partial}{\partial x}\left(x^2+y^2+z^2\right) \\
&= -\frac{2x}{(x^2+y^2+z^2)^2} \\
&= -2f(x,y,z)^2 \cdot x.
\end{aligned}$$

Collecting yields

$$\nabla f \;=\; -2f(x,y,z)(x,y,z).$$

Question: #8b: Compute the equation of the tangent planes for $f(x,y,z) = x^3 - 2y^3 + z^3 = 0$ ***at*** $(1,1,1)$**.**

Solution: First, we note that the point $(1,1,1)$ *is* on the surface. The tangent plane is given by equation (1) on page 167. Explicitly, it is all (x,y,z) satisfying

$$(\nabla f)(1,1,1) \cdot (x-1, y-1, z-1).$$

As

$$\nabla f \;=\; (3x^2, -6y^2, 3z^2),$$

we have

$$(\nabla f)(1,1,1) \;=\; (3,-6,3),$$

which implies the tangent plane is

$$(3,-6,3) \cdot (x-1, y-1, z-1) \;=\; 0,$$

or

$$3x - 6y + 3z \;=\; 0.$$

Question: #19: A function** $f:\mathbb{R}^n\to\mathbb{R}$ **is said to be* even *if $f(\overrightarrow{x})=f(-\overrightarrow{x})$ ***for all*** $\overrightarrow{x}$***. If*** f ***is differentiable and even, find*** $(Df)(\overrightarrow{0})$**.**

Solution: Whenever we have to prove something in several variables, it is not a bad idea to look at some examples from one-variable calculus to build up our intuition. We first recall some even, differentiable functions: x^2, x^4, x^{2n}, $\cos x$. All of these have first derivative equal to 0 at the origin, and thus it is natural to guess that $(Df)(\overrightarrow{0})=\overrightarrow{0}$.

One way to prove this is by using the Chain Rule. Let $g(\overrightarrow{x})=-\overrightarrow{x}$ (so $g:\mathbb{R}^n\to\mathbb{R}^n$). Then

$$A(x)\ =\ f(\overrightarrow{x})\ =\ f(g(\overrightarrow{x})),$$

so

$$(DA)(\overrightarrow{x})\ =\ (Df)(\overrightarrow{x})\ =\ (Df)(g(\overrightarrow{x}))(Dg)(\overrightarrow{x}).$$

As $g(\overrightarrow{x})=-\overrightarrow{x}$, unwinding this we find

$$g(x_1,\dots,x_n)\ =\ (-x_1,\dots,-x_n),$$

which implies

$$(Dg)(\overrightarrow{x})\ =\ (\nabla g)(\overrightarrow{x})\ =\ -I,$$

where I is the $n\times n$ identity matrix which is 1 along the main diagonal and 0 elsewhere. The reason this is the answer is that g has n inputs *and* n outputs. Thus (Dg) is a matrix with n rows and n columns. The first row is Dg_1 or ∇g_1, where $g_1(x_1,\dots,x_n)=-x_1$, while the last row is Dg_n or ∇g_n.

At the origin, $g(\overrightarrow{0})=\overrightarrow{0}$ and $(Dg)(\overrightarrow{0})=-I$, and thus

$$(Df)(\overrightarrow{0})\ =\ (Df)(-\overrightarrow{0})(Dg)(\overrightarrow{0})$$

becomes

$$(Df)(\overrightarrow{0})\ =\ (Df)(\overrightarrow{0})\,(-I)\ =\ -(Df)(\overrightarrow{0}).$$

We thus have an equation of the form $\overrightarrow{u}=-\overrightarrow{u}$; the only solution is $\overrightarrow{u}=\overrightarrow{0}$, or in other words since $(Df)(\overrightarrow{0})$ equals its own negative, it must be the zero vector.

2.7 Review Exercises - Page 173

Question: #22: Find the direction in which the function $w(x,y)=x^2+xy$ ***increases most rapidly at the point*** $(-1,1)$***. What is the magnitude of*** ∇w ***at this point.***

Solution: We have $\nabla w = (2x + y, x)$, so $(\nabla w)(-1, 1) = (-1, -1)$. As the directional derivative in the direction $\overrightarrow{v}$ at $(-1, 1)$ is $(\nabla w)(-1, 1) \cdot \overrightarrow{v}$, which is maximized when $\overrightarrow{v} = (\nabla w)(-1, 1) = (-1, 1)$.

Question: #24: Let $z(x, y) = f(x - y)/y$ (where f is differentiable and $y \neq 0$), show that the identity $z + y\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$.

Solution: We have

$$\frac{\partial z}{\partial y} \;=\; \frac{f'(x-y)(-1) \cdot y - f(x-y)}{y^2}, \quad \frac{\partial z}{\partial x} \;=\; \frac{f'(x-y)}{y}.$$

Thus

$$z + y\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} \;=\; \frac{f(x-y)}{y} + f'(x-y) + \frac{-f'(x-y)y - f(x-y)}{y} \;=\; 0.$$

In the arguments above, we frequently used the one-variable chain rule. For example,

$$\frac{\partial}{\partial y} f(x-y) \;=\; \frac{\partial}{\partial y} f(g(y)),$$

where $g(y) = x - y$. We can now use the one-variable chain rule. As x is fixed, the answer is just $f'(g(y)) \cdot g'(y)$, which is $f'(x-y) \cdot (-1)$.

Question: #44: Verify the chain rule for the function $f(x, y) = x^2/(2 + \cos y)$ and the path $c(t) = (x(t), y(t)) = (e^t, e^{-t})$.

Solution: Setting $A(t) = f(c(t))$, we have $(DA)(t) = (Df)(c(t))c'(t)$. We have $c'(t) = (e^t, -e^{-t})$ (which should really be written as a column vector). For Df, we have

$$Df \;=\; \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \;=\; \left(\frac{2x}{2 + \cos y}, -\frac{x^2 \sin y}{(2 + \cos y)^2}\right);$$

however, we want $(Df)(c(t))$, which is

$$(Df)(c(t)) \;=\; \left(\frac{2e^t}{2 + \cos e^{-t}}, -\frac{e^{2t} \sin e^{-t}}{(2 + \cos e^t)^2}\right).$$

Taking the dot product, we obtain

$$(DA)(t) \;=\; \frac{2e^t}{2 + \cos e^{-t}} \cdot e^t + \frac{e^{2t} \sin e^{-t}}{(2 + \cos e^t)^2} \cdot e^{-t}.$$

We can also compute this derivative directly, as

$$A(t) \;=\; f(c(t)) \;=\; \frac{e^{2t}}{(2+\cos e^{-t})}.$$

Taking the derivative yields

$$A'(t) \;=\; \frac{2e^{2t}(2+\cos e^{-t}) + e^{2t}\sin e^{-t}}{(2+\cos e^{-t})^2},$$

which does agree with the Chain Rule.

3 Higher-order derivatives; maxima and minima

3.1 Iterated partial derivatives

Question: #3: Compute $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}$ ***and*** $\frac{\partial^2 f}{\partial y^2}$ ***for*** $f(x,y) \;=\; \cos(xy^2)$***, and verify the equality of the mixed derivatives.***

Solution: We have

$$\begin{aligned}
\frac{\partial f}{\partial x} &= -y^2\sin(xy^2) \\
\frac{\partial^2 f}{\partial x^2} &= -y^4\cos(xy^2) \\
\frac{\partial^2 f}{\partial y \partial x} &= -2y\sin(xy^2) - 2xy^3\cos(xy^2).
\end{aligned}$$

Similarly, we find

$$\begin{aligned}
\frac{\partial f}{\partial y} &= -2xy\sin(xy^2) \\
\frac{\partial^2 f}{\partial y^2} &= -2x\sin(xy^2) - 4x^2y^2\cos(xy^2) \\
\frac{\partial^2 f}{\partial x \partial y} &= -2y\sin(xy^2) - 2xy^3\cos(xy^2).
\end{aligned}$$

Note that we do have $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$.

Question: #8b: Find all the second partial derivatives of $z(x, y) = x^2y^2e^{2xy}$***.***

Solution: We have

$$\begin{aligned}
\frac{\partial z}{\partial x} &= 2xy^2e^{2xy} + 2x^2y^3e^{2xy} \\
\frac{\partial^2 z}{\partial x^2} &= 2y^2e^{2xy} + 4xy^3e^{2xy} + 4xy^3e^{2xy} + 4x^2y^4e^{2xy} \\
\frac{\partial^2 z}{\partial y \partial x} &= 4xye^{2xy} + 4x^2y^2e^{2xy} + 6x^2y^2e^{2xy} + 4x^3y^3e^{2xy} \\
\frac{\partial z}{\partial y} &= 2x^2ye^{2xy} + 2x^3y^2e^{2xy} \\
\frac{\partial^2 z}{\partial y^2} &= 2x^2e^{2xy} + 4x^3ye^{2xy} + 4x^3ye^{2xy} + 4x^4y^2e^{2xy}.
\end{aligned}$$

Question: Supplemental problem related to #11: Use the fact that the derivative of a sum is the sum of the derivatives to prove that the derivative of a sum of three terms is the sum of the three derivatives.

