

Notes on Vector Calculus (following Apostol, Schey, and Feynman)

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1. Dot Product, Cross Product, Scalar Triple Product.

The standard inner product in \mathbb{R}^n is the “dot product,” defined as follows. If

$$\mathbf{a} = (a_1, a_2, \dots, a_n) \quad \text{and} \quad \mathbf{b} = (b_1, b_2, \dots, b_n)$$

then

$$\mathbf{a} \cdot \mathbf{b} \equiv \sum_{i=1}^n a_i b_i. \quad (1.1)$$

The standard norm in \mathbb{R}^n is defined in terms of the dot product as

$$\|\mathbf{a}\| \equiv \sqrt{\mathbf{a} \cdot \mathbf{a}} = \sqrt{a_1^2 + a_2^2 + \dots + a_n^2}. \quad (1.2)$$

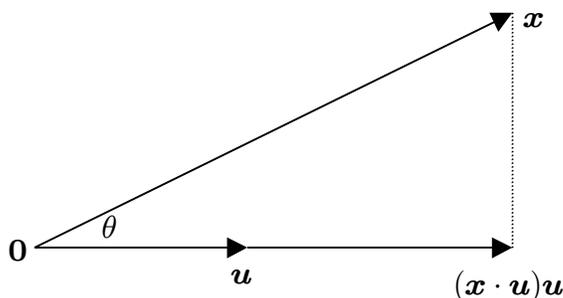
In \mathbb{R}^2 and \mathbb{R}^3 the norm of a vector is its length. If $\|\mathbf{u}\| = 1$, then \mathbf{u} is said to be a *unit* vector. Some special notation is used in \mathbb{R}^2 and \mathbb{R}^3 . A point in \mathbb{R}^2 is sometimes written (x, y) and a point in \mathbb{R}^3 is sometimes written (x, y, z) . Alternatively, the symbols x , y , and z replace the indices 1, 2, and 3. For example, we might write $\mathbf{a} = (a_x, a_y, a_z)$ for a vector in \mathbb{R}^3 . The symbols \mathbf{i} , \mathbf{j} , and \mathbf{k} denote the three standard unit coordinate vectors. Hence

$$(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \quad \text{and} \quad \mathbf{a} = (a_x, a_y, a_z) = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}.$$

Suppose that \mathbf{u} is a unit vector. For any vector \mathbf{x} ,

$$\mathbf{x} \cdot \mathbf{u} = \|\mathbf{x}\|\|\mathbf{u}\|\cos\theta = \|\mathbf{x}\|\cos\theta$$

where θ is the angle between \mathbf{u} and \mathbf{x} . This means that $\mathbf{x} \cdot \mathbf{u}$ is the component of \mathbf{x} in the direction of \mathbf{u} , and $(\mathbf{x} \cdot \mathbf{u})\mathbf{u}$ is the orthogonal projection of \mathbf{x} onto the subspace of scalar multiples of \mathbf{u} (called the subspace of vectors “spanned” by \mathbf{u}). This is illustrated below.



The **cross product** of two vectors $\mathbf{a} = (a_x, a_y, a_z)$ and $\mathbf{b} = (b_x, b_y, b_z)$ in \mathbb{R}^3 is defined as

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &\equiv (a_y b_z - a_z b_y)\mathbf{i} + (a_z b_x - a_x b_z)\mathbf{j} + (a_x b_y - a_y b_x)\mathbf{k} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}.\end{aligned}\tag{1.3}$$

It may be shown that $\|\mathbf{a} \times \mathbf{b}\|$ equals the area of the parallelogram determined by \mathbf{a} and \mathbf{b} (i.e., $\|\mathbf{a}\|\|\mathbf{b}\|\sin\theta$ where θ is the angle between \mathbf{a} and \mathbf{b} , $0 \leq \theta \leq \pi$), and that $\mathbf{a} \times \mathbf{b}$ is orthogonal to the plane determined by \mathbf{a} and \mathbf{b} . More precisely, the direction of $\mathbf{a} \times \mathbf{b}$ is determined by the “right-hand” rule as follows: if the right hand is held with the thumb stuck out and with the fingers curled in the direction of rotation of \mathbf{a} into \mathbf{b} , then the thumb points in the direction of $\mathbf{a} \times \mathbf{b}$. In other words, if the index finger of the right hand is pointed forward and shows the direction of \mathbf{a} , and if the middle finger is bent to show the direction of \mathbf{b} , and if the thumb is perpendicular to the plane determined by the index and middle finger, then the thumb points in the direction of $\mathbf{a} \times \mathbf{b}$. Because $\mathbf{a} \times \mathbf{b}$ is orthogonal to both \mathbf{a} and \mathbf{b} , it follows that

$$\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\mathbf{a} \times \mathbf{b}) = 0.$$

Also, it's clear that $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ for any vector \mathbf{a} . Cross products have the following algebraic properties.

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= -(\mathbf{b} \times \mathbf{a}) \\ \mathbf{a} \times (\mathbf{b} + \mathbf{c}) &= (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c}) \\ \xi(\mathbf{a} \times \mathbf{b}) &= (\xi\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (\xi\mathbf{b})\end{aligned}$$

(where ξ is any scalar). It may be shown that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}\tag{1.4}$$

for any three vectors \mathbf{a} , \mathbf{b} , and \mathbf{c} .

The **scalar triple product** of any three vectors \mathbf{a} , \mathbf{b} , and \mathbf{c} is defined as the scalar

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}.$$

It may be shown that $|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}|$ is the volume of the parallelepiped determined by \mathbf{a} , \mathbf{b} , and \mathbf{c} . This suggests (and it may be shown to be true) that a cyclic permutation of the three vectors does not affect the scalar triple product; that is,

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} = (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b}.\tag{1.5}$$

The commutativity of the dot product then implies that the dot and cross products in a scalar triple product may be interchanged:

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}).\tag{1.6}$$

Finally, it may be shown that the scalar triple product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ may be written in terms of a determinant as follows:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}. \quad (1.7)$$

2. The Gradient.

Let D be a subset of \mathbb{R}^n . **Definition:** a *scalar field* on D is a mapping from D into \mathbb{R} ; a *vector field* on D is a mapping from D into \mathbb{R}^n .

Suppose that D be a subset of \mathbb{R}^n and that φ is a differentiable scalar field defined on D . For any point $\mathbf{r} = (x_1, x_2, \dots, x_n)$ in D , the n -tuple

$$\nabla\varphi(\mathbf{r}) \equiv \text{grad } \varphi(\mathbf{r}) \equiv \left(\frac{\partial\varphi}{\partial x_1}, \frac{\partial\varphi}{\partial x_2}, \dots, \frac{\partial\varphi}{\partial x_n} \right) \quad (2.1)$$

(where each partial derivative is evaluated at \mathbf{r}) is called the **gradient** of φ . We'll write $\nabla\varphi$ or $\text{grad } \varphi$ if the point where the partial derivatives are to be evaluated is clear. The collection of vectors $\nabla\varphi(\mathbf{r})$ constitutes a vector field over D .

Example 1. Let $\varphi(\mathbf{r}) \equiv \|\mathbf{r}\| \equiv r = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$. Then

$$\frac{\partial\varphi}{\partial x_i} = \frac{\partial r}{\partial x_i} = \frac{1}{2}(2x_i)(x_1^2 + x_2^2 + \dots + x_n^2)^{-\frac{1}{2}} = \frac{x_i}{r}. \quad (2.2)$$

It follows that

$$\nabla\varphi(\mathbf{r}) = \|\mathbf{r}\|^{-1}\mathbf{r}, \quad (2.3)$$

a unit vector in the direction of \mathbf{r} .

As φ is differentiable, the derivative of φ at \mathbf{r} with respect to any vector \mathbf{u} exists and is denoted $\varphi'(\mathbf{r}; \mathbf{u})$. It may be shown that

$$\varphi'(\mathbf{r}; \mathbf{u}) = \nabla\varphi(\mathbf{r}) \cdot \mathbf{u}. \quad (2.4)$$

If \mathbf{u} is a unit vector, $\varphi'(\mathbf{r}; \mathbf{u})$ is said to be a *directional derivative*; it's the rate of change of φ with respect to distance in the direction of \mathbf{u} . In this case,

$$\varphi'(\mathbf{r}; \mathbf{u}) = \|\nabla\varphi(\mathbf{r})\|\cos\theta \quad (2.5)$$

where θ is the angle between $\nabla\varphi(\mathbf{r})$ and \mathbf{u} . That is, $\nabla\varphi(\mathbf{r}) \cdot \mathbf{u}$ is the component of $\nabla\varphi(\mathbf{r})$ in the direction of \mathbf{u} . As $\cos\theta$ is maximal when $\theta = 0$, it follows that $\nabla\varphi$ points in the direction at which φ increases fastest, and $\|\nabla\varphi(\mathbf{r})\|$ gives the rate of change of φ in that direction.

