

Three trouble spots on the spherical Earth, located at

$$P_1 = \left(\frac{5}{13}, \frac{12}{13}, 0 \right) \quad P_2 = \left(\frac{12}{13}, \frac{5}{13}, 0 \right) \quad \text{and}$$

$$P_3 = \left(\frac{3}{13}, \frac{4}{13}, \frac{12}{13} \right)$$

Method 1: Use "barrowing" distance squared

$$D_1(x, y, z) = 13 \sum_{i=1}^3 \|(\mathbf{x}, \mathbf{y}, \mathbf{z}) - \mathbf{P}_i\|^2$$

↳ factor 13 clears denomins, harmless

↳ working with squares a real issue

$$\nabla D_1 = (78x - 80, 78y - 42, 78z - 24)$$

$$\text{Constraint: } g(x, y, z) = x^2 + y^2 + z^2 = 1$$

$$\nabla g = (2x, 2y, 2z)$$

$$\text{so } \nabla D_1 = \lambda \nabla g$$

$$\hookrightarrow \text{yields } \lambda = 38 - \sqrt{985}$$

$$x = 4 \sqrt{\frac{5}{187}} \approx .637$$

$$\lambda = 38 + \sqrt{985}$$

$$x = -4 \sqrt{\frac{5}{187}}$$

$$y = \frac{-21}{\sqrt{985}} \approx .669$$

$$y = -\frac{-21}{\sqrt{985}}$$

$$z = \frac{12}{\sqrt{985}} \approx .382$$

$$z = -\frac{12}{\sqrt{985}}$$

$$\text{yields } D_1 \approx 15.23$$

$$\text{yields } D_1 \approx 140.77$$

$$\text{Note } \sqrt{\frac{D_1}{13}} \approx 1.08$$

Method 2: Don't use distance squared, use
distance traveling through the Earth.

Quickly leads to an algebraic nightmare! Square-root
all throughout the denominator.

Numerically approximate answer:

$$\text{Using } D_2(x, y, z) = \sum_{i=1}^3 \|(\mathbf{x}, \mathbf{y}, \mathbf{z}) - \mathbf{P}(i)\|$$

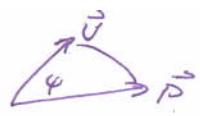
$$\hookrightarrow x \approx .6588$$

$$y \approx .7232$$

$$z \approx .2071$$

Maximum value is 1.8.

Method 2: Use $\vec{P} \cdot \vec{U} = \cos \varphi$ on sphere



And note the arclength is $\varphi = \arccos(\vec{P} \cdot \vec{U})$

As \arccos is a pain to work with, so we will minimize

$$D_3(x, y, z) = 2[(1 - \cos \varphi_1) + (1 - \cos \varphi_2) + (1 - \cos \varphi_3)]$$

For small angles, $2(1 - \cos \varphi) \approx \varphi^2 \approx \text{arclength squared!}$

$$D_3(x, y, z) = 2 \sum_{i=1}^3 [-(x, y, z) \cdot \vec{P}(i)] * 13$$

$\hookrightarrow 13$ harmless - clear denominators

$$= 28 - 40x - 42y - 28z$$

$$\nabla D_3 = (-40, -42, -28) = \lambda(2x, 2y, 2z) = \lambda \nabla g$$

$$\text{Thus } x = -\frac{20}{\lambda}, \quad y = -\frac{21}{\lambda}, \quad z = -\frac{12}{\lambda}$$

$$\text{and } x^2 + y^2 + z^2 = 1 \Rightarrow \frac{20^2 + 21^2 + 12^2}{\lambda^2} = 1 \Rightarrow \lambda = \pm \sqrt{985}$$

Clearly minimum is when $\lambda = -\sqrt{985}$

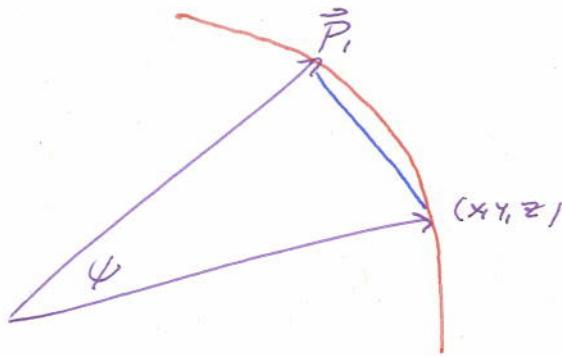
$$\text{Thus } x = \frac{20}{\sqrt{985}} \approx .637$$

$$y = \frac{21}{\sqrt{985}} \approx .669$$

$$z = \frac{12}{\sqrt{985}} \approx .382$$

$$D_3 = 15.2306$$

Why is Method 3 the same as Method 1?



distance from Earth, by law of cosines, is

$$d^2 = l^2 + l^2 - 2 \cdot l \cdot l \cdot \cos \varphi$$

Thus $d^2 = 2 - 2 \cos \varphi$

$$d^2 = 2(1 - \cos \varphi)$$

And hence our two functions, D_1 and D_3 , are in fact the same!