Solution: The idea to solve this problem is quite useful in mathematics (and may be useful to attacking #11). We know that for any two functions $f(x)$ and $g(x)$ that $\frac{d}{dx}(f(x) + g(x)) = \frac{df}{dx} + \frac{dg}{dx}$. We now use this result to show a similar claim holds for the sum of three functions. We have

$$\begin{aligned}
A(x) &= f(x) + g(x) + h(x) \\
\frac{dA}{dx} &= \frac{d}{dx}(f(x) + g(x) + h(x)) \\
&= \frac{d}{dx}(B(x) + h(x)), \quad B(x) \ = \ f(x) + g(x) \\
&= \frac{dB}{dx} + \frac{dh}{dx} \\
&= \left(\frac{df}{dx} + \frac{dg}{dx}\right) + \frac{dh}{dx} \\
&= \frac{df}{dx} + \frac{dg}{dx} + \frac{dh}{dx},
\end{aligned}$$

where we constantly used the fact that the derivative of a sum of two functions is the sum of the two derivatives.

Question: #11: Use Theorem 1 to show that if $f(x, y, z)$ ***is of class*** C^3 ***then*** $\frac{\partial^3 f}{\partial x \partial y \partial z} = \frac{\partial^3 f}{\partial y \partial z \partial x}$**.**
Hint: Slowly switch orders of differentiation. For example, we know $\frac{\partial f}{\partial z} = \frac{\partial f}{\partial z}$, and so we may differentiate both sides with respect to x, obtaining $\frac{\partial^2 f}{\partial x \partial z} = \frac{\partial^2 f}{\partial x \partial z}$, and then we may rewrite the right hand side as $\frac{\partial^2 f}{\partial z \partial x}$. We now differentiate both sides with respect to y, and keep switching orders.

3.2 Taylor's theorem

Question: #1: Find the second order Taylor series expansion for $f(x, y) = (x + y)^2$ ***about*** $(x_0, y_0) = (0, 0)$**.**

Solution: We give two solutions. The first is the standard solution. We have

$$(\nabla f)(x, y) \ = \ \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \ = \ (2(x + y), 2(x + y))$$

and

$$(Hf)(x, y) \ = \ \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} \ = \ \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}.$$

Thus the second order expansion is

$$f(0, 0) + (\nabla f)(0, 0) \cdot (x, y) + \frac{1}{2}(x, y)(Hf)(0, 0) \begin{pmatrix} x \\ y \end{pmatrix},$$

which is

$$\begin{aligned} 0 + \frac{1}{2}(xy) \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad &= \quad \frac{1}{2}(xy) \begin{pmatrix} 2x + 2y \\ 2x + 2y \end{pmatrix} \\ &= \quad \frac{1}{2}\left[x(2x + 2y) + y(2x + 2y)\right] \\ &= \quad x(x + y) + y(x + y) \ = \ (x + y)^2. \end{aligned}$$

We now present another solution. The Taylor series expansion of $g(u) = u^2$ is simply $0 + 0u + u^2$, and thus taking $x + y$ for u gives the second order Taylor series is just $(x + y)^2$.

It isn't surprising that this is the answer – f is a polynomial of degree 2, and thus its second order Taylor series should equal itself!

Question: #4: Find the second order Taylor series expansion for $f(x,y) = e^{-(x^2+y^2)}\cos(xy)$ ***about*** $(x_0,y_0) = (0,0)$**.**

Solution: The long way to do this is to compute

$$(\nabla f)(x,y) \ = \ \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

and

$$(Hf)(x,y) \ = \ \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix},$$

and then use our result that the second order expansion is

$$f(0,0) + (\nabla f)(0,0) \cdot (x,y) + \frac{1}{2}(x,y)(Hf)(0,0)\begin{pmatrix} x \\ y \end{pmatrix}.$$

This is not pleasant; for instance,

$$\frac{\partial f}{\partial x} \ = \ -2xe^{-(x^2+y^2)}\cos(xy) - ye^{-(x^2+y^2)}\sin(xy).$$

There is a faster way. Rolling up our sleeves and doing the work, we find

$$f(0,0) \ = \ 1,$$
$$(\nabla f)(0,0) \ = \ (0,0)$$

and after even more work we find

$$(Hf)(0,0) \ = \ \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix};$$

we can make our life a little easier by noting that f is of class $\mathcal{C}^2$, and thus $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$. Thus we have one fewer painful derivative to take.

By Taylor's theorem, the second order approximation is just

$$f(0,0) + (\nabla f)(0,0) \cdot (x,y) + \frac{1}{2}(x,y)(Hf)(0,0)\begin{pmatrix} x \\ y \end{pmatrix}.$$

Substituting gives

$$\begin{aligned} & 1 + (0,0)\cdot(x,y) + \frac{1}{2}(x,y)\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \\ = \ & 1 + \frac{1}{2}(x,y)\begin{pmatrix} -2x \\ -2y \end{pmatrix} \\ = \ & 1 - x^2 - y^2. \end{aligned}$$

We can determine the Taylor series very easily using our trick. We have

$$e^u = 1 + u + \frac{u^2}{2} + \cdots,$$

so

$$e^{-(x^2+y^2)} = 1 - (x^2 + y^2) + \cdots;$$

we stopped at this term as this term is already of order 2 in x and y, and thus there is no need to keep further terms (as we only want up to second order). Similarly we find

$$\cos(w) = 1 - \frac{w^2}{2} + \cdots,$$

so

$$\cos(xy) = 1 - \cdots;$$

here we only kept one term as the next term would be $w^2/2 = x^2y^2/2$, which is a fourth order (and not a second order) term. We thus find the Taylor series expansion of order 2 at the origin is simply

$$1 - (x^2 + y^2),$$

and this was obtained with significantly less work!

You of course need to know how to compute a Taylor series in general, but this trick will work in most of the problems you need.

3.3 Extrema of real-valued functions

Question: #4: Find the critical points of $f(x, y) = x^2 + y^2 + 3xy$**.**

Solution: We must solve $\nabla f = \overrightarrow{0}$. We have

$$(\nabla f) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (2x + 3y, 2y + 3x).$$

Thus an extremum occurs when

$$2x + 3y = 0, \quad 3x + 2y = 0.$$

There are several ways to proceed. Note, however, that at this point it is no longer a calculus problem, but rather an algebra one. A common approach is to solve for one variable in terms of the other (i.e., the substitution method). Another is to multiply the equations by various constants and combine.

Let's solve for y in terms of x. We have $y = -2x/3$ from the first equation and $y = -3x/2$ from the second. Thus the only solution is $x = y = 0$.

Another way of arranging the algebra is to find $y = -2x/3$ from the first equation, and then substitute this into the second, which becomes $3x + 2(-2x/3) = 0$, which clearly implies $x = 0$.

Alternatively, note $5x + 5y = 0$ so $x = -y$ and then $-2y + 3y = 0$ yields $y = 0$.

Question: #6: Find the critical points of $f(x, y) = x^2 - 3xy + 5x - 2y + 6y^2 + 8$**.**

Solution: The critical points are where $\nabla f = \overrightarrow{0}$. For our function we have

$$(\nabla f)(x, y) \ = \ \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \ = \ (2x - 3y + 5, -3x - 2 + 12y),$$

so in order for this to equal the zero vector we must have

$$2x - 3y + 5 \ = \ 0, \quad \text{and} \quad -3x + 12y - 2 \ = \ 0.$$

These are two equations in two unknowns. We have

$$2x - 3y \ = \ -5, \quad -3x + 12y \ = \ 2.$$

There are lots of ways to solve this. We could multiply the first equation by 4 and add it to the second. This will cancel all the y terms, and leave us with $8x - 3x = -20 + 2$, or $5x = -18$ or $x = -18/5$. As $y = \frac{2x+5}{3}$, this implies $y = -\frac{11}{15}$.

Another way to solve this system of equations is to isolate y as a function of x using the first equation, and substitute this into the second. We find $2x - 3y = -5$, so $y = \frac{2x+5}{3}$. Substituting this into the second equation yields

$$-3x - 2 + 12\frac{2x + 5}{3} \ = \ 0,$$

which implies

$$-3x - 2 + 8x + 20 \ = \ 0,$$

or

$$x \ = \ -\frac{18}{5},$$

exactly as before.