Suppose, now, that \mathbf{r} is a differentiable vector-valued function that maps an interval of real numbers $[a, b]$ into $D \subseteq \mathbb{R}^n$. For any $t \in [a, b]$ we write

$$\mathbf{r}(t) = (x_1(t), x_2(t), \dots, x_n(t)).$$

The parameter t is commonly interpreted as *time*. The vector $\mathbf{r}(t)$ traces out a curve or “path” in \mathbb{R}^n as t varies over $[a, b]$. The vector of derivatives

$$\mathbf{r}'(t) = (x_1'(t), x_2'(t), \dots, x_n'(t))$$

is called the *velocity vector* and is tangent to the curve at each point. The norm of the velocity vector $\|\mathbf{r}'(t)\|$ measures the *speed* at which the curve is traversed. The *unit tangent vector* $\mathbf{T}(t)$ is defined as

$$\mathbf{T} \equiv \mathbf{T}(t) \equiv \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}. \quad (2.6)$$

The *arc-length* function s is given by

$$s(t) = \int_a^t \|\mathbf{r}'(\tau)\| d\tau \quad (2.7)$$

with derivative given by

$$s'(t) = \|\mathbf{r}'(t)\|. \quad (2.8)$$

Combining eqs. (2.6) and (2.8), we find that the unit tangent vector may be interpreted as the rate of change of \mathbf{r} with respect to s :

$$\mathbf{T} \equiv \frac{\mathbf{r}'(t)}{s'(t)} = \frac{d\mathbf{r}/dt}{ds/dt} = \frac{d\mathbf{r}}{ds} = \left(\frac{dx_1}{ds}, \dots, \frac{dx_n}{ds} \right). \quad (2.9)$$

In \mathbb{R}^3 , we write $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, so

$$\mathbf{T} = \frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j} + \frac{dz}{ds}\mathbf{k}.$$

Now suppose that φ is a differentiable scalar field defined on D . Let $g \equiv \varphi \circ \mathbf{r}$. Then $g: [a, b] \rightarrow \mathbb{R}$ and for each $t \in [a, b]$

$$g(t) \equiv \varphi[\mathbf{r}(t)].$$

Under these assumptions, the function g is differentiable, and the derivative $g'(t)$ is given by the following *chain rule*:

$$g'(t) = \nabla\varphi[\mathbf{r}(t)] \cdot \mathbf{r}'(t) = \sum_{i=1}^n \frac{\partial\varphi}{\partial x_i} x_i'(t) \quad (2.10)$$

where each partial derivative is evaluated at $\mathbf{r}(t)$. The dot product

$$\nabla\varphi[\mathbf{r}(t)] \cdot \mathbf{T}'(t)$$

is called the *directional derivative of φ along the curve*. Some authors write $d\varphi/ds$ for this directional derivative, as

$$\nabla\varphi \cdot \mathbf{T} = \sum_{i=1}^n \frac{\partial\varphi}{\partial x_i} \cdot \frac{dx_i}{ds} = \frac{d\varphi}{ds}.$$

Potential functions. The meaning of “potential function” varies from author to author. Broadly speaking, there are two definitions, one used by mathematicians and the other used by physicists.

Mathematicians. Let \mathbf{F} be a vector field defined on a set $D \subseteq \mathbb{R}^n$. If there exists a scalar field φ defined on D such that $\mathbf{F} = \nabla\varphi$, then φ is said to be a *potential function* for \mathbf{F} .

Physicists. Let \mathbf{F} be a vector field defined on a set $D \subseteq \mathbb{R}^n$. If there exists a scalar field U defined on D such that $\mathbf{F} = -\nabla U$, then U is said to be a *potential function* for \mathbf{F} . In mechanics, the notion of a potential function is applied almost exclusively to *force fields*. A vector field \mathbf{F} is said to be a **force field** if $\mathbf{F}(\mathbf{r})$ may be interpreted as the *force* acting on a particle at the point \mathbf{r} . If there exists a potential function for a force field \mathbf{F} , then \mathbf{F} is said to be *conservative* (for reasons that will be explained later). In other parts of physics, the use of “potential function” is broadened. For example, in electrostatics the force on a charge q is given by $q\mathbf{E}$ where \mathbf{E} is a vector field called the “electric field.” If there exists a scalar field U such that $\mathbf{E} = -\nabla U$, then U is said to be an *electrostatic potential*.

The two notions of “potential function” differ principally in a sign convention; clearly $U(\mathbf{r}) = -\varphi(\mathbf{r})$. I will attempt to use the symbols φ and U consistently to denote, respectively, the mathematician's and physicist's meaning of “potential function.”

Example 2. In \mathbb{R}^3 , let $U(\mathbf{r}) = r^L$, where L is an integer and $r \equiv \|\mathbf{r}\|$. Generalizing Example 1, it may be shown that

$$-\nabla U = -Lr^{L-2}\mathbf{r}$$

Hence U is a potential function of the force field $\mathbf{F} = -Lr^{L-2}\mathbf{r}$. The equipotential surfaces of U are concentric spheres centered at the origin.

Example 3: The Newtonian potential. Newton's law of gravitation says that the force which a particle of mass M exerts on a particle of mass m is a vector of norm GmM/r^2 and directed from the particle of mass m towards the particle of mass M , where G is a proportionality constant and r is the distance between the two particles. Hence, if the particle of mass M is placed at the origin and the particle of mass m is located at $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then the force acting on the particle of mass m is given by

$$\mathbf{F} = -\frac{GmM}{r^3}\mathbf{r} \quad \text{where} \quad r \equiv \|\mathbf{r}\|.$$

Using Example 2, we see that $\mathbf{F} = -\nabla U$ where $U(\mathbf{r}) = -GmMr^{-1}$. It follows that

$$U(\mathbf{r}) \equiv -\frac{GmM}{r}$$

is a potential function for Newtonian gravity.

Example 4: Central forces. A central or “radial” force field \mathbf{F} in \mathbb{R}^3 is one that can be written in the form

$$\mathbf{F}(\mathbf{r}) = h(r)\mathbf{e}_r$$

where $r \equiv \|\mathbf{r}\|$ and $\mathbf{e}_r \equiv r^{-1}\mathbf{r}$ is a unit vector in the direction of \mathbf{r} . That is, a central force is directed radially, either towards the origin (if $h(r) < 0$), or away from the origin (if $h(r) > 0$), and the magnitude of the force at any point depends only on the distance from the center to that point. **Proposition.** Every central force field is conservative. **Proof.** Define

$$H(r) = \int h(r) dr \quad \text{and} \quad U(\mathbf{r}) = -H(\|\mathbf{r}\|).$$

Then

$$\frac{\partial U}{\partial x} = -H'(r)\frac{\partial r}{\partial x} = -h(r)\frac{x}{r}$$

from eq. (2.2). Similar results holds for $\partial U/\partial y$ and $\partial U/\partial z$. Therefore,

$$-\nabla U(\mathbf{r}) = \frac{h(r)}{r}\mathbf{r} = h(r)\mathbf{e}_r = \mathbf{F}(\mathbf{r})$$

as required.

3. Divergence and Curl

The symbol ∇ is called “del” or “nabla.” It is useful to think of ∇ as a vector *operator*:

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n} \right).$$

In \mathbb{R}^3 , we write

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}.$$

“Multiplication” by $\partial/\partial x_i$ means “take the partial derivative with respect to x_i .” That is, if

$\varphi(\mathbf{r}) = \varphi(x_1, x_2, \dots, x_n)$ is a scalar field,

$$\left(\frac{\partial}{\partial x_i} \right) \varphi \equiv \frac{\partial \varphi}{\partial x_i}.$$

Playing with this operator as if it were a real vector often (but not always) yields results that turn out to be true. For the true results, then, this device has heuristic utility.

For example, suppose that \mathbf{F} is a vector field defined on $D \subseteq \mathbb{R}^3$. For any point $\mathbf{r} \equiv x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ in D we'll write

$$\mathbf{F}(\mathbf{r}) = F_x(\mathbf{r})\mathbf{i} + F_y(\mathbf{r})\mathbf{j} + F_z(\mathbf{r})\mathbf{k}.$$

Operating in a purely formal manner, we may form both a dot product and a cross product of ∇ and \mathbf{F} . These operations yield a scalar

$$\nabla \cdot \mathbf{F} \equiv \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \quad (3.1)$$

and a vector

$$\begin{aligned} \nabla \times \mathbf{F} &\equiv \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} \\ &= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \mathbf{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \mathbf{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \mathbf{k}, \end{aligned} \quad (3.2)$$

where all partial derivatives are to be evaluated at the point \mathbf{r} . Amazingly, both these objects are meaningful and useful. The scalar $\nabla \cdot \mathbf{F}$ is called the **divergence** of \mathbf{F} and is also written “div \mathbf{F} .” The vector $\nabla \times \mathbf{F}$ is called the **curl** of \mathbf{F} and is also written “curl \mathbf{F} .”

We will give geometric interpretations of $\nabla \cdot \mathbf{F}$ and $\nabla \times \mathbf{F}$ after our discussion of line and surface integrals. However, two simple examples at this stage will start to give the reader some idea of the meaning of the divergence and the curl.

Example 1. Suppose $\mathbf{F}(\mathbf{r}) \equiv \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. That is, this vector field is radially directed, and $\|\mathbf{F}(\mathbf{r})\| = \|\mathbf{r}\|$, the distance from the origin to \mathbf{r} . Hence,

$$\operatorname{div} \mathbf{F}(\mathbf{r}) = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3.$$

Example 2. Consider a rigid circular disk rotating around an axis through its center and perpendicular to the plane of the disk. Without loss of generality, we may set up the coordinate system so that the disk rotates in the xy -plane, and the axis of rotation coincides with the z coordinate. Let ω denote the angular *speed* of the disk (in radians per second). Physicists find it convenient to let $\boldsymbol{\omega} = \omega\mathbf{k}$ denote the angular *velocity* of the disk: that is,

angular velocity is a vector with magnitude ω directed along the z axis. Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ denote a point on the disk. The speed of that point depends on ω and on $r \equiv \|\mathbf{r}\|$ according to the equation $v = \omega r$. In more detail, the *velocity* \mathbf{v} of that point (a vector) is given by

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega \\ x & y & 0 \end{vmatrix} = \omega(-y\mathbf{i} + x\mathbf{j})$$

where $v \equiv \|\mathbf{v}\|$. Note that \mathbf{v} is a vector field. We now ask: what is the curl of \mathbf{v} ? From eq. (3.2),

$$\text{curl } \mathbf{v} = \nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\omega y & \omega x & 0 \end{vmatrix} = 2\omega\mathbf{k} = 2\boldsymbol{\omega}.$$

In words, the curl of linear velocity is just twice the angular velocity of the disk.