Method 4: Cross Product and Sines

We have, since everything is a unit vector,

$$\|P_i \times (x, y, z)\| = \sin \varphi \approx \varphi \text{ for small } \varphi$$

$$D_\varphi(x, y, z) = 3 \sum_{i=1}^3 \|P_i \times (x, y, z)\|^2 \\ = \{329x^2 - 264xy + 322y^2 - 72xz - 96yz + 363z^2\}$$

In Mathematica, $\text{Norm}((x, y, z) \times P(i))^2$ was bad, involving abs values; better to code as $((x, y, z) \times P(i)) \cdot ((x, y, z) \times P(i))$

$$\nabla D_\varphi = (658x - 264y - 72z, -264x + 644y - 96z, -72x - 96y + 726z)$$

$$\text{Solve } \nabla D_\varphi = \lambda \nabla g$$

$$\text{Minimum is } x \approx .659$$

$$y \approx .689$$

$$z \approx .301$$

$$\text{Value } D_\varphi \approx 174.6$$

Lagrange Multipliers

Military Trouble Spots Example

(* Three unit vectors on the sphere *)

p[1] = {5/13, 12/13, 0};

p[2] = {12/13, 5/13, 0};

p[3] = {3/13, 4/13, 12/13};

{Norm[p[1]], Norm[p[2]], Norm[p[3]]}

{1, 1, 1}

(* First we do burrowing through
the Earth distance-squared *)

D1[x_, y_, z_] := 13 Sum[

(x - p[i][[1]])^2 + (y - p[i][[2]])^2 +
(z - p[i][[3]])^2, {i, 1, 3}]

D[x, y, z]

Expand[D1[x, y, z]]

0

39 - 40 x + 39 x^2 - 42 y + 39 y^2 - 24 z + 39 z^2

Expand[{D[D1[x, y, z], x],

D[D1[x, y, z], y], D[D1[x, y, z], z]}]

{-40 + 78 x, -42 + 78 y, -24 + 78 z}

Solve[{{{-40 + 78 x, -42 + 78 y, -24 + 78 z} ==
2 w {x, y, z},
x^2 + y^2 + z^2 == 1}, {w, x, y, z}]

$$\left\{ \left\{ w \rightarrow 39 - \sqrt{985}, x \rightarrow 4 \sqrt{\frac{5}{197}}, y \rightarrow \frac{21}{\sqrt{985}}, z \rightarrow \frac{12}{\sqrt{985}} \right\}, \left\{ w \rightarrow 39 + \sqrt{985}, x \rightarrow -4 \sqrt{\frac{5}{197}}, y \rightarrow -\frac{21}{\sqrt{985}}, z \rightarrow -\frac{12}{\sqrt{985}} \right\} \right\}$$

$$D1 \left[4 \cdot \sqrt{\frac{5}{197}}, \frac{21}{\sqrt{985}}, \frac{12}{\sqrt{985}} \right]$$

$$D1 \left[-4 \cdot \sqrt{\frac{5}{197}}, -\frac{21}{\sqrt{985}}, -\frac{12}{\sqrt{985}} \right]$$

$$\text{Sqrt} \left[D1 \left[4 \cdot \sqrt{\frac{5}{197}}, \frac{21}{\sqrt{985}}, \frac{12}{\sqrt{985}} \right] \right] / 13$$

15.2306

140.769

1.0824

N[p[1]]

N[p[2]]

N[p[3]]

$$\text{opt1} = \text{N}\left[\left\{4 \cdot \sqrt{\frac{5}{197}}, \frac{21}{\sqrt{985}}, \frac{12}{\sqrt{985}}\right\}\right]$$