3.4 Constrained extrema and Lagrange multipliers

Question: #1: Find the extrema of $f(x,y,z) = x - y + z$ ***subject to*** $g(x,y,z) = x^2 + y^2 + z^2 = 2$**.**

Solution: By the method of Lagrange multipliers, we need $(\nabla f)(x,y,z) = \lambda(\nabla g)(x,y,z)$ for (x,y,z) to be an extremum. We have

$$\nabla f \ = \ (1,-1,1)$$

and

$$\nabla g \ = \ (2x,2y,2z).$$

Thus we are searching for a λ and a point (x,y,z) where

$$(1,-1,1) \ = \ \lambda(2x,2y,2z).$$

We find

$$2\lambda x \ = \ 1, \quad 2\lambda y \ = \ -1, \quad 2\lambda z \ = \ 1.$$

As $\lambda \neq 0$ (if $\lambda = 0$ then there is no way to have the two gradients equal), we have $x = z = -y$. We still have another equation to use, namely $g(x,y,z) = 2$. There are several ways to proceed. We can solve and find $x = z = 1/2\lambda$, $y = -1/2\lambda$, and thus

$$\frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} \ = \ 2,$$

which implies $3/4\lambda^2 = 2$ or $\lambda^2 = 3/8$, which yields $\lambda = \pm\sqrt{3/8}$. There are thus two points where f may have an extremum, namely

$$(1/2\sqrt{3/8}, -1/2\sqrt{3/8}, 1/2\sqrt{3/8}), \quad (-1/2\sqrt{3/8}, 1/2\sqrt{3/8}, -1/2\sqrt{3/8}).$$

Evaluating f at the first point gives $1/2\sqrt{3/8}$, while evaluating f at the second point gives $-1/2\sqrt{3/8}$.

Question: #4: Find the extrema of $f(x,y) = x$ ***subject to*** $g(x,y) = x^2 + 2y^2 = 3$**.**

Solution: We have $\nabla f = (1,0)$ and $\nabla g = (2x,4y)$. Thus at an extremum the point (x,y) must satisfy, for some λ, the equation

$$(1,0) \ = \ \lambda(2x,4y).$$

This implies $1 = 2\lambda x$ and $0 = 4\lambda y$. We must therefore have $y = 0$, but at this point we cannot determine x and λ, only their product (which is 1/2). All is not lost, however, as we know $x^2 + 2y^2 = 3$. As $y = 0$, we then find $x^2 = 3$ so $x = \pm\sqrt{3}$. We could now easily determine λ (it is just $\pm 1/2\sqrt{3}$); however, there is no need to. The only reason we care about λ is that it is supposed to help us in finding where f has an extremum. As we already know the x and y coordinates, we have all the information we need. Thus the extrema occur at $x = \pm\sqrt{3}$.

We could have predicted this answer in the beginning. We have our function depending only on x and constrained to lie on an ellipse. We thus naturally want the x-extension as large as possible, which means taking $y = 0$ and being at the extremes of the major-axis.

Question: Find the maximum value of $f(x, y, z) = xyz$ ***given that*** $g(x, y, z) = x + y + z = 3$ ***and*** $x, y, z \geq 0$***.***

Solution: We may interpret this problem as saying we have a bar three units in length, and we can fold it twice at right angles to give a skeleton of part of a box; how should we divide it so that the volume is maximized? While it seems clear that the answer should be $x = y = z = 1$, we must prove this. The main constraint is $g(x, y, z) = 3$; we need the other constraint so as to eliminate possible solutions such as $(-100)(-100)(203)$.

Using Lagrange multipliers, we want $\nabla f = \lambda \nabla g$. As

$$\nabla f = (yz, xz, xy)$$

and

$$\nabla g = (1, 1, 1),$$

this means

$$(yz, xz, xy) = \lambda(1, 1, 1).$$

If $\lambda = 0$ then at least one of x, y and z equals zero, and the volume xyz is zero; thus this clearly cannot be the maximum. We may thus assume $\lambda \neq 0$. We have

$$yz = xz = xy = \lambda,$$

and we may assume none of x, y or z vanish. As $yz = xz$, since $z \neq 0$ we have $y = x$. Looking at the other equality yields $y = z$, and hence $x = y = z$. As $g(x, y, z) = x + y + z = 3$, since the three variables are equal we must have each of them equal to 1.

More generally, if we have n non-negative numbers with a fixed sum, then their product is maximized when they are all equal. The next, more advanced question we could ask is what n maximizes the product for a given sum. This question is related to what base we should use in building computers. Interestingly, this implies that if we are primarily concerned with data storage, we should work in base 3 and not base 2. The answer is related to e, and the fact that 3 is closer to e than 2. This was an extra credit problem earlier in the semester; for a non-multivariable calculus solution, see

```
http://www.williams.edu/go/math/sjmiller/public_html/
            105/extracredit/ExtraCredit_SummandsN.pdf
```

Question: Maximize the function $f(x,y,z) = xy + yz + xz$ ***on the unit sphere*** $g(x,y,z) = x^2 + y^2 + z^2 = 1$***. Note this is a hard problem, but looking through the arguments below will give you a great grounding in how to handle the algebra that can arise.***

Solution: We need $\nabla f = \lambda \nabla g$. Differentiating yields

$$\nabla f \ = \ (y+z, x+z, x+y) \ = \ \lambda(2x, 2y, 2z) \ = \ \lambda \nabla g.$$

We thus have four equations in four unknowns:

$$\begin{aligned} y + z &= 2\lambda x \\ x + z &= 2\lambda y \\ x + y &= 2\lambda z \\ x^2 + y^2 + z^2 &= 1. \end{aligned}$$

There are many ways to solve these equations. We describe a few. First, note that if we take ratios of any two of the first three equations that the λ disappears. (Note λ cannot equal 0. If it did, we would have $x + z = 0$ and $x + y = 0$. This would force y to equal z, which when substituted into $y + z = 0$ would give $y = z = 0$. We would then have $x = 0$, and hence the constraint $g(x,y,z) = 1$ could not be satisfied.) Dividing the first equation by the second gives $\frac{y+z}{x+z} = \frac{x}{y}$. Cross multiplying gives $y^2 + yz = x^2 + xz$. Looking at the ratio of the second and the third equations gives $\frac{x+z}{x+y} = \frac{y}{z}$, or $z^2 + xz = y^2 + xy$. We thus have

$$x^2 + xz \ = \ y^2 + yz, \quad z^2 + xz \ = \ y^2 + xy.$$

Subtracting these two equations from each other gives

$$x^2 - z^2 \ = \ yz - xy,$$

or

$$(x-z)(x+z) \ = \ y(z-x).$$

There are thus two solutions: either $x-z=0$ or $y=-(x+z)$. We leave the rest of this approach to the reader.

Another way to attack this problem is to add the first three equations to each other, which gives

$$(y+z)+(x+z)+(x+y) \ = \ 2\lambda x+2\lambda y+2\lambda z,$$

or equivalently

$$2(x+y+z) \ = \ 2\lambda(x+y+z).$$

Thus either $x+y+z=0$ or $\lambda=1$. If $\lambda=1$ then squaring the first three equations gives

$$(y+z)^2+(x+z)^2+(x+y)^2 \ = \ 4x^2+4y^2+4z^2 \ = \ 4,$$

where the last follows from the fact that $x^2+y^2+z^2=1$. If we expand the squares we find

$$y^2+2yz+z^2+x^2+2xz+z^2+x^2+2xy+y^2 \ = \ 4.$$

Note the left hand side has $2(x^2+y^2+z^2)$, which is 2. Thus we have

$$2+2yz+2xz+2xy \ = \ 4,$$

or

$$yz+xz+yz \ = \ 1.$$

Note, however, that $yz+xz+yz$ is just our function $f(x,y,z)$! We leave the rest of the details of this problem to the reader.

4 Vector-valued functions

5 Double and Triple Integrals

5.1 Introduction

Question: #1b: Find $\int_0^{\pi/2}\int_0^1(y\cos x+2)dydx$.

Solution: We first do the y-integral, and then the x-integral. We have

$$\begin{aligned}\int_0^{\pi/2}\int_0^1 (y\cos x+2)dydx &= \int_0^{\pi/2}\left[\int_0^1 (y\cos x+2)dy\right]dx\\ &= \int_0^{\pi/2}\left[\frac{y^2}{2}\bigg|_0^1 \cos x + 2y\bigg|_0^1\right]dx\\ &= \int_0^{\pi/2}\left(\frac{\cos x}{2}+2\right)dx\\ &= \left[\frac{\sin x}{2}+2x\right]\bigg|_0^{\pi/2}\\ &= \frac{1}{2}+\pi.\end{aligned}$$

Question: #1d: Find $\int_{-1}^0\int_1^2(-x\ln y)dydx$.