So far in our play with ∇ we've only considered first derivatives. When we consider second derivatives, four of the possible combinations turn out to be meaningful and useful.

(1) $\nabla \cdot (\nabla\varphi) = \text{div}(\text{grad } \varphi)$. Working formally, we find

$$\begin{aligned} \nabla \cdot (\nabla\varphi) &= \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) \cdot \left(\mathbf{i} \frac{\partial\varphi}{\partial x} + \mathbf{j} \frac{\partial\varphi}{\partial y} + \mathbf{k} \frac{\partial\varphi}{\partial z} \right) \\ &= \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}. \end{aligned}$$

It turns out that this scalar field is very useful in physics. The operation $\nabla \cdot (\nabla\varphi)$ is called the *Laplacian* of φ and is written $\nabla^2\varphi$. If $\nabla^2\varphi(\mathbf{r}) = 0$ for all \mathbf{r} in some volume D , the scalar function φ is said to be *harmonic*. The Laplacian of a vector field \mathbf{F} is defined “component-wise”: if $\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$ then

$$\nabla^2\mathbf{F} = \nabla^2 F_x\mathbf{i} + \nabla^2 F_y\mathbf{j} + \nabla^2 F_z\mathbf{k}.$$

(2) $\nabla \times (\nabla\varphi) = \text{curl}(\text{grad } \varphi)$. For any vector \mathbf{v} and scalar ξ , we know that $\mathbf{v} \times (\mathbf{v}\xi) = (\mathbf{v} \times \mathbf{v})\xi = \mathbf{0}$. This suggests that $\nabla \times (\nabla\varphi)$, the curl of a gradient, should equal $\mathbf{0}$. This turns out to be true under some weak conditions: if φ is a scalar field with continuous second-order mixed partial derivatives, then $\text{curl}(\text{grad } \varphi) = \mathbf{0}$. Conversely, it may be shown that if $\text{curl } \mathbf{F} = \mathbf{0}$ for all points \mathbf{x} in an open convex set D , then there exists a scalar field φ defined on D such that $\mathbf{F} = \nabla\varphi$.

(3) $\nabla \cdot (\nabla \times \mathbf{F}) = \text{div}(\text{curl } \mathbf{F})$. For any vectors \mathbf{a} and \mathbf{b} , we know that $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0$. This suggests that $\nabla \cdot (\nabla \times \mathbf{F}) = \text{div}(\text{curl } \mathbf{F}) = 0$. This is in fact the case: if all the mixed partial derivatives of a vector field \mathbf{F} are continuous, then $\nabla \cdot (\nabla \times \mathbf{F}) = \text{div}(\text{curl } \mathbf{F}) = 0$. Conversely, if D is an open interval in \mathbb{R}^3 , and $\nabla \cdot \mathbf{G} = 0$ throughout

D , then $\mathbf{G} = \text{curl } \mathbf{F}$ for some vector field \mathbf{F} . [An “open interval” in \mathbb{R}^3 is the Cartesian product of open intervals. That is, an open interval in \mathbb{R}^3 has the form $(a_x, b_x) \times (a_y, b_y) \times (a_z, b_z)$ where $a_x < b_x$, $a_y < b_y$, and $a_z < b_z$.]

(4) $\nabla \times (\nabla \times \mathbf{F}) = \text{curl}(\text{curl } \mathbf{F})$. Equation (1.4) may be written $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$. If we substitute ∇ for \mathbf{a} and \mathbf{b} , and \mathbf{F} for \mathbf{c} , we obtain

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - (\nabla \cdot \nabla)\mathbf{F} = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F} \quad (3.3)$$

which holds if all mixed partial derivatives are continuous. In other words,

$$\text{curl}(\text{curl } \mathbf{F}) = \text{grad}(\text{div } \mathbf{F}) - \nabla^2 \mathbf{F}.$$

(There are other ways the right-hand side of eq. (1.4) may be written, but these lead to meaningless formulae when ∇ is substituted for \mathbf{a} and \mathbf{b} .)

4. Line Integrals.

Let \mathbf{r} be a vector-valued function that maps an interval of real numbers $[a, b]$ into $D \subseteq \mathbb{R}^n$. If \mathbf{r} is continuous on $[a, b]$, then \mathbf{r} is said to be a *continuous path* in n -space. The path is said to be *smooth* if \mathbf{r}' exists and is continuous in (a, b) . The path is said to be *piecewise smooth* if $[a, b]$ can be partitioned into a finite number of subintervals in each of which the path is smooth.

Let \mathbf{r} be a piecewise smooth path in n -space defined on an interval $[a, b]$, and let \mathbf{F} be a vector field defined and bounded on the graph of \mathbf{r} . The line integral of \mathbf{F} along \mathbf{r} is denoted by the symbol $\int \mathbf{F} \cdot d\mathbf{r}$ and is defined by the equation

$$\int \mathbf{F} \cdot d\mathbf{r} \equiv \int_a^b \mathbf{F}[\mathbf{r}(t)] \cdot \mathbf{r}'(t) dt, \quad (4.1)$$

whenever the integral on the right exists, either as a proper or improper integral.

Other notations for line integrals. If C denotes the graph of \mathbf{r} , the line integral $\int \mathbf{F} \cdot d\mathbf{r}$ is also written as $\int_C \mathbf{F} \cdot d\mathbf{r}$ and is called *the integral of \mathbf{F} along C* . If $\mathbf{a} = \mathbf{r}(a)$ and $\mathbf{b} = \mathbf{r}(b)$, then the line integral is sometimes written as $\int_a^b \mathbf{F} \cdot d\mathbf{r}$ or $\int_a^b \mathbf{F}$ and is called *the line integral of \mathbf{F} from \mathbf{a} to \mathbf{b} along \mathbf{r}* . When the notation $\int_a^b \mathbf{F}$ is used it should be kept in mind that the integral depends not only on the end points \mathbf{a} and \mathbf{b} but also (in general) on the path \mathbf{r} joining them. When $\mathbf{a} = \mathbf{b}$ the path is said to be *closed*. The symbol \oint is often used to indicate integration along a closed path. When \mathbf{F} and \mathbf{r} are expressed in terms of their components, say

$$\mathbf{F}(\mathbf{r}) = (F_1(\mathbf{r}), F_2(\mathbf{r}), \dots, F_n(\mathbf{r})) \quad \text{and} \quad \mathbf{r}(t) = (x_1(t), x_2(t), \dots, x_n(t))$$

then the integral on the right in eq. (4.1) becomes the integral of a sum (and a sum of integrals):

$$\int \mathbf{F} \cdot d\mathbf{r} = \int_a^b \sum_{i=1}^n F_i[\mathbf{r}(t)] x'_i(t) dt = \sum_{i=1}^n \int_a^b F_i[\mathbf{r}(t)] x'_i(t) dt.$$

In this case, the line integral is also written as $\int F_1 dx_1 + F_2 dx_2 + \cdots + F_n dx_n$. In \mathbb{R}^2 the path \mathbf{r} is usually written as a pair of parametric equations $(x(t), y(t))$, and the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is written $\int_C F_x dx + F_y dy$. Similarly, in \mathbb{R}^3 the path \mathbf{r} is usually written as a triple of parametric equations $(x(t), y(t), z(t))$ and the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is written $\int_C F_x dx + F_y dy + F_z dz$.

Basic properties of line integrals. Line integrals share many of the fundamental properties of ordinary integrals. For example, they have a *linearity property* with respect to the integrand:

$$\int (\alpha \mathbf{F} + \beta \mathbf{G}) \cdot d\mathbf{r} = \alpha \int \mathbf{F} \cdot d\mathbf{r} + \beta \int \mathbf{G} \cdot d\mathbf{r}$$

and an additive property with respect to the path of integration:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

where the two curves C_1 and C_2 make up the curve C .

Change of parameter. As evaluation of the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ makes use of the parametric representation $\mathbf{r}(t)$, it might seem that an alternative parameterization of the curve C would yield a different value of $\int_C \mathbf{F} \cdot d\mathbf{r}$. In fact, the value of $\int_C \mathbf{F} \cdot d\mathbf{r}$ is invariant with respect to the parameterization of C up to a change of sign. Let \mathbf{r} be a continuous path in n -space defined on an interval $[a, b]$, and let g be a differentiable real-valued function defined on an interval $[c, d]$ such that (1) g' is never zero on $[c, d]$, and (2) g maps $[c, d]$ onto $[a, b]$. Then the function $\tilde{\mathbf{r}}: [c, d] \rightarrow \mathbb{R}^n$ defined by

$$\tilde{\mathbf{r}}(u) \equiv \mathbf{r}[g(u)]$$

is a continuous path having the same graph C as \mathbf{r} . Two paths \mathbf{r} and $\tilde{\mathbf{r}}$ so related are said to be *equivalent*. If $g' > 0$ everywhere on $[c, d]$, we say that \mathbf{r} and $\tilde{\mathbf{r}}$ trace out C in the *same direction*, and if $g' < 0$ everywhere on $[c, d]$, we say that \mathbf{r} and $\tilde{\mathbf{r}}$ trace out C in *opposite directions*. In the first case, the change of parameter function g is said to be *orientation-preserving*, and in the second case it is said to be *orientation-reversing*.