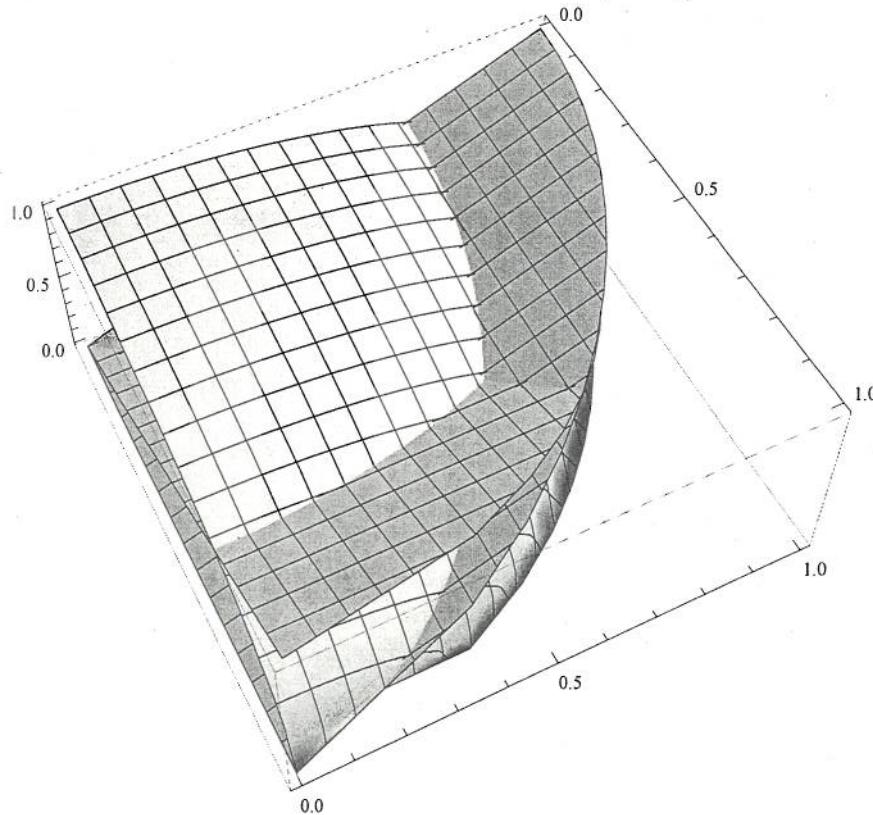
{0.384615, 0.923077, 0.}

{0.923077, 0.384615, 0.}

{0.230769, 0.307692, 0.923077}

{0.637253, 0.669116, 0.382352}

```
Plot3D[ {x, y, Abs[Sqrt[1 - x^2 - y^2]]},
{x, 0, 1}, {y, 0, Abs[Sqrt[1 - x^2]]}]
```



(* Second we do burrowing through
the Earth distance- what a mess! *)

```
D2[x_, y_, z_] := Sum[Sqrt[

$$(x - p[i][[1]])^2 + (y - p[i][[2]])^2 +$$


$$(z - p[i][[3]])^2], \{i, 1, 3\}]$$

Expand[\{D[D2[x, y, z], x],
D[D2[x, y, z], y], D[D2[x, y, z], z]\}]
```

$$\left\{ - \frac{3}{13 \sqrt{\left(-\frac{3}{13} + x\right)^2 + \left(-\frac{4}{13} + y\right)^2 + \left(-\frac{12}{13} + z\right)^2}} + \right.$$

$$\frac{x}{\sqrt{\left(-\frac{3}{13} + x\right)^2 + \left(-\frac{4}{13} + y\right)^2 + \left(-\frac{12}{13} + z\right)^2}} -$$

$$\frac{5}{13 \sqrt{\left(-\frac{5}{13} + x\right)^2 + \left(-\frac{12}{13} + y\right)^2 + z^2}} +$$

$$\frac{x}{\sqrt{\left(-\frac{5}{13} + x\right)^2 + \left(-\frac{12}{13} + y\right)^2 + z^2}} -$$

$$\frac{12}{13 \sqrt{\left(-\frac{12}{13} + x\right)^2 + \left(-\frac{5}{13} + y\right)^2 + z^2}} +$$

$$\begin{aligned}
 & \frac{x}{\sqrt{\left(-\frac{12}{13} + x\right)^2 + \left(-\frac{5}{13} + y\right)^2 + z^2}}, \\
 & - \frac{4}{13 \sqrt{\left(-\frac{3}{13} + x\right)^2 + \left(-\frac{4}{13} + y\right)^2 + \left(-\frac{12}{13} + z\right)^2}} + \\
 & \frac{y}{\sqrt{\left(-\frac{3}{13} + x\right)^2 + \left(-\frac{4}{13} + y\right)^2 + \left(-\frac{12}{13} + z\right)^2}} - \\
 & \frac{12}{13 \sqrt{\left(-\frac{5}{13} + x\right)^2 + \left(-\frac{12}{13} + y\right)^2 + z^2}} + \\
 & \frac{y}{\sqrt{\left(-\frac{5}{13} + x\right)^2 + \left(-\frac{12}{13} + y\right)^2 + z^2}}
 \end{aligned}$$

$$\frac{5}{13 \sqrt{\left(-\frac{12}{13} + x\right)^2 + \left(-\frac{5}{13} + y\right)^2 + z^2}} +$$