Solution: Again, we do the y-integral first, followed by the x-integral. We need to find a function whose derivative is $\ln y$. It is natural (forgive the pun) to try $y\ln y$. Why is this a reasonable guess? When we take the derivative, we use the product rule and the first piece is just $1\cdot\ln y$. Thus this is close to what we want, though not quite the correct answer. The problem is the full derivative is

$$1\cdot\ln y + y\cdot\frac{1}{y} \;=\; \ln y + 1;$$

again, this is almost correct, but we are off by 1. We may interpret this as saying our guess is off by a function whose derivative is 1; one example of such a function is y. If we subtract this from our original guess, we should end up with the correct anti-derivative. Specifically,

$$(y\ln y - y)' \;=\; 1\cdot\ln y + y\cdot\frac{1}{y} \;-\; 1 \;=\; \ln y;$$

we have thus found the sought-after anti-derivative. This is the Method of *Guess and Check*, and it is a powerful way to find anti-derivatives.

Armed with the anti-derivative for $\ln y$, we can solve the problem. We have

$$\begin{aligned}
\int_{-1}^{0}\int_{1}^{2}(-x\ln y)dydx &= \int_{-1}^{0}\left[\int_{1}^{2}(-x\ln y)dy\right]dx \\
&= -\int_{-1}^{0}x\left[\int_{1}^{2}\ln y dy\right]dx \\
&= -\int_{-1}^{0}x\left[y\ln y - y\right]\Big|_{1}^{2}dx \\
&= -\int_{-1}^{0}x\left[(2\ln 2 - 2)-(1\ln 1 - 1)\right]dx \\
&= -\int_{-1}^{0}x(2\ln 2 - 1)dx \\
&= -(2\ln 2 - 1)\int_{-1}^{0}xdx \\
&= -(2\ln 2 - 1)\left[\frac{x^2}{2}\Big|_{-1}^{0}\right] \\
&= -(2\ln 2 - 1)\left[-\frac{1}{2}\right] \\
&= \frac{2\ln 2 - 1}{2}.
\end{aligned}$$

5.2 The Double Integral over a Rectangle

Question: #1c: Evaluate $\int\int_R (xy)^2\cos(x^3)dA$***, where*** $R = [0,1]\times[0,1]$.

Solution: We need to choose whether or not we want to integrate first with respect to x or with respect to y. For this problem, it does not matter as we can write the integral as $f(x)g(y)$ for some functions f and g (here $f(x) = x^2\cos(x^3)$

and $g(y) = y^2$). Let's do the integration with respect to y first. We have

$$\begin{aligned}
\int\int_R (xy)^2 \cos(x^3) dA &= \int_0^1 \left[\int_0^1 x^2 y^2 \cos(x^3) dy\right] dx \\
&= \int_0^1 \cos(x^3) x^2 \left[\int_0^1 y^2 dy\right] dx \\
&= \int_0^1 \cos(x^3) x^2 \left[\left.\frac{y^3}{3}\right|_0^1\right] dx \\
&= \int_0^1 \cos(x^3) x^2 \frac{1}{3} dx
\end{aligned}$$

$$\frac{1}{9}\int_0^1 \cos(x^3) 3x^2 dx;$$

where we multiplied by 1 in the form 3/3 to facilitate the application of u-substitution below (though of course this is not needed). Let $u = x^3$. Then $du = 3x^2 dx$, and as $x : 0 \to 1$ we have $u : 0 \to 1$. (Note it is very important that our function $u = x^3$ is monotonic or strictly increasing in this domain). Thus we have

$$\begin{aligned}
\int\int_R (xy)^2 \cos(x^3) dA &= \frac{1}{9}\int_0^1 \cos u du \\
\left.\frac{\sin u}{9}\right|_0^1 & \\
&= \frac{\sin 1}{9}.
\end{aligned}$$

Question: Compute $\int_0^1 \int_0^1 y\cos(xy) dA$**.**

Solution: We have a choice as to whether or not we want to integrate with respect to x first or with respect to y. Note the integrand is $y\cos(xy)$. If we integrate with respect to x first, then everything will work out nicely through u-substitution; if we do the y integral first we have to use the method of *Guess and Check* to figure out an anti-derivative (with respect to y) of $y\cos(xy)$. Thus let's integrate with respect to x first. We have

$$\begin{aligned}\int_0^1\int_0^1 y\cos(xy)dA &= \int_0^1\left[\int_0^1 y\cos(xy)dx\right]dy\\ &= \int_0^1\left[\int_0^1 \cos(xy)ydx\right]dy.\end{aligned}$$

Let $u = xy$, so $du = ydx$ and $x : 0 \to 1$ means $u : 0 \to y$. We find

$$\begin{aligned}\int_0^1\int_0^1 y\cos(xy)dA &= \int_0^1\left[\int_0^y \cos u du\right]dy\\ &= \int_0^1\left[\sin u\Big|_0^y\right]dy\\ &= \int_0^1 \sin y\\ &= [-\cos y]\Big|_0^1\\ &= (-\cos 1)-(-1)\\ &= 1-\cos 1.\end{aligned}$$

5.3 The Double and Triple Integral Over More General Regions

Question: #1a: Evaluate the iterated integral

$$\int_0^1\int_0^{x^2} dydx,$$

state whether or not the region is x-simple, y-simple or simple. Draw the region.

Solution: Solution: The region is drawn in Figure 1.

The region is y-simple, as for $0 \le y \le 1$ we have $\phi_1(y) \le x \le \phi_2(y)$ with $\phi_1(y) = 0$ and $\phi_2(y) = x^2$. Similarly we see the region is x-simple. For $0 \le y \le 1$ we have $\sqrt{y} \le x \le 1$; we take $\psi_1(y) = \sqrt{y}$ and $\psi_2(y) = 1$. As the region is both x-simple and y-simple, it is simple.

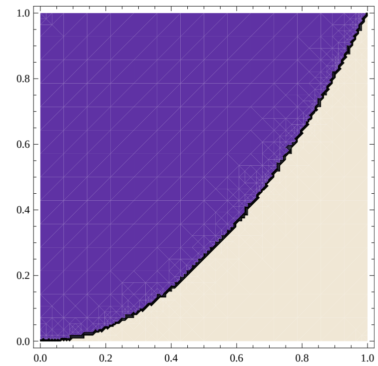

Figure 1: Region corresponding to $0 \leq x \leq 1$ and $0 \leq y \leq x^2$.

We now evaluate the integral. We have

$$\begin{aligned}\int_0^1 \int_0^{x^2} dydx &= \int_0^1 \left[\int_0^{x^2} 1dy\right] dx \\ &= \int_0^1 \left[y\Big|_0^{x^2}\right] dx \\ &= \int_0^1 x^2 dx \\ &= \frac{x^3}{3}\Big|_0^1.\end{aligned}$$

Question: #1a: Evaluate the iterated integral

$$\int_{-3}^{2} \int_0^{y^2} (x^2 + y)dxdy,$$

state whether or not the region is x-simple, y-simple or simple. Draw the region.

Solution: The region is drawn in Figure 2.

The region is clearly x-simple, as for $-3 \leq y \leq 2$ we have $\psi_1(y) \leq x \leq \psi_2(y)$, where $\psi_1(y) = 0$ and $\psi_2(y) = y^2$ (and of course $\psi_1(y) \leq \psi_2(y)$. The region is not y-simple (and hence it is not simple). The reason it is not y-simple

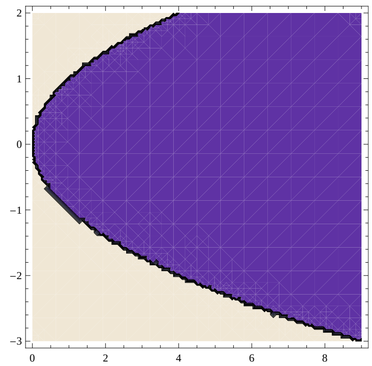

Figure 2: Region corresponding to $-3 \leq y \leq 2$ and $0 \leq x \leq y^2$.

is that for $x \in [0,1]$ the values of y vary discontinuously. For example, if $x = 2$ then $-3 \leq y \leq -\sqrt{2}$ *and* $\sqrt{2} \leq y \leq 2$.