Theorem 4.1. Let \mathbf{r} and $\tilde{\mathbf{r}}$ be equivalent piecewise smooth paths. Then we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot d\tilde{\mathbf{r}}$$

if \mathbf{r} and $\tilde{\mathbf{r}}$ trace out C in the same direction, and

$$\int_C \mathbf{F} \cdot d\mathbf{r} = - \int_C \mathbf{F} \cdot d\tilde{\mathbf{r}}$$

if \mathbf{r} and $\tilde{\mathbf{r}}$ trace out C in opposite directions.

Line integrals with respect to arc length. In some circumstances the arc-length function s provides a natural and convenient parameterization of C , the graph of \mathbf{r} . Suppose that h is a scalar field defined and bounded on C . The *line integral of h with respect to arc length* along C is denoted by $\int_C h ds$ and defined by

$$\int_C h ds \equiv \int_a^b h[\mathbf{r}(t)] s'(t) dt, \quad (4.2)$$

whenever this integral exists. In particular, consider the scalar field given by

$$h[\mathbf{r}(t)] \equiv \mathbf{F}[\mathbf{r}(t)] \cdot \mathbf{T}(t),$$

the dot product of a vector field \mathbf{F} defined on C and the unit tangent vector. In this case, the integral with respect to arc length $\int_C h ds$ is identical to the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ because

$$\mathbf{F}[\mathbf{r}(t)] \cdot \mathbf{r}'(t) = \mathbf{F}[\mathbf{r}(t)] \cdot \mathbf{T}(t) s'(t) = h[\mathbf{r}(t)] s'(t).$$

The integral in (4.2) is naturally viewed as the limiting value of a Riemann sum

$$\sum_{j=1}^N h(\mathbf{r}_j) \Delta s_j$$

obtained when the curve C is partitioned into N segments, where the j th segment is of length Δs_j and contains the point \mathbf{r}_j .

If C is a closed path, the line integral

$$\oint_C \mathbf{F} \cdot \mathbf{T} ds = \oint_C \mathbf{F} \cdot d\mathbf{r}$$

is called the **circulation** of \mathbf{F} around C .

The concept of work in mechanics. Consider a particle which moves along a curve in \mathbb{R}^3 under the action of a force field \mathbf{F} . If the curve is the graph of a piecewise smooth path \mathbf{r} , then the **work** done by \mathbf{F} is defined to be the line integral $\int \mathbf{F} \cdot d\mathbf{r}$.

The principle of work and energy. Suppose a particle of mass m moves freely through space under the action of a force field \mathbf{F} . If the speed of the particle at time t is $v(t)$, then its kinetic energy is defined to be $\frac{1}{2}mv(t)^2$. We may show that the change in the particle's kinetic energy in any time interval is equal to the work done by \mathbf{F} during that time interval.

Proof. Let $\mathbf{r}(t)$ denote the position of the particle at time t , for all $t \in [a, b]$. We want to

show that

$$\int_{\mathbf{r}(a)}^{\mathbf{r}(b)} \mathbf{F} \cdot d\mathbf{r} = \frac{1}{2}mv(b)^2 - \frac{1}{2}mv(a)^2. \quad (4.3)$$

The motion of the particle at any time is governed by Newton's second law of motion, which says

$$\mathbf{F}[\mathbf{r}(t)] = m\mathbf{r}''(t) = m\mathbf{v}'(t)$$

where $\mathbf{v}(t)$ denotes the velocity vector at time t , and $v(t) \equiv \|\mathbf{v}(t)\|$. Hence

$$\mathbf{F}[\mathbf{r}(t)] \cdot \mathbf{r}'(t) = m\mathbf{v}'(t) \cdot \mathbf{v}(t) = \frac{1}{2}m \frac{d}{dt}(\mathbf{v}(t) \cdot \mathbf{v}(t)) = \frac{1}{2}m \frac{d}{dt}(v(t)^2).$$

Integrating from a to b we obtain

$$\int_{\mathbf{r}(a)}^{\mathbf{r}(b)} \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}[\mathbf{r}(t)] \cdot \mathbf{r}'(t) dt = \frac{1}{2}m [v(t)^2]_a^b = \frac{1}{2}mv(b)^2 - \frac{1}{2}mv(a)^2,$$

as was to be shown.

Independence of the path. Suppose that \mathbf{F} is a vector field that is continuous on an open connected set $D \subseteq \mathbb{R}^n$. [For the definition of “open connected set” see Apostol, pp. 332-333.] In general, the line integral

$$\int_a^b \mathbf{F} \cdot d\mathbf{r}$$

depends not only on the end points \mathbf{a} and \mathbf{b} , but also on the path $\mathbf{r}(\cdot)$ that connects them. For some vector fields \mathbf{F} , however, $\int_a^b \mathbf{F} \cdot d\mathbf{r}$ doesn't depend on $\mathbf{r}(\cdot)$, and in this case we say the integral is *independent of the path from \mathbf{a} to \mathbf{b}* . If the integral $\int_a^b \mathbf{F} \cdot d\mathbf{r}$ is independent of the path from \mathbf{a} to \mathbf{b} for all \mathbf{a} and \mathbf{b} in D , then we'll say that $\int_a^b \mathbf{F} \cdot d\mathbf{r}$ is *independent of the path in D* .

Let C be a piecewise smooth closed path in D , where D is an open connected set in \mathbb{R}^n . Let \mathbf{a} and \mathbf{b} be two distinct points on the path C . If the integral $\int_a^b \mathbf{F} \cdot d\mathbf{r}$ is independent of the path from \mathbf{a} to \mathbf{b} , then the circulation of \mathbf{F} around C is zero:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0.$$

If the integral $\int_a^b \mathbf{F} \cdot d\mathbf{r}$ is independent of the path from \mathbf{a} to \mathbf{b} for every pair of points \mathbf{a} and \mathbf{b} , then the circulation of \mathbf{F} around C is zero,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0, \quad (4.4)$$

for every piecewise smooth closed path C in D . Conversely, if eq. (4.4) holds for every piecewise smooth closed path C in D , then $\int_a^b \mathbf{F} \cdot d\mathbf{r}$ is independent of the path in D .

The second fundamental theorem of calculus for real functions states that:

$$\int_a^b \varphi'(x) dx = \varphi(b) - \varphi(a)$$

provided that φ' is continuous on some open interval containing both a and b . An analogous result holds for line integrals. **Theorem 4.2** (The second fundamental theorem of calculus for line integrals). Let φ be a differentiable scalar field with a continuous gradient $\nabla\varphi$ on an open connected set $D \subseteq \mathbb{R}^n$. For any two points \mathbf{a} and \mathbf{b} joined by a piecewise smooth path $\mathbf{r}(\cdot)$ in D we have

$$\int_a^b \nabla\varphi \cdot d\mathbf{r} = \varphi(\mathbf{b}) - \varphi(\mathbf{a}). \quad (4.5)$$

Corollary. Equation (4.5) implies that $\int_a^b \nabla\varphi \cdot d\mathbf{r}$ is independent of the path from \mathbf{a} to \mathbf{b} . As eq. (4.5) holds for every pair of points \mathbf{a} and \mathbf{b} in D , it follows that $\int_a^b \nabla\varphi \cdot d\mathbf{r}$ is independent of the path in D . Hence,

$$\oint_C \nabla\varphi \cdot d\mathbf{r} = 0 \quad (4.6)$$

for every piecewise smooth closed path C in D . In words, the circulation of a gradient around any piecewise smooth closed path in D is zero.

The conservation of mechanical energy in a conservative force field. Suppose a particle of mass m moves freely through space under the action of a force field \mathbf{F} . We have previously shown that the work done by \mathbf{F} over an interval of time equals the change in the kinetic energy of the particle during that time interval. To be precise, if $[a, b]$ is the time interval, then

$$\int_{\mathbf{r}(a)}^{\mathbf{r}(b)} \mathbf{F} \cdot d\mathbf{r} = \frac{1}{2}mv(b)^2 - \frac{1}{2}mv(a)^2 \quad (4.3)$$

where $\mathbf{r}(t)$ denotes the location and $v(t)$ denotes the speed of the particle at any time $t \in [a, b]$. Assume now that \mathbf{F} is a *conservative* force field, so $\mathbf{F} = -\nabla U$ for some potential function U . Then

$$\int_{\mathbf{r}(a)}^{\mathbf{r}(b)} \mathbf{F} \cdot d\mathbf{r} = -\int_{\mathbf{r}(a)}^{\mathbf{r}(b)} \nabla U \cdot d\mathbf{r} = U[\mathbf{r}(a)] - U[\mathbf{r}(b)]. \quad (4.7)$$

Combining equations (4.3) and (4.7) and rearranging, we find

$$\frac{1}{2}mv(b)^2 + U[\mathbf{r}(b)] = \frac{1}{2}mv(a)^2 + U[\mathbf{r}(a)]. \quad (4.8)$$

The function $U(\mathbf{r})$ gives the **potential energy** of the particle at \mathbf{r} . Equation (4.8), then, says that the sum of the kinetic and potential energy of a particle is a conserved quantity if the particle moves under the action of a conservative force field. (This explains why such force fields are said to be “conservative.”)

A converse of Theorem 4.2 is also true. **Theorem 4.3.** Let \mathbf{F} be a continuous vector field defined on an open connected set $D \subseteq \mathbb{R}^n$. If the line integral $\int_a^b \mathbf{F} \cdot d\mathbf{r}$ is independent of the path in D [or if eq. (4.4) holds for every piecewise smooth closed path C in D] then there exists a differentiable scalar field φ on D such that $\mathbf{F}(\mathbf{r}) = \nabla\varphi(\mathbf{r})$.