$$\frac{y}{\sqrt{\left(-\frac{12}{13} + x\right)^2 + \left(-\frac{5}{13} + y\right)^2 + z^2}},$$

$$-\frac{12}{13 \sqrt{\left(-\frac{3}{13} + x\right)^2 + \left(-\frac{4}{13} + y\right)^2 + \left(-\frac{12}{13} + z\right)^2}} +$$

$$\frac{z}{\sqrt{\left(-\frac{3}{13} + x\right)^2 + \left(-\frac{4}{13} + y\right)^2 + \left(-\frac{12}{13} + z\right)^2}} +$$

$$\frac{z}{\sqrt{\left(-\frac{5}{13} + x\right)^2 + \left(-\frac{12}{13} + y\right)^2 + z^2}} +$$

$$\frac{z}{\sqrt{\left(-\frac{12}{13} + x\right)^2 + \left(-\frac{5}{13} + y\right)^2 + z^2}} \}$$

? FindMinimum

FindMinimum[f , x] searches for a local minimum in f , starting from an automatically selected point.
 FindMinimum[f , $\{x, x_0\}$] searches for a local minimum in f , starting from the point $x = x_0$.
 FindMinimum[f , $\{\{x, x_0\}, \{y, y_0\}, \dots\}$] searches for a local minimum in a function of several variables.
 FindMinimum[$\{f, cons\}$, $\{\{x, x_0\}, \{y, y_0\}, \dots\}$] searches for a local minimum subject to the constraints $cons$.
 FindMinimum[$\{f, cons\}$, $\{x, y, \dots\}$] starts from a point within the region defined by the constraints. >>

```
FindMinimum[
  {D2[x, y, z], x^2 + y^2 + z^2 == 1},
  {{x, .6}, {y, .6}, {z, .3}}]
{1.80631, {x → 0.658882,
  y → 0.723174, z → 0.207108} }
```

(* Third method:
 measure distance by $1 - \cos(\text{angle})$ *)

```
D3[x_, y_, z_] := 2 * 13 *
  Sum[(1 - {x, y, z}.p[i]), {i, 1, 3}]
```

```
Expand[D3[x, y, z]]
```

78 - 40 x - 42 y - 24 z

```
lambda = Sqrt[20^2 + 21^2 + 12^2]
```

$\sqrt{985}$

```
20. / Sqrt[985]
```

0.637253

D3 [20. / Sqrt[985],
 21. / Sqrt[985], 12. / Sqrt[985]]

15.2306

(* Third method:
 measure distance by $1 - \cos(\text{angle})$ *)

D4 [x_, y_, z_] :=

$13^2 * \text{Sum}[\text{Cross}[\{x, y, z\}, p[i]],$

$\text{Cross}[\{x, y, z\}, p[i]], \{i, 1, 3\}]$

Expand[D4[x, y, z]]

$$329 x^2 - 264 x y + 322 y^2 - 72 x z - 96 y z + 363 z^2$$

Expand[{D[D4[x, y, z], x],

$D[D4[x, y, z], y], D[D4[x, y, z], z]]$

$$\{658 x - 264 y - 72 z,$$

$$-264 x + 644 y - 96 z, -72 x - 96 y + 726 z\}$$

NSolve[

$$\{ \{ 658 x - 264 y - 72 z, -264 x + 644 y - 96 z,$$

$$-72 x - 96 y + 726 z \} == 2 w \{x, y, z\},$$

$$x^2 + y^2 + z^2 == 1 \}, \{w, x, y, z\}]$$

$$\{ \{ w \rightarrow 458.27, x \rightarrow 0.69845,$$

$$y \rightarrow -0.709518, z \rightarrow 0.0935518 \},$$

$$\{ w \rightarrow 458.27, x \rightarrow -0.69845,$$

$$y \rightarrow 0.709518, z \rightarrow -0.0935518 \},$$

$$\{ w \rightarrow 381.095, x \rightarrow -0.27837,$$

$$y \rightarrow -0.148919, z \rightarrow 0.948859 \},$$

$$\{ w \rightarrow 381.095, x \rightarrow 0.27837,$$

$$y \rightarrow 0.148919, z \rightarrow -0.948859 \},$$

$$\{ w \rightarrow 174.635, x \rightarrow -0.659301,$$

$$y \rightarrow -0.688772, z \rightarrow -0.301521 \},$$

$$\{ w \rightarrow 174.635, x \rightarrow 0.659301,$$

$$y \rightarrow 0.688772, z \rightarrow 0.301521 \} \}$$

D4 [0.6984499726061706` ,
-0.7095179344272434` ,
0.09355178508471927`]

D4 [0.6593007671819094` ,
0.6887724929072444` ,
0.30152106295860254`]

458.27

174.635