We now evaluate the integral. We have

$$\begin{aligned}
\int_{-3}^{2}\int_{0}^{y^2}(x^2+y)dxdy &= \int_{-3}^{2}\left[\int_{0}^{y^2}(x^2+y)dx\right]dy \\
&= \int_{-3}^{2}\left[\frac{x^3}{3}\bigg|_{0}^{y^2} + xy\bigg|_{0}^{y^2}\right]dy \\
&= \int_{-3}^{2}\left(\frac{y^6}{3}+y^3\right)dy \\
&= \frac{y^7}{21}\bigg|_{-3}^{2} + \frac{y^4}{4}\bigg|_{-3}^{2} \\
&= \frac{128}{21}+\frac{729}{7}+4-\frac{81}{4} \\
&= \frac{7895}{84}.
\end{aligned}$$

5.4 Changing the order of integration

Question: #1a: Sketch the region and evaluate $\int_0^1\int_x^1 xydydx$ ***both ways.***

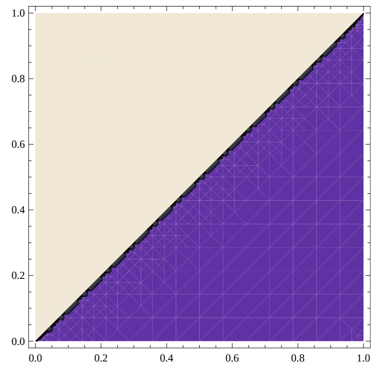

Figure 3: Region corresponding to $x \leq y \leq 1$ and $0 \leq x \leq 1$.

Solution: See Figure 3 for a sketch of the region.
We have

$$\begin{aligned}
\int_0^1 \int_x^1 xydydx &= \int_0^1 x \left[\int_x^1 ydy \right] dx \\
&= \int_0^1 x \left[\frac{y^2}{2} \bigg|_x^1 \right] dx \\
&= \int_0^1 x \left[\frac{1}{2} - \frac{x^2}{2} \right] dx \\
&= \int_0^1 \left(\frac{x}{2} - \frac{x^3}{2} \right) dx \\
&= \frac{x^2}{4} \bigg|_0^1 - \frac{x^4}{8} \bigg|_0^1 \\
&= \frac{1}{4} - \frac{1}{8} = \frac{1}{8}.
\end{aligned}$$

We now do the integration in the opposite order. We fix y now, and y varies from 0 to 1. It is now x that varies, and x ranges from 0 to y. We thus find

$$\begin{aligned}
\int_0^1 \int_x^1 xydydx &= \int_0^1 \int_0^y xydxdy \\
&= \int_0^1 y \left[\int_0^y xdx \right] dy \\
&= \int_0^1 y \left[\left. \frac{x^2}{2} \right|_0^y \right] dy \\
&= \int_0^1 y \left(\frac{y^2}{2} \right) dy \\
&= \left. \frac{y^4}{8} \right|_0^1 = \frac{1}{8}.
\end{aligned}$$

Note that the two orders of integration lead to the same answer for this problem.

5.5 Mathematical Modeling: Baseball / Sabermetrics Lecture

The following are some problems related to the ones from the baseball lecture. The slides are online at

`http://www.williams.edu/go/math/sjmiller/public_html`

`/105/talks/PythagWLTalk_GeneralCalcVersion.pdf`

Question: #1: Let $f(x) = 6x(1-x)$ ***for*** $0 \le x \le 1$ ***and 0 otherwise, and let*** $g(y) = 12y^2(1-y)$ ***for*** $0 \le y \le 1$ ***and zero otherwise. Prove*** f ***and*** g ***are probability distributions. Let*** X ***be a random variable whose probability density of taking on the value*** x ***is*** $f(x)$***, and let*** Y ***be a random variable whose probability density of taking on the value*** y ***is*** $g(y)$***. Compute the probability that*** $X > Y$ ***(assuming, of course, that*** X ***and*** Y ***are independent).***

Solution: To prove that f and g are probability distributions, we must show that each is non-negative and integrates to 1. Both are clearly non-negative; we are left with showing each integrates to one. The fastest way to do this is to note that it suffices to study the integral from 0 to 1 of $x^k(1-x)$. We have

$$\begin{aligned}
\int_0^1 x^k(1-x) &= \int_0^1 \left[x^k - x^{k+1}\right] dx \\
&= \left.\frac{x^{k+1}}{k+1}\right|_0^1 - \left.\frac{x^{k+2}}{k+2}\right|_0^1 \\
&= \frac{1}{k+1} - \frac{1}{k+2} \\
&= \frac{1}{(k+1)(k+2)}.
\end{aligned}$$

Thus

$$h(x) = \begin{cases} (k+1)(k+2)x^k(1-x) & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

is a probability distribution for any positive integer k. In particular, if we take $k = 1$ we see that we should have $6x(1-x)$, while if $k = 2$ we should have $12x^2(1-x)$, which we do. This thus verifies *both* distributions are probability distributions simultaneously.

We now compute the probability that $X > Y$. We are integrating over the triangle $0 \le y \le x \le 1$, and have

$$\begin{aligned}
\text{Prob}(X \ge Y) &= \int_{x=0}^1 \int_{y=0}^x f(x)g(y)dydx \\
&= \int_{x=0}^1 f(x) \left[\int_{y=0}^x g(y)dy\right] dx \\
&= \int_{x=0}^1 6x(1-x) \left[\int_{y=0}^x 12y^2(1-y)dy\right] dx \\
&= \int_{x=0}^1 6x(1-x)12 \left[\int_{y=0}^x (y^2-y^3)dy\right] dx \\
&= \int_{x=0}^1 72(x-x^2) \left[\left.\frac{y^3}{3}\right|_0^x - \left.\frac{y^4}{4}\right|_0^x\right] dx \\
&= \int_{x=0}^1 72(x-x^2) \left[\frac{x^3}{3} - \frac{x^4}{4}\right] dx.
\end{aligned}$$

The simplest way to evaluate this is to expand, and we find

$$\begin{aligned}
\text{Prob}(X \geq Y) &= 72 \int_0^1 \left[\frac{x^6}{4} - \frac{7x^5}{12} + \frac{x^4}{3} \right] dx \\
&= \left. \frac{x^7}{28} \right|_0^1 - \left. \frac{7x^6}{72} \right|_0^1 + \left. \frac{x^5}{15} \right|_0^1 \\
&= \frac{13}{2520}.
\end{aligned}$$

6 Change of variable formula and applications of integration

6.1

6.2 The Change of Variable Theorem

Question: #1: Consider the change of variables $u = 2x + 3y$ ***and*** $v = 4y$***. Show that this map takes the unit square*** $[0, 1] \times [0, 1]$ ***(i.e., the set of points*** (x, y) ***with*** $0 \leq x, y \leq 1$***) to a parallelogram. Use the change of variables formula to find the area of the parallelogram.***

Solution: The unit square is mapped to the parallelogram shown in Figure 4. To see this, look and see where each vertex of the unit square is sent. We have $(0, 0)$ goes to $(0, 0)$, we have $(1, 0)$ goes to $(2, 0)$, $(0, 1)$ goes to $(3, 4)$ and finally $(1, 1)$ goes to $(5, 4)$. More generally, if we take a point of the form $(x, 0)$ it is mapped to the point $(2x, 0)$, so we see the interval $[0, 1]$ on the x-axis is mapped to the interval $[0, 2]$ in the u-axis. A similar analysis shows all the other lines of the unit square are mapped to lines in the uv-plane. For example, consider the line $(x, 1)$ with $0 \leq x \leq 1$. This is mapped to the line $(2x + 3, 4)$ in the uv-plane, or equivalently the line from $(3, 4)$ (corresponding to $x = 0$) to the point $(5, 4)$ (corresponding to $x = 1$).

We need the inverse transformation T^{-1}, which gives us the x and y corresponding to a choice of u and v. We have to invert the relations

$$u = 2x + 3y, \quad v = 4y.$$

The second is the easiest; we clearly need to have $y = v/4$. Knowing this, we then find

$$u = 2x + \frac{3v}{4},$$

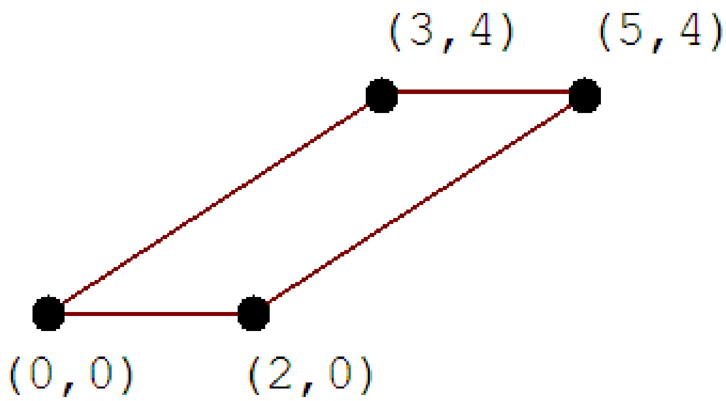

Figure 4: Region the unit square is mapped to under $T(x, y) = (2x + 3y, 4y)$.

or

$$x \;=\; \frac{u}{2} - \frac{3v}{4}.$$

In other words, we have

$$T^{-1}(u, v) \;=\; (x(u, v), y(u, v)) \;=\; \left(\frac{u}{2} - \frac{3v}{4}, \frac{v}{4}\right).$$