Suppose that \mathbf{F} is a vector field that is continuous on an open connected set $D \subseteq \mathbb{R}^n$. Theorems 4.2 and 4.3 give us necessary and sufficient conditions for \mathbf{F} to be a gradient: \mathbf{F} is a gradient if and only if $\int_a^b \mathbf{F} \cdot d\mathbf{r}$ is independent of the path in D . Equivalently, \mathbf{F} is a gradient if and only if the circulation of \mathbf{F} around any piecewise smooth closed path in D is zero. These necessary conditions are not very useful, however, because they're generally impossible to check. The following theorem provides a set of necessary conditions for \mathbf{F} to be a gradient that are readily checked.

Theorem 4.4. Let $\mathbf{F}(\mathbf{r}) = (F_1(\mathbf{r}), F_2(\mathbf{r}), \dots, F_n(\mathbf{r}))$ be a continuously differentiable vector field defined on an open connected set $D \subseteq \mathbb{R}^n$. If \mathbf{F} is a gradient, then

$$\frac{\partial F_i}{\partial x_j}(\mathbf{r}) = \frac{\partial F_j}{\partial x_i}(\mathbf{r}) \quad (4.7)$$

for all i and j in $\{1, 2, \dots, n\}$ and all $\mathbf{r} \in D$. **Proof.** Suppose that $\mathbf{F} = \nabla\varphi$ for some scalar field φ defined on D . Then

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial}{\partial x_j} \left(\frac{\partial \varphi}{\partial x_i} \right) = \frac{\partial^2 \varphi}{\partial x_j \partial x_i} \quad \text{and} \quad \frac{\partial F_j}{\partial x_i} = \frac{\partial}{\partial x_i} \left(\frac{\partial \varphi}{\partial x_j} \right) = \frac{\partial^2 \varphi}{\partial x_i \partial x_j}.$$

The conditions of the theorem guarantee the equality of these two “mixed partials.” (See Apostol, page 278.)

As a corollary of Theorem 4.4, we have formula (2) of Section 3: for any continuously twice differentiable scalar field φ defined on an open connected set $D \subseteq \mathbb{R}^3$,

$$\text{curl}(\nabla\varphi) = \nabla \times \nabla\varphi = \mathbf{0}.$$

The proof is left to the reader.

If the set D of Theorem 4.4 is assumed to be *convex*, then eq. (4.7) gives *sufficient* conditions for \mathbf{F} to be a gradient. **Theorem 4.5.** Let $\mathbf{F}(\mathbf{r}) = (F_1(\mathbf{r}), F_2(\mathbf{r}), \dots, F_n(\mathbf{r}))$ be a continuously differentiable vector field defined on a convex open connected set $D \subseteq \mathbb{R}^n$. If

$$\frac{\partial F_i}{\partial x_j}(\mathbf{r}) = \frac{\partial F_j}{\partial x_i}(\mathbf{r})$$

for all i and j in $\{1, 2, \dots, n\}$ and all $\mathbf{r} \in D$, then there exists a scalar field φ defined on D such that $\mathbf{F} = \nabla\varphi$. For the proof, see Apostol, pp. 351-352. **Corollary:** Suppose that \mathbf{F} is a continuously differentiable vector field defined on a convex open connected set D in \mathbb{R}^3 . If

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \mathbf{0}$$

everywhere in D , then there exists a scalar field φ defined on D such that $\mathbf{F} = \nabla\varphi$.

5. Surface Integrals

There are several ways to specify a “surface” in \mathbb{R}^3 . (1) **Implicit representation.** The set of all points (x, y, z) that satisfy an equation of the form $F(x, y, z) = 0$. (2) **Explicit representation.** Sometimes one can solve $F(x, y, z) = 0$ for one of the variables in terms of the other two. For example, suppose it's possible to solve for z in terms of x and y . The solution $z = f(x, y)$ is said to be an *explicit* representation of the surface. (3) **Parametric representation.** We have 3 equations expressing x , y , and z as functions of two parameters u and v :

$$x = X(u, v), \quad y = Y(u, v), \quad \text{and} \quad z = Z(u, v) \quad (5.1)$$

where (u, v) is allowed to vary over some connected set T in the uv -plane. Sometimes we'll write the three parametric equations of eq. (5.1) in a single *vector* form:

$$\mathbf{r}(u, v) = X(u, v)\mathbf{i} + Y(u, v)\mathbf{j} + Z(u, v)\mathbf{k}. \quad (5.2)$$

The image of T under the mapping \mathbf{r} is called a *parametric surface* and is denoted $\mathbf{r}(T)$. We assume that X , Y , and Z are continuous. If the mapping \mathbf{r} is one-to-one, the image $\mathbf{r}(T)$ is called a *simple parametric surface*. Note that an explicit representation of a surface is obtained from a parametric representation with the functions $X(u, v) = u$, $Y(u, v) = v$, and $Z(u, v) = f(u, v)$.

The fundamental vector product. If X , Y , and Z are differentiable on T , we consider the two vectors

$$\frac{\partial \mathbf{r}}{\partial u} = \frac{\partial X}{\partial u}\mathbf{i} + \frac{\partial Y}{\partial u}\mathbf{j} + \frac{\partial Z}{\partial u}\mathbf{k}$$

and

$$\frac{\partial \mathbf{r}}{\partial v} = \frac{\partial X}{\partial v}\mathbf{i} + \frac{\partial Y}{\partial v}\mathbf{j} + \frac{\partial Z}{\partial v}\mathbf{k}.$$

The cross product of these two vectors is referred to as the *fundamental vector product* of

the representation \mathbf{r} .

$$\begin{aligned}
 \mathbf{N} &\equiv \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial X}{\partial u} & \frac{\partial Y}{\partial u} & \frac{\partial Z}{\partial u} \\ \frac{\partial X}{\partial v} & \frac{\partial Y}{\partial v} & \frac{\partial Z}{\partial v} \end{vmatrix} \\
 &= \begin{vmatrix} \frac{\partial Y}{\partial u} & \frac{\partial Z}{\partial u} \\ \frac{\partial Y}{\partial v} & \frac{\partial Z}{\partial v} \end{vmatrix} \mathbf{i} + \begin{vmatrix} \frac{\partial Z}{\partial u} & \frac{\partial X}{\partial u} \\ \frac{\partial Z}{\partial v} & \frac{\partial X}{\partial v} \end{vmatrix} \mathbf{j} + \begin{vmatrix} \frac{\partial X}{\partial u} & \frac{\partial Y}{\partial u} \\ \frac{\partial X}{\partial v} & \frac{\partial Y}{\partial v} \end{vmatrix} \mathbf{k} \\
 &= \frac{\partial(Y, Z)}{\partial(u, v)} \mathbf{i} + \frac{\partial(Z, X)}{\partial(u, v)} \mathbf{j} + \frac{\partial(X, Y)}{\partial(u, v)} \mathbf{k}.
 \end{aligned} \tag{5.3}$$

If (u, v) is a point in T at which both $\partial \mathbf{r} / \partial u$ and $\partial \mathbf{r} / \partial v$ are continuous and $\mathbf{N} \neq \mathbf{0}$, then the image point $\mathbf{r}(u, v)$ is said to be a *regular point* of \mathbf{r} . If $\mathbf{r}(u, v)$ is not a regular point, then it is said to be a *singular point* of \mathbf{r} . A surface $\mathbf{r}(T)$ is said to be *smooth* if all of its points are regular points.

In the case of an explicitly represented surface

$$\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + f(x, y)\mathbf{k}$$

we have

$$\frac{\partial \mathbf{r}}{\partial x} = \mathbf{i} + \frac{\partial f}{\partial x} \mathbf{k} \quad \text{and} \quad \frac{\partial \mathbf{r}}{\partial y} = \mathbf{j} + \frac{\partial f}{\partial y} \mathbf{k}$$

so

$$\mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \partial f / \partial x \\ 0 & 1 & \partial f / \partial y \end{vmatrix} = -\frac{\partial f}{\partial x} \mathbf{i} - \frac{\partial f}{\partial y} \mathbf{j} + \mathbf{k}. \tag{5.4}$$

Note that

$$\|\mathbf{N}\| = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} \tag{5.5}$$

in this case.

As each vector $\partial \mathbf{r} / \partial u$ and $\partial \mathbf{r} / \partial v$ is tangent to the surface $\mathbf{r}(T)$, it follows that \mathbf{N} is “normal” (i.e., perpendicular) to the surface at $\mathbf{r}(u, v)$. Hence, if $\|\mathbf{N}\| \neq 0$,

$$\mathbf{n} \equiv \frac{\mathbf{N}}{\|\mathbf{N}\|} \tag{5.6}$$

is a unit vector that is normal to the surface at $\mathbf{r}(u, v)$.

Let R denote a rectangle with base Δu and height Δv in T , where Δu and Δv are “small.” The image $\mathbf{r}(R)$ is approximately a parallelogram with sides

$$\frac{\partial \mathbf{r}}{\partial u} \Delta u \quad \text{and} \quad \frac{\partial \mathbf{r}}{\partial v} \Delta v.$$

The area of this parallelogram is

$$\left\| \frac{\partial \mathbf{r}}{\partial u} \Delta u \times \frac{\partial \mathbf{r}}{\partial v} \Delta v \right\| = \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| \Delta u \Delta v.$$

Hence

$$\|\mathbf{N}\| = \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\|$$

may be thought of as a local magnification factor for areas.

The area of a parametric surface. Let $S \equiv \mathbf{r}(T)$. The computation given above suggests the following definition. The area of S , denoted $A(S)$, is defined by the double integral

$$A(S) \equiv \iint_T \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| du dv. \quad (5.7)$$

If S is defined explicitly, this integral becomes

$$A(S) \equiv \iint_T \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} dx dy$$

where T is the projection of S onto the xy -plane.