We now find the determinant of the derivative. First we compute

$$(DT^{-1})(u, v) \;=\; \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \;=\; \begin{pmatrix} \frac{1}{2} & -\frac{3}{4} \\ 0 & \frac{1}{4} \end{pmatrix}.$$

The determinant is

$$\det((DT^{-1})(u, v)) \;=\; \frac{1}{2} \cdot \frac{1}{4} + \frac{3}{4} \cdot 0 \;=\; \frac{1}{8},$$

and thus the absolute value of the determinant is

$$\left|\det((DT^{-1})(u, v))\right| \;=\; \frac{1}{8},$$

which means

$$dxdy \quad \longrightarrow \quad \left|\det((DT^{-1})(u, v))\right| dudv \;=\; \frac{1}{8} dudv.$$

By the Change of Variables formula, if S is the original unit square in xy-space and $P = T(S)$ is the parallelogram in uv-space, we have

$$\int\int_S 1dxdy \;=\; \int\int_{T(S)} 1\left|\det((DT^{-1})(u,v))\right| dudv \;=\; \int\int_{T(S)} 1\cdot\frac{1}{8}dudv.$$

As 1/8 is constant, we can pull it out of the integral and find

$$\int\int_S 1dxdy \;=\; \frac{1}{8}\int\int_{T(S)} 1dudv;$$

the left double integral is the area of the unit square, while the right double integral is the area of our parallelogrma. We thus find

$$\text{Area}(S) \;=\; \frac{1}{8}\text{Area}(T(S)) \;=\; \frac{1}{8}\text{Area}(P),$$

or equivalently that the area of the parallelogram is 8:

$$\text{Area}(P) \;=\; 8\text{Area}(S) \;=\; 8\cdot 1 \;=\; 8.$$

We could consider more general maps from squares to parallelograms, but this illustrates the principle and proves a nice, known result: the area of a parallelogram is base times height. For our parallelogram, the has has length 2 and the height is 4, which do multiply to give an area of 8.

Notice that we are able to deduce the formula for the parallelogram's area by knowing the area of the square *because* the absolute value of the determinant of the derivative matrix is constant (i.e., independent of u and v). This allows us to pull out that common factor of 1/8 and leaves us with the integral of 1 over the parallelogram, which is thus its area. Whenever we have a change of variables where the determinant is constant, these calculations can often allow us to deduce the area of one region from knowing another. This is true in the homework problem, where you are asked to find the area of an ellipse knowing the area of another region. For that problem, consider the ellipse

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 \;\leq\; 1.$$

Consider the change of variables $u = x/a$ and $v = y/b$, so

$$T(x,y) \;=\; (u(x,y), v(x,y)) \;=\; (x/a, y/b)$$

or equivalently the inverse map T^{-1} would be

$$T^{-1}(u,v) \;=\; (x(u,v), y(u,v)) \;=\; (au, bv).$$

Note this maps the ellipse to the unit disk

$$u^2 + v^2 \;\leq\; 1,$$

and we know the area of the unit disk is just $\pi 1^2 = \pi$!

Question: #1: This is a slight modification of Problem #1 from Section 6.2: Let D ***be the unit disk*** $x^2 + y^2 \leq 1$***. Consider the integral***

$$\int\int_D \cos(x^2 + y^2) dx dy.$$

Evaluate this using polar coordinates.

Solution: We have $dxdy$ goes to $rdrd\theta$, and the unit disk becomes $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$. We replace $f(x,y)$ with $f(r\cos\theta, r\sin\theta)$, and thus find

$$\begin{aligned} \int\int_D \cos(x^2+y^2)dxdy &= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} \cos(r^2\cos^2\theta + r^2\sin^2\theta) r dr d\theta \\ &= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} \cos(r^2) r dr d\theta \\ &= \int_{\theta=0}^{2\pi} \frac{1}{2}\left[\int_{r=0}^{1} \cos(r^2) 2r dr\right] d\theta \\ &= \int_{\theta=0}^{2\pi} \frac{1}{2}\left[\sin(r^2)\Big|_0^1\right] d\theta \\ &= \int_{\theta=0}^{2\pi} \frac{\sin 1}{2} d\theta \\ &= \frac{\sin 1}{2}\cdot 2\pi \;=\; \pi\sin 1. \end{aligned}$$

Question: #13: This is a problem similar to Problem #13 from Section 6.2. Consider the cylinder C ***given by*** $x^2 + y^2 \leq 9$ ***and*** $-1 \leq z \leq 2$***. Evaluate***

$$\int\int\int_C f(x,y,z) dx dy dz$$

where

$$f(x,y,z) \;=\; z\sqrt{x^2+y^2}.$$

Solution: If we wanted to write down the integral explicitly in Cartesian coordinates, we would have

$$\int_{z=-1}^{2}\int_{y=-3}^{3}\int_{x=-\sqrt{9-y^2}}^{\sqrt{9-y^2}} z\sqrt{x^2+y^2}dxdydz.$$

To see this, note that on the boundary $x^2+y^2=9$, so if we have chosen a value of y then x ranges from $-\sqrt{9-y^2}$ to $\sqrt{9-y^2}$; these are not integrals we desire to evaluate! For cylindrical coordinates, we have

$$dxdydz \quad\longrightarrow\quad rdrd\theta dz,$$

and

$$x \;=\; r\cos\theta, \quad y \;=\; r\sin\theta, \quad z \;=\; z.$$

Our function $f(x,y,z)$ becomes $f(r\cos\theta, r\sin\theta, z)$, or in our case

$$z\sqrt{x^2+y^2} \quad\longrightarrow\quad z\sqrt{r^2\cos^2\theta+r^2\sin^2\theta} \;=\; zr.$$

The bounds of integration are z ranges from -1 to 2, θ ranges from 0 to 2π, and r

ranges from 0 to 3. We thus have

$$\begin{aligned}
\int\int\int_C f(x,y,z)dxdydz &= \int_{z=-1}^{2}\int_{\theta=0}^{2\pi}\int_{r=0}^{3} f(r\cos\theta, r\sin\theta, z)rdrd\theta dz \\
&= \int_{z=-1}^{2}\int_{\theta=0}^{2\pi}\int_{r=0}^{3} z\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}rdrd\theta dz \\
&= \int_{z=-1}^{2}\int_{\theta=0}^{2\pi}\int_{r=0}^{3} zr\cdot rdrd\theta dz \\
&= \int_{z=-1}^{2} z\int_{\theta=0}^{2\pi}\left[\int_{r=0}^{3} r^2 dr\right] d\theta dz \\
&= \int_{z=-1}^{2} z\int_{\theta=0}^{2\pi}\left[\left.\frac{r^3}{3}\right|_0^3\right] d\theta dz \\
&= \int_{z=-1}^{2} z\int_{\theta=0}^{2\pi}\frac{27}{3}d\theta dz \\
&= 9\int_{z=-1}^{2} z\left[\int_{\theta=0}^{2\pi} d\theta\right] dz \\
&= 9\int_{z=-1}^{2} z2\pi dz \\
&= 18\pi\int_{z=-1}^{2} zdz \\
&= 18\pi\left.\frac{z^2}{2}\right|_{-1}^{2} \\
&= 18\pi\left[\frac{4}{2}-\frac{1}{2}\right] \\
&= 18\pi\cdot\frac{3}{2} \\
&= 27\pi.
\end{aligned}$$

Question: #21: This is a problem similar to Problem #21 from Section 6.2. Consider the unit sphere S given by $x^2+y^2+z^2\leq 1$. Evaluate

$$\int\int\int_S f(x,y,z)dxdydz$$

for

$$f(x,y,z) \;=\; \frac{1}{(x^2+y^2+z^2)}.$$

If we were to write the integral out explicitly in Cartesian coordinates, we would find it equals

$$\int_{z=-1}^{1}\int_{y=-\sqrt{1-z^2}}^{\sqrt{1-z^2}}\int_{x=-\sqrt{1-y^2-z^2}}^{\sqrt{1-y^2-z^2}} f(x,y,z)dxdydz,$$

and these bounds of integration should look horrible! We now convert to spherical coordinates. We have

$$x \;=\; \rho\sin\phi\cos\theta, \quad y \;=\; \rho\sin\phi\sin\theta, \quad z \;=\; \rho\cos\phi,$$

with

$$0 \;\leq\; \rho \;\leq\; 1, \quad 0 \;\leq\; \theta \;\leq\; 2\pi, \quad 0 \;\leq\; \phi \;\leq\; \pi.$$

Our function $f(x,y,z)$ becomes

$$f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi) \;=\; \frac{1}{\rho^2}$$

after some simple algebra. Finally,

$$dxdydz \quad\longrightarrow\quad \rho^2\sin\phi d\rho d\theta d\phi.$$

Note: other textbooks change the role of θ and ϕ, especially physics books. We thus have

$$
\begin{aligned}
& \int\int\int_S f(x,y,z)dxdydz \\
= \;& \int_{\phi=0}^{\pi}\int_{\theta=0}^{2\pi}\int_{\rho=0}^{1} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi d\rho d\theta d\phi \\
= \;& \int_{\phi=0}^{\pi}\int_{\theta=0}^{2\pi}\int_{\rho=0}^{1} \frac{1}{\rho^2}\rho^2\sin\phi d\rho d\theta d\phi \\
= \;& \int_{\phi=0}^{\pi}\int_{\theta=0}^{2\pi}\int_{\rho=0}^{1} \sin\phi d\rho d\theta d\phi \\
= \;& \int_{\phi=0}^{\pi}\int_{\theta=0}^{2\pi} \sin\phi \left[\int_{\rho=0}^{1} d\rho\right] d\theta d\phi \\
= \;& \int_{\phi=0}^{\pi} \sin\phi \left[\int_{\theta=0}^{2\pi} d\theta\right] d\phi \\
= \;& 2\pi\int_{\phi=0}^{\pi} \sin\phi d\phi \\
= \;& 2\pi\left[-\cos\phi\Big|_0^{\pi}\right] \\
= \;& 2\pi\left[(-\cos\pi)-(-\cos 0)\right] \\
= \;& 2\pi(1+1) \\
= \;& 4\pi.
\end{aligned}
$$

7 Sequences and Series

7.1 Page 10.3

Question: #2: Give an example of a sequence that does not have a limit.

Solution: Let $a_n = (-1)^n$. Clearly this sequence does not have a limit, as half the time the sequence is 1 and half the time the sequence is -1. For another example, consider $a_n = n!$, which clearly grows without bound.

Question: #4: Compute the limit of the sequence $a_n = 3/n^2$ ***or explain why it does not converge.***

Solution: Note $\lim_{n\to\infty} 3 = 3$ and $\lim_{n\to\infty} n^2 = \infty$. Technically we cannot use the limit of a quotient is the quotient of the limit as the denominator tends to infinity and thus doesn't converge; however, as the numerator is bounded (it is in fact always 3) and the denominator becomes arbitrarily large, we can see that the sequence does converge to 0. For example, if $n \geq 55$ then $3/n^2 \leq 1/1000$, if $n \geq 174$ then $3/n^2 \leq 1/10000$, and if $n \geq 548$ then $3/n^2 \leq 1/100000$.

Question: Similar Problem to #5: Find the limit of $a_n = \frac{n^3+2n^2-n-2}{3n^3+n-11}$***, or prove it does not exist.***

Solution: There are several ways to do this. We cannot use the limit of a quotient is the quotient of the limits, as both the numerator and denominator tend to infinity as $n \to \infty$. One approach is to pull out the largest power of n in the numerator and denominator:

$$a_n \ = \ \frac{n^3+2n^2-n-2}{3n^3+n-11} \ = \ \frac{n^3\left(1+\frac{2}{n}-\frac{1}{n^2}-\frac{2}{n^3}\right)}{n^3\left(3+\frac{1}{n^2}-\frac{11}{n^3}\right)} \ = \ \frac{1+\frac{2}{n}-\frac{1}{n^2}-\frac{2}{n^3}}{3+\frac{1}{n^2}-\frac{11}{n^3}}.$$

After pulling out the n^3, we see the numerator tends to 1 as $n \to \infty$ and the denominator tends to 3 as $n \to \infty$. We can now use the limit of a quotient is the quotient of the limit, and find

$$\lim_{n\to\infty} a_n \ = \ \lim_{n\to\infty} \frac{1+\frac{2}{n}-\frac{1}{n^2}-\frac{2}{n^3}}{3+\frac{1}{n^2}-\frac{11}{n^3}} \ = \ \frac{\lim_{n\to\infty}\left(1+\frac{2}{n}-\frac{1}{n^2}-\frac{2}{n^3}\right)}{\lim_{n\to\infty}\left(3+\frac{1}{n^2}-\frac{11}{n^3}\right)} \ = \ \frac{1}{3}.$$

Alternatively, we can use L'Hopital's rule to evaluate the limit; we keep taking derivatives until we no longer have infinity over infinity:

$$\begin{aligned}\lim_{n\to\infty} \frac{n^3+2n^2-n-2}{3n^3+n-11} \quad &= \quad \lim_{n\to\infty} \frac{3n^2+4n-1}{9n^2+1} \\ &= \quad \lim_{n\to\infty} \frac{6n+4}{18n} \\ &= \quad \lim_{n\to\infty} \frac{6}{18} \ = \ \frac{1}{3}.\end{aligned}$$

Question: Similar Problem to #7: Find the limit of $a_n = \frac{4n^2-11n+1}{5n^6+12}$***, or prove the limit does not exist.***

Solution: The limit is zero. One way to see this is to pull out the highest power of n from the numerator and the denominator; it is n^2 for the numerator and n^6 for the denominator. We have

$$a_n \;=\; \frac{4n^2 - 11n + 1}{5n^6 + 12} \;=\; \frac{n^2\left(4 - \frac{11}{n} + \frac{1}{n^2}\right)}{n^6\left(\frac{5}{+}\frac{12}{n^6}\right)} \;=\; \frac{4 - \frac{11}{n} + \frac{1}{n^2}}{n^4\left(\frac{5}{+}\frac{12}{n^6}\right)}.$$

Note the numerator tends to 4 as $n \to \infty$ while the denominator tends to infinity; thus the ratio tends to 0.

Alternatively, we could use L'Hopital's rule, taking derivatives until we no longer have infinity over infinity:

$$\begin{aligned} \lim_{n\to\infty} \frac{4n^2 - 11n + 1}{5n^6 + 12} &= \lim_{n\to\infty} \frac{8n - 11}{30n^5} \\ &= \lim_{n\to\infty} \frac{8}{150n^4} \;=\; 0. \end{aligned}$$

7.2 Page 10.6

Question: Similar Problem to #8: Find the limit of the series $\sum_{n=0}^{\infty} \frac{2^n}{3^n}$ or prove it does not exist.

Solution: Note that this sum is the same as

$$\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n;$$

this is the same as a geometric series with ratio $r = 2/3$, which is less than 1 in absolute value. We know the geometric series $\sum_{n=0}^{\infty} r^n$ converges if $|r| < 1$; thus this series converges.

Question: #10: Find n such that $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} > 10^6$.

Solution: The n^{th} harmonic number, H_n, is defined by

$$H_n \;=\; 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

We have $H_{100} \approx 5.2$, $H_{1000} \approx 7.5$, $H_{10^{10}} \approx 23.6$, $H_{10^{100}} \approx 230.8$, $H_{10^{1000}} \approx 2303.2$, and so on. Note how slowly this grows! In fact, $H_n \approx \ln n$ for n large. The sought after value of n is about 10^{434295}, which is quite large!

There are ways to find n that will work without knowing $H_n \approx \ln n$. One way is to note that $1/3+1/4 \geq 1/2$, $1/5+1/6+1/7+1/8 \geq 1/2$, $1/9+\cdots+1/16 \geq 1/2$ and so on. Thus we can keep getting at least 1/2....

7.3 Page 10.7

Question: Similar Problem to #13: Find all p ***such that the sequence*** $a_n = \frac{1}{n \ln^p n}$ ***converges.***

Solution: For any fixed p, once n is large the sequence is strictly decreasing and we can use the integral test. Thus the series converges or diverges depending on whether or not

$$\int_{x=\text{big}}^{\infty} \frac{1}{x \ln^p x}\, dx$$

converges or diverges; we write 'big' to indicate that the lower bound does not really matter – what matters is the behavior at infinity. We integrate by parts. Let $u = \ln x$ so $du = dx/x$, and thus our integral becomes

$$\int_{u=\ln(\text{big})}^{\infty} u^{-p} du.$$

The integral of u^{-p} is $\frac{u^{1-p}}{p}$ if $p \neq 1$ and $\ln u$ if $p = 1$. Thus the integral converges if $p > 1$ and diverges $p \leq 1$.

7.4 Page 10.8

Question: Similar Problem to #14: Determine if the series $\sum_{n=1}^{\infty} \frac{1}{2^n+\sqrt{n}}$ ***converges or diverges.***

Solution: We use the comparison test. Note that while $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges, $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges. As $2^n + \sqrt{n} \geq 2^n$, we have

$$0 \leq \frac{1}{2^n + \sqrt{n}} \leq \frac{1}{2^n}.$$

Thus the series converges by the comparison test.

Question: Similar Problem to #15: Determine if the series $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{5^n}$ converges or diverges.