Definition. Let $S \equiv \mathbf{r}(T)$ be a parametric surface described by a differentiable function \mathbf{r} defined on a region T in the uv -plane, and let g be a scalar field defined and bounded on S . The surface integral of g over S is denoted by the symbol $\iint_S g dS$ [or by $\iint_{\mathbf{r}(T)} g(x, y, z) dS$],

and is defined by

$$\iint_{\mathbf{r}(T)} g dS \equiv \iint_T g[\mathbf{r}(u, v)] \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| du dv \quad (5.8)$$

whenever the double integral on the right exists.

Note: the symbol dS used in a surface integral always denotes a differential element of surface area, whereas the symbol ds used in a line integral always denotes a differential element of arc length. (Later we'll use the symbol dV to denote a differential volume element. That is, dV is just shorthand for $dx dy dz$.)

Any surface S may be represented parametrically in different ways. It may be shown that the value of $\int_S g dS$ does not depend on the parameterization.

Although it's necessary to go back to eq. (5.8) to actually calculate a surface integral, intuitively we may think of this surface integral as the limiting value of a Riemann sum. Suppose we approximate the surface S by a polyhedron of L faces, where the ℓ th face has area ΔS_ℓ and is tangent to S at (x_ℓ, y_ℓ, z_ℓ) . Now consider the sum

$$\sum_{\ell=1}^L g(x_\ell, y_\ell, z_\ell) \Delta S_\ell.$$

If we let $L \rightarrow \infty$ in such a way that $\max\{\Delta S_\ell\} \rightarrow 0$, this Riemann sum approaches $\int_S g dS$.

If the surface S is represented explicitly by $z = f(x, y)$, the surface integral may be written

$$\int_S g dS = \int_T \int g[x, y, f(x, y)] \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} dx dy$$

The flux of a vector field through a surface. Let $S = \mathbf{r}(T)$ be a simple parametric surface, let \mathbf{n} be the unit normal vector to S defined by eq. (5.6), and let \mathbf{F} be a vector field defined on S . At any point on S the dot product $\mathbf{F} \cdot \mathbf{n}$ is the component of \mathbf{F} in the direction of \mathbf{n} . The surface integral

$$\int_S \mathbf{F} \cdot \mathbf{n} dS = \int_T \int \mathbf{F} \cdot \mathbf{n} \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| du dv = \int_T \int \mathbf{F} \cdot \mathbf{N} du dv \quad (5.9)$$

is called the *flux* of \mathbf{F} through the surface. This kind of surface integral occurs frequently in applications. The flux of a vector field through a surface is meaningful regardless of the nature of \mathbf{F} , but perhaps the situation where flux is easiest to interpret is when

$$\mathbf{F}(x, y, z) = \rho(x, y, z)\mathbf{v}(x, y, z)$$

where $\rho(x, y, z)$ and $\mathbf{v}(x, y, z)$ denote the *density* and the *velocity*, respectively, of a fluid at the point (x, y, z) . Then the flux measures the mass of fluid passing through the surface per unit time. See the discussions in Feynman and Schey for more on the intuitive meaning of “flux.”

Suppose that S is represented explicitly by $z = f(x, y)$. From eq. (5.4) we have

$$\mathbf{N} = -\frac{\partial f}{\partial x} \mathbf{i} - \frac{\partial f}{\partial y} \mathbf{j} + \mathbf{k}.$$

Now write the vector field \mathbf{F} in terms of its components:

$$\mathbf{F}(x, y, z) = F_x(x, y, z)\mathbf{i} + F_y(x, y, z)\mathbf{j} + F_z(x, y, z)\mathbf{k}.$$

It follows that the flux integral in this case may be written

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iint_T \left[-F_x \frac{\partial f}{\partial x} - F_y \frac{\partial f}{\partial y} + F_z \right] dx \, dy$$

where F_x , F_y , and F_z are evaluated at $(x, y, f(x, y))$.

6. The Divergence Theorem

Mathematical solids. To a mathematician, a “solid” is a particular kind of subset of \mathbb{R}^3 . As my mathematical dictionary quaintly defines it, a “geometric solid” is “[a]ny portion of space which is occupied conceptually by a physical solid; e.g., a cube or a sphere.” The key word here is “conceptually.” A mathematical solid, unlike a physical solid, has no *solidity*. For example, a spherical bubble trapped in a block of ice is a mathematical solid. In this document we’ll implicitly assume various things about the solids of interest. In particular, we’ll assume of any solid of interest V that (1) V is a connected set, (2) V is bounded, and (3) the boundary of V is a regular surface in the sense of section 5, or the union of several such surfaces. In addition, this surface must be “orientable”; for a definition, see Apostol, page 456. The boundary of a solid partitions \mathbb{R}^3 into two parts, an *interior* (the solid) and an *exterior*, and it’s not possible to pass from the interior to the exterior along a continuous path without going through the surface.

Open and closed surfaces. Vector calculus deals with two different kinds of surface: *open*, and *closed*. An *open* surface is bounded by an edge that we’ll assume is a piecewise smooth curve. For example, a piece of paper is an open surface. A *closed* surface is not bounded by an edge, but itself forms the boundary of a solid. The surface of a beach ball is an example of a closed surface.

Unit normal vectors. Suppose $\mathbf{r}: T \rightarrow \mathbb{R}^3$ is a parametric representation of a surface S . At any regular point $\mathbf{r}(u, v)$ there are *two* unit vectors that are normal to the surface:

$$\mathbf{n}_1 \equiv \frac{\mathbf{N}}{\|\mathbf{N}\|} \quad \text{where} \quad \mathbf{N} \equiv \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}$$

and

$$\mathbf{n}_2 \equiv -\mathbf{n}_1.$$

In calculating a “flux integral”

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS$$

it is necessary to specify which of these two unit normal vectors is to be used. If the surface S is closed, a universal convention is that the *outward facing* normal unit vector is used.

Let \mathbf{F} be a differentiable vector field defined on $D \subseteq \mathbb{R}^3$, say

$$\mathbf{F}(\mathbf{r}) = F_x(\mathbf{r})\mathbf{i} + F_y(\mathbf{r})\mathbf{j} + F_z(\mathbf{r})\mathbf{k} \quad \text{where} \quad \mathbf{r} \equiv x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

We've defined the divergence of \mathbf{F} by eq. (3.1):

$$\operatorname{div} \mathbf{F} \equiv \nabla \cdot \mathbf{F} \equiv \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}. \quad (3.1)$$

This expression may seem totally dependent on the chosen system of coordinates. Miraculously enough, it turns out that $\operatorname{div} \mathbf{F}$ has a physical interpretation that is completely independent of the coordinate system.

To explain this interpretation, we need to introduce the concept of “flux density” at a point. Let V denote a mathematical solid with surface S , let ΔV denote the volume of V , and suppose that \mathbf{r} is in the interior of V . For example, V could be a sphere, or a rectangular parallelepiped. By the “flux density of \mathbf{F} over V ” I mean the ratio of the flux of \mathbf{F} through S to the volume of V :

$$\frac{1}{\Delta V} \iint_S \mathbf{F} \cdot \mathbf{n} \, dS.$$

By “the flux density of \mathbf{F} at \mathbf{r} ” I mean the limit of this ratio as the solid V is allowed to shrink down to the singleton set $\{\mathbf{r}\}$:

$$\text{Flux density of } \mathbf{F} \text{ at } \mathbf{r} \equiv \lim_{\Delta V \rightarrow 0} \frac{1}{\Delta V} \iint_S \mathbf{F} \cdot \mathbf{n} \, dS.$$

The discerning reader may object that this limit apparently depends on how the set of contracting solids V are chosen, so “flux density” appears not to be well-defined by this formula. It's a remarkable fact, however, that this is not the case: the limiting value of this ratio does not depend on how the contracting solids V are chosen. To fix ideas, it's convenient to let V be a rectangular parallelepiped with dimensions Δx , Δy , and Δz , and centered around $\mathbf{r} = (x, y, z)$. An instructive and easy calculation then shows that

$$\lim_{\Delta V \rightarrow 0} \frac{1}{\Delta V} \iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}.$$

In words, the flux density of \mathbf{F} at \mathbf{r} is just the divergence of \mathbf{F} at \mathbf{r} :

$$\operatorname{div} \mathbf{F} = \lim_{\Delta V \rightarrow 0} \frac{1}{\Delta V} \iint_S \mathbf{F} \cdot \mathbf{n} \, dS. \quad (6.1)$$

Some authors simply *define* $\operatorname{div} \mathbf{F}$ to be what I've called the flux density. While this approach has some conceptual advantages, it also complicates the exposition.

The Divergence Theorem. The Divergence Theorem, also called Gauss' Theorem, relates a triple integral over the interior of a solid to an integral over the surface of that solid.

Theorem 6.1 (The Divergence Theorem). Let V be a solid in \mathbb{R}^3 bound by an orientable closed surface S , and let \mathbf{n} be the outwardly directed unit vector on S . If \mathbf{F} is a continuously differentiable vector field defined on V , then

$$\iiint_V (\operatorname{div} \mathbf{F}) dV = \iint_S \mathbf{F} \cdot \mathbf{n} dS. \quad (6.2)$$

For a proof of the Divergence Theorem, see Apostol, pp. 457-459. Given that we may interpret $\operatorname{div} \mathbf{F}$ as a “flux density,” we see that eq. (6.2) says just that the flux of \mathbf{F} through the surface of a solid V is the integral of the flux density of \mathbf{F} over the interior of V .

Exercise. Prove the following proposition. Let V be a solid in \mathbb{R}^3 bound by an orientable closed surface S , and let \mathbf{n} be the outwardly directed unit vector on S . If \mathbf{F} is a continuously differentiable vector field defined on V , then

$$\iint_S (\operatorname{curl} \mathbf{F}) \cdot \mathbf{n} dS = 0. \quad (6.3)$$

Exercise. Use the Divergence Theorem to prove eq. (6.1).

We can gain some insight into the Divergence Theorem if we combine eq. (6.1) with what I call the “shared surface” theorem¹. Let V be a solid bounded by a surface S . Suppose we divide V into two solids V_1 and V_2 by inserting a surface S_{12} , which becomes part of the surface of both V_1 and V_2 . We'll say that S_{12} is a “shared surface.” Let S_1 denote the part of S that still bounds V_1 , and let S_2 denote the part of S that still bounds V_2 . Hence, $V = V_1 \cup V_2$, $S = S_1 \cup S_2$, the boundary surface of V_1 is $S_1 \cup S_{12}$, and the boundary surface of V_2 is $S_2 \cup S_{12}$. For example, let V be the rectangular parallelepiped

$$V \equiv \{(x, y, z): 0 \leq x \leq 2, 0 \leq y \leq 1, 0 \leq z \leq 1\}$$

and insert the square surface

$$S_{12} \equiv \{(x, y, z): x = 1, 0 \leq y \leq 1, 0 \leq z \leq 1\}.$$

Then V_1 and V_2 are cubes:

$$V_1 \equiv \{(x, y, z): 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1\}$$

¹ As the Divergence Theorem is used to prove eq. (6.1), this analysis may seem more than a little ass-backwards. Point taken! But this analysis has heuristic utility as it increases our insight into why the Divergence Theorem is true.

and

$$V_2 \equiv \{(x, y, z): 1 \leq x \leq 2, 0 \leq y \leq 1, 0 \leq z \leq 1\}$$

that share the face S_{12} . Returning to the general case, if \mathbf{F} is a vector field defined over V , then the flux of \mathbf{F} out of V_1 may be written

$$\iint_{S_1 \cup S_{12}} \mathbf{F} \cdot \mathbf{n}_1 dS = \iint_{S_1} \mathbf{F} \cdot \mathbf{n}_1 dS + \iint_{S_{12}} \mathbf{F} \cdot \mathbf{n}_1 dS$$

where \mathbf{n}_1 denotes an outwardly directed unit normal vector for V_1 . Similarly, the flux of \mathbf{F} out of V_2 may be written

$$\iint_{S_2 \cup S_{12}} \mathbf{F} \cdot \mathbf{n}_2 dS = \iint_{S_2} \mathbf{F} \cdot \mathbf{n}_2 dS + \iint_{S_{12}} \mathbf{F} \cdot \mathbf{n}_2 dS$$

where \mathbf{n}_2 denotes an outwardly directed unit normal vector for V_2 . As \mathbf{n}_1 and \mathbf{n}_2 are outward normal vectors to V_1 and V_2 , respectively, it follows that $\mathbf{n}_2 = -\mathbf{n}_1$ on S_{12} . Hence, the flux out of V_2 through the shared face S_{12} is just the negative of the flux out of V_1 through S_{12} . In symbols,

$$\iint_{S_{12}} \mathbf{F} \cdot \mathbf{n}_2 dS = - \iint_{S_{12}} \mathbf{F} \cdot \mathbf{n}_1 dS.$$

Hence the sum of the fluxes out of the two solids V_1 and V_2 is given by

$$\iint_{S_1} \mathbf{F} \cdot \mathbf{n}_1 dS + \iint_{S_2} \mathbf{F} \cdot \mathbf{n}_2 dS. \quad (6.4)$$

The flux of \mathbf{F} out of the whole solid V is given by

$$\iint_S \mathbf{F} \cdot \mathbf{n} dS$$

where \mathbf{n} is a unit normal vector on S . But this flux can be rewritten as

$$\iint_S \mathbf{F} \cdot \mathbf{n} dS = \iint_{S_1} \mathbf{F} \cdot \mathbf{n}_1 dS + \iint_{S_2} \mathbf{F} \cdot \mathbf{n}_2 dS \quad (6.5)$$

because $\mathbf{n} = \mathbf{n}_1$ on S_1 and $\mathbf{n} = \mathbf{n}_2$ on S_2 . Comparing eqs. (6.4) and (6.5), we conclude: the flux of \mathbf{F} out of the whole solid V is equal to the sum of the fluxes out of the two component solids V_1 and V_2 , and this is true because the fluxes from V_1 and V_2 across the shared surface cancel.

This conclusion holds if the original solid V is partitioned into any number of component solids V_1, V_2, \dots, V_N . **The shared surface theorem.** Suppose a mathematical solid V with

surface S is partitioned into any number of component solids V_1, V_2, \dots, V_N with surfaces S_1, S_2, \dots, S_N . If \mathbf{F} is a vector field defined on V , then

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \sum_{\ell=1}^N \iint_{S_\ell} \mathbf{F} \cdot \mathbf{n}_\ell \, dS \quad (6.6)$$

where \mathbf{n} is a unit normal vector on S and \mathbf{n}_ℓ is a unit normal vector on S_ℓ for $\ell = 1, \dots, N$. In words, the flux of \mathbf{F} out of the original solid V is equal to the sum of the fluxes out of the N component solids. This conclusion follows from the fact that partitioning V into component subsolids creates internal shared surfaces, and all the fluxes across shared surfaces cancel.

We may combine eqs. (6.1) and (6.6) to gain some insight into the Divergence Theorem. Let V be a solid in \mathbb{R}^3 bound by an orientable closed surface S , let \mathbf{n} be the outwardly directed unit vector on S , and let \mathbf{F} be a continuously differentiable vector field defined on V . The expression on the right-hand side of eq. (6.2)

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS$$

is the flux of \mathbf{F} out of V . We now partition V into a large number N of component “subsolids” V_1, V_2, \dots, V_N with surfaces S_1, S_2, \dots, S_N . From the shared surface theorem,

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \sum_{\ell=1}^N \iint_{S_\ell} \mathbf{F} \cdot \mathbf{n}_\ell \, dS \quad (6.6)$$

where the terms in this equation are explained above. Let ΔV_ℓ denote the volume of V_ℓ for $\ell = 1, 2, \dots, N$. If ΔV_ℓ is small enough, it follows from eq. (6.1) that

$$\iint_{S_\ell} \mathbf{F} \cdot \mathbf{n}_\ell \, dS \approx (\operatorname{div} \mathbf{F}(\mathbf{r}_\ell)) \Delta V_\ell \quad (6.7)$$

where \mathbf{r}_ℓ is any point in V_ℓ . Combining eqs. (6.6) and (6.7), we find

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS \approx \sum_{\ell=1}^N (\operatorname{div} \mathbf{F}(\mathbf{r}_\ell)) \Delta V_\ell. \quad (6.8)$$

This approximation becomes an equality if we let $N \rightarrow \infty$ and $\max\{\Delta V_\ell\} \rightarrow 0$. But the sum on the right-hand side of eq. (6.8) is just a Riemann sum for $\iiint_V (\operatorname{div} \mathbf{F}) \, dV$. This completes our heuristic “proof” of the Divergence Theorem.

7. Stokes' Theorem.

Let \mathbf{F} be a vector field defined on $D \subseteq \mathbb{R}^3$. In the previous section we used the concept of a “flux density” to give a geometric meaning to $\operatorname{div} \mathbf{F}$. In this section we'll introduce the concept of “circulation density” to give a geometric meaning to $\operatorname{curl} \mathbf{F}$. To be precise, let $\mathbf{r} \equiv (x, y, z)$ be a point in D , let \mathbf{n} be a unit vector in \mathbb{R}^3 , let $\Pi(\mathbf{r}, \mathbf{n})$ be the plane through \mathbf{r} that is normal to \mathbf{n} , let C be a piecewise smooth closed path in $\Pi(\mathbf{r}, \mathbf{n})$ that encloses \mathbf{r} , and let ΔS denote the area of the region enclosed by C . Now consider the “circulation integral”

$$I \equiv \oint_C \mathbf{F} \cdot \mathbf{T} \, ds. \quad (7.1)$$

By convention, in calculating I the path C is traversed in a counterclockwise direction as viewed from the tip of \mathbf{n} when \mathbf{n} is based at \mathbf{r} . The circulation I given by eq. (7.1) is a *scalar* defined as an integral over the *whole path* C , whereas the curl of \mathbf{F} at \mathbf{r} is a *vector* defined at the *point* \mathbf{r} alone. What can I possibly tell us about $\nabla \times \mathbf{F}(\mathbf{r})$? Our strategy will be to examine the limiting behavior of I as we let the curve C contract down to the point \mathbf{r} . The circulation I necessarily decreases to zero as ΔS goes to zero, but the limit of the ratio of I to ΔS , the circulation per unit area, is more interesting. We'll define the “circulation density of \mathbf{F} around \mathbf{n} at \mathbf{r} ” to be

$$\gamma(\mathbf{r}, \mathbf{n}) \equiv \lim_{\Delta S \rightarrow 0} \frac{1}{\Delta S} \oint_C \mathbf{F} \cdot \mathbf{T} \, ds.$$

An astute reader might object that this limit apparently depends on how the family of curves C that contract down to \mathbf{r} are chosen. As it turns out, this isn't the case; it may be shown that the limit of $I/\Delta S$ as $\Delta S \rightarrow 0$ does not depend on how the curves C are chosen.

To appreciate the utility of $\gamma(\mathbf{r}, \mathbf{n})$, it's best to see some examples. First, let $\mathbf{n} = \mathbf{k}$, so $\Pi(\mathbf{r}, \mathbf{n})$ is parallel to the xy -plane. Let C be the rectangle with base Δx , height Δy , and centered around \mathbf{r} in the plane $\Pi(\mathbf{r}, \mathbf{k})$. An instructive and easy calculation shows that

$$\gamma(\mathbf{r}, \mathbf{k}) = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}.$$

From eq. (3.2), this is the \mathbf{k} component of $\operatorname{curl} \mathbf{F}$ at \mathbf{r} . Hence,

$$[\nabla \times \mathbf{F}(\mathbf{r})] \cdot \mathbf{k} = \gamma(\mathbf{r}, \mathbf{k}).$$

Similar calculations with planes parallel to the xz -plane and the yz -plane yield

$$[\nabla \times \mathbf{F}(\mathbf{r})] \cdot \mathbf{i} = \gamma(\mathbf{r}, \mathbf{i}) \quad \text{and} \quad [\nabla \times \mathbf{F}(\mathbf{r})] \cdot \mathbf{j} = \gamma(\mathbf{r}, \mathbf{j}).$$

These results suggest (but don't exactly prove) the following: for any unit vector \mathbf{n} ,

$$[\nabla \times \mathbf{F}(\mathbf{r})] \cdot \mathbf{n} = \gamma(\mathbf{r}, \mathbf{n}). \quad (7.2)$$

In words, $\nabla \times \mathbf{F}(\mathbf{r})$ is a vector whose component in the direction of \mathbf{n} is equal to $\gamma(\mathbf{r}, \mathbf{n})$, the circulation density of \mathbf{F} at \mathbf{r} in the plane $\Pi(\mathbf{r}, \mathbf{n})$.

Jordan curves. A path Γ in \mathbb{R}^n is specified parametrically by a continuous vector valued function $\gamma: [a, b] \rightarrow \mathbb{R}^n$. If $\gamma(a) = \gamma(b)$, the path is *closed*. If Γ is closed and $\gamma(t_1) \neq \gamma(t_2)$ for every $t_1 \neq t_2$ in $(a, b]$, then Γ is said to be a *simple* closed curve. Geometrically, a simple closed curve doesn't intersect itself. A simple closed curve in a plane is called a *Jordan curve*. Every Jordan curve Γ partitions the plane into two disjoint open connected sets having Γ as their common boundary. One of these sets is bounded and is called the *interior* of Γ . The other is unbounded and is called the *exterior* of Γ .

“Counterclockwise” traversal. Let Γ be a Jordan curve in the xy -plane, and let R denote the interior of Γ . We need to define (somewhat informally) what it means to traverse Γ in a “counterclockwise” direction. First, we define “upright” to mean: in the direction of *positive* values of z . **Definition:** an upright pedestrian walking on Γ is moving in a *counterclockwise* direction if R is on his or her *left*.

Green's Theorem (for a plane region bounded by a piecewise smooth Jordan curve). Let P and Q be scalar fields that are continuously differentiable on an open set S in the xy -plane. Let C be a piecewise smooth Jordan curve, and let R denote the union of C and its interior. Assume that $R \subseteq S$. Then the following equation is true:

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \oint_C P dx + Q dy \quad (7.3)$$

where the line integral is taken around C in the counterclockwise direction.

Stokes' Theorem is a direct generalization of Green's Theorem. Let S be a surface in \mathbb{R}^3 bounded by a curve C , and let \mathbf{F} be a vector field defined on S . Stokes' Theorem states that the circulation of \mathbf{F} around C is equal to the surface integral of $(\text{curl } \mathbf{F}) \cdot \mathbf{n}$ over S , where \mathbf{n} is a suitably chosen unit normal vector at each point of S .

Stokes' Theorem. Let S be a smooth simple parametric surface, say $S = \mathbf{r}(T)$, where T is a region in the uv -plane bounded by a piecewise smooth Jordan curve Γ . Assume also that \mathbf{r} is a one-to-one mapping whose components have continuous second-order partial derivatives on some open set containing $T \cup \Gamma$. Let C denote the image of Γ under \mathbf{r} , and let \mathbf{F} be a continuously differentiable vector field defined on S . Then

$$\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n} dS = \oint_C \mathbf{F} \cdot \mathbf{T} ds \quad (7.4)$$

where \mathbf{n} is the unit normal vector defined by eq. (5.6), and the path Γ is traversed in the counterclockwise direction when the line integral is evaluated.

This statement of Stokes' theorem is taken from Apostol, where a proof may be found.

Remark 1. This statement of Stokes' theorem makes explicit use of the parameterization $S = \mathbf{r}(T)$ and the parameterization of Γ . As noted previously, the value of a surface

integral doesn't depend on how the surface is parameterized, and the value of a line integral doesn't depend on parameterization up to a sign. Hence eq. (7.4) is true almost regardless of how S and Γ are parameterized. It might seem preferable, therefore, to state Stokes' theorem in a way that makes no explicit reference to a particular parameterization. Significant complications arise, however, if one attempts to rephrase Stokes' theorem without making explicit use of these parameterizations. In particular, eq. (5.6) gives us a convenient way to ensure that the normal unit vectors \mathbf{n} are all on the same "side" of S , and it's difficult to see how this condition could be guaranteed without using the parameterization. Also, it's much easier to define "counterclockwise" for the Jordan curve Γ in the uv -plane than for the closed path C in \mathbb{R}^3 .

Remark 2. Stokes' theorem reduces to Green's theorem if S is a region in the xy -plane. To see this, write

$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}.$$

If S is a region in the xy -plane, then $\mathbf{n} = \mathbf{k}$ everywhere on S , and hence

$$(\text{curl } \mathbf{F}) \cdot \mathbf{n} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}.$$

Also, the closed curve C lies entirely in the xy -plane, so the line integral in eq. (7.4) becomes

$$\oint_C \mathbf{F} \cdot \mathbf{T} ds = \int_C P dx + Q dy.$$

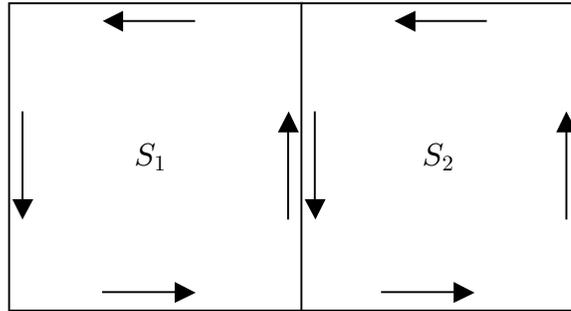
Remark 3. The surface S is said to be a "capping surface" of the closed curve C . For any given closed curve C , there are an infinite number of capping surfaces. Some are as tight and "minimal energy" as a soap film on a wire frame. Others billow out to Betelgeuse or beyond. To me, the most amazing thing about Stokes' theorem is that it says that the value of the surface integral

$$\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n} dS$$

is *invariant* over all surfaces S that cap C , so long as \mathbf{F} is defined and continuously differentiable on S . Now imagine a *large* capping surface on a *small* closed path C . If we let C shrink down to a point, the circulation $\oint_C \mathbf{F} \cdot \mathbf{T} ds$ necessarily decreases to zero. This gives us another way to prove eq. (6.3). (I learned of this method from Feynman.)

Remark 4. Equation (7.2) tells us that $(\text{curl } \mathbf{F}) \cdot \mathbf{n}$ may be interpreted as a "circulation density." On the other hand, we recognize that the line integral in Stokes' theorem is the circulation of \mathbf{F} around C . Hence, Stokes' theorem tells us that the circulation of \mathbf{F} around C is equal to the integral of circulation density over any surface that caps C .

There are several ways to extend the conclusion of Stokes' theorem (eq. (7.4)) to more general surfaces than are stated in the hypotheses of the theorem. In particular, one may knit several surfaces together along their edges, so long as the directions of integration along any edge shared by two surfaces is opposite. This is easiest to explain by an illustration. Consider the two rectangles S_1 and S_2 with a common edge shown below.



Let C_1 and C_2 denote the borders of S_1 and S_2 , respectively, let $C_{12} = C_1 \cap C_2$ denote the shared edge, let $S \equiv S_1 \cup S_2$, and let $C = (C_1 \cup C_2) - C_{12}$ denote the border of S . We want to compute the sum of the circulations of \mathbf{F} around C_1 and C_2 , i.e.,

$$\oint_{C_1} \mathbf{F} \cdot \mathbf{T} ds + \oint_{C_2} \mathbf{F} \cdot \mathbf{T} ds, \quad (7.5)$$

where the direction of integration is counterclockwise (as indicated by the arrows shown in the figure), and compare this sum to the circulation of \mathbf{F} around C , i.e.,

$$\oint_C \mathbf{F} \cdot \mathbf{T} ds,$$

also integrated in a counterclockwise direction. Now consider the contributions to $\oint_{C_1} \mathbf{F} \cdot \mathbf{T} ds$ and $\oint_{C_2} \mathbf{F} \cdot \mathbf{T} ds$ attributable to integration along the shared edge C_{12} . Because $\oint_{C_1} \mathbf{F} \cdot \mathbf{T} ds$ and $\oint_{C_2} \mathbf{F} \cdot \mathbf{T} ds$ are integrated in opposite directions along the shared edge, we see that their contributions to the sum (7.5) just cancel (see Theorem 4.1), so

$$\oint_C \mathbf{F} \cdot \mathbf{T} ds = \oint_{C_1} \mathbf{F} \cdot \mathbf{T} ds + \oint_{C_2} \mathbf{F} \cdot \mathbf{T} ds. \quad (7.6)$$

Now, Stokes' theorem applies to both S_1 and S_2 :

$$\oint_{C_1} \mathbf{F} \cdot \mathbf{T} ds = \iint_{S_1} (\text{curl } \mathbf{F}) \cdot \mathbf{n}_1 dS \quad (7.7)$$

and

$$\oint_{C_2} \mathbf{F} \cdot \mathbf{T} \