Solution: We have $\sqrt{n} \le 2^n$. Thus

$$\frac{\sqrt{n}}{5^n} \le \frac{2^n}{5^n} = \left(\frac{2}{5}\right)^n.$$

Our series is thus bounded term by term by the geometric series with ratio $2/5$, and thus converges by the comparison.

Question: Similar Problem to #16: Determine if the series $\sum_{n=1}^{\infty} \frac{2^n \cdot n!}{\sqrt{n}+11}$ converges or diverges.

Solution: This series diverges. Note the numerator is growing much faster than the denominator. The easiest way to see this is that the denominator is at most n for large n, and $n!/n = (n-1)!$. In other words, the terms in the sequence tend to infinity, and thus the sum cannot converge.

7.5 Page 10.10

Question: #20: Determine if $\sum_{k=0}^{n} \frac{3^{2k+1}}{10^k}$ converges or diverges.

Solution: We can re-write the terms in a more illuminating manner. We have

$$\begin{aligned} \sum_{k=0}^{n} \frac{3^{2k+1}}{10^k} &= 3\sum_{k=0}^{n} \frac{3^{2k}}{10^k} \\ &= 3\sum_{k=0}^{n} \frac{9^k}{10^k} \\ &= 3\sum_{k=0}^{n} \left(\frac{9}{10}\right)^k. \end{aligned}$$

Note the above sum is just three times the geometric series with ratio 3, and thus converges.

Instead of using the comparison test we can also use the Ratio Test. We look at a_{k+1}/a_k, which for us is

$$\frac{a_{k+1}}{a_k} \;=\; \frac{3^{2k+3}/10^{k+1}}{3^{2k+1}/10^k} \;=\; \frac{9}{10}.$$

Thus the limit of a_{k+1}/a_k is $9/10 < 1$, and the series therefore converges by the Ratio Test.

Question: #22: Determine if $\sum_{k=0}^{n} \frac{3^k(k^4+k+1)}{5^k}$ converges or diverges.

Solution: If we didn't have the factor $k^4 + k + 1$, it would be straightforward, as the series would just be the geometric series with ratio $3/5$. As k^4 grows polynomially but 3^k and 5^k grow exponentially, we expect the series to still converge. Thus we look for an upper bound for the numerator such that, even when multiplied by 3^k, it grows slower than the denominator by a significant margin.

For example, let's try and show the numerator is bounded by $C \cdot 4^k$ for some constant C. We want to show for k large that

$$3^k(k^4 + k + 1) \;\le\; C \cdot 4^k,$$

or

$$k^4 + k + 1 \;\le\; C \cdot (4/3)^k.$$

We have $k^4 + k + 1 \le 3k^4$, and thus if we take $C = 3$ we need only show, for k large, that $k^4 \le (4/3)^k$. While this is not true for small k, it is true for large k and thus the series is bounded by the geometric series with ratio $4/5$, and hence converges.

We now provide an alternative proof using the ratio test. We look at a_{k+1}/a_k, which for us is

$$\frac{a_{k+1}}{a_k} \;=\; \frac{(3/5)^{k+1}((k+1)^4 + k + 2)}{(3/5)^k(k^4 + k + 1)} \;=\; \frac{3}{5} \cdot \frac{(k+1)^4 + k + 2}{k^4 + k + 1}.$$

If we take the limit as $k \to \infty$, we see that the limit is $3/5$. As this is less than 1, by the Ratio Test the series converges.

8 From path integrals to Stokes' Theorem

The final homework assignment of the semester is: *Section 4.2: #1 (see formula at the bottom of the page for help). Section 4.4: #1, #14. Section 7.1: #3b. Section 7.2: #1c. Section 8.1: #3a.* The problems below are similarly chosen problems.

Question: Section 4.2: #2: Find the arc length of the curve $c(t) = (1, 3t^2, t^3)$ ***for*** $0 \leq t \leq 1$***.***

Solution: The answer is

$$\int_0^1 ||c'(t)||\, dt,$$

where

$$c'(t) \ = \ (0, 6t, 3t^2)$$

so

$$||c'(t)|| \ = \ \sqrt{36t^2 + 9t^4} \ = \ 3t\sqrt{4 + t^2};$$

this will lead to a straightforward integral because of the factor of t outside the square-root. We have

$$\begin{aligned} \int_0^1 ||c'(t)||\, dt &= \int_0^1 3t\sqrt{4+t^2}dt \\ &= \frac{3}{2}\int_0^1 (4+t^2)^{1/2} 2tdt \\ &= (4+t^2)^{3/2}\Big|_0^1 \\ &= 5^{3/2} - 4^{3/2}. \end{aligned}$$

Question: Section 4.4: #2: Find the divergence and the curl of $V(x, y, z) = (yz, xz, xy) = (V_1(x, y, z), V_2(x, y, z), V_3(x, y, z))$***.***

Solution: The divergence is

$$\mathrm{div}(V) \ = \ \nabla \cdot V \ = \ \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z} \ = \ 0;$$

the fact that the divergence is zero has physical interpretations. For the curl, we have

$$\mathrm{curl}(V) \ = \ \nabla \times V \ = \ \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_2 \end{vmatrix}.$$

Expanding gives

$$\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}, \frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}, \frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right) \ = \ (0, 0, 0)\,;$$

thus this vector field has both zero curl and zero divergence!

Question: Section 7.1: #3a: Find the path integral $\int_c f(x,y,z)ds$ ***where*** $f(x,y,z) = \exp(\sqrt{z})$ ***and*** $c(t) = (1,2,t^2)$ ***for*** $0 \leq t \leq 1$**.**

Solution: The path integral is

$$\int_0^1 f(c(t))||c'(t)||\, dt.$$

We have

$$c'(t) \;=\; (0,0,2t), \quad ||c'(t)|| \;=\; 2|t|$$

(which is $2t$ as $t \geq 0$). Further,

$$f(c(t)) \;=\; f(1,2,t^2) \;=\; \exp(\sqrt{t^2}) \;=\; \exp(t).$$

Thus the path integral is

$$\int_0^1 f(c(t))||c'(t)||\, dt \quad = \quad \int_0^1 \exp(t)2tdt.$$

The integral (or anti-derivative) of $\exp(t)t$ is just $\exp(t)(t-1)$, and thus we have

$$\int_0^1 f(c(t))||c'(t)||\, dt \;=\; 2\exp(t)(t-1)\Big|_0^1 \;=\; 2.$$

Question: Section 7.2: #1b: Let $F(x,y,z) = (x,y,z)$**.** ***Evaluate the integral of*** F ***along the path*** $c(t) = (\sin t, 0, \cos t)$ ***for*** $0 \leq t \leq 2\pi$**.**

Solution: We have

$$c'(t) \;=\; (\cos t, 0, -\sin t)$$

,

$$F(c(t)) \;=\; F(\sin t, 0, \cos t) \;=\; (\sin t, 0, \cos t).$$

Thus the line integral is

$$\begin{aligned} \int_0^{2\pi} F(c(t)) \cdot c'(t)dt \quad &= \quad \int_0^1 (\sin t, 0, \cos t)(\cos t, 0, -\sin t)dt \\ &= \quad \int_0^{2\pi} 0dt \;=\; 0. \end{aligned}$$

Question: Section 8.1: #3b: Verify Green's theorem for the disk with center $(0,0)$ ***and radius*** R ***and the functions*** $P(x,y) = x + y$***,*** $Q(x,y) = y$***.***

Solution: We have

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \;=\; 0 - 1 \;=\; -1;$$

thus

$$\int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy \;=\; \int\int_D -1 dxdy \;=\; -\mathrm{Area}(D);$$

of course, the area of the disk is πR^2 so this double integral is $-\pi R^2$.

For the other part of Green's theorem, we note the boundary curve is

$$c(t) \;=\; (R\cos t, R\sin t), \quad 0 \le t \le 2\pi$$

(remember we must travel so that the region D is on our left). Thus

$$c'(t) \;=\; (-R\sin t, R\cos t).$$

Further,

$$\overrightarrow{F}(c(t)) \;=\; \overrightarrow{F}(R\cos t, R\sin t) \;=\; (R\cos t + R\sin t, R\sin t),$$

and hence

$$\begin{aligned}
\int_c \overrightarrow{F} \cdot d\overrightarrow{s} \quad &= \quad \int_0^{2\pi} \overrightarrow{F}(c(t)) \cdot c'(t) dt \\
&= \quad \int_0^{2\pi} (R\cos t + R\sin t, R\sin t) \cdot (-R\sin t, R\cos t) dt \\
&= \quad \int_0^{2\pi} -R^2 \sin^2 t dt.
\end{aligned}$$

We evaluated the sine-integral many ways, and found it equals π; one could also use trig identities and find $\sin^2 t = \frac{1-\cos(2t)}{2}$. Thus the integral equals π times $-R^2$, which does match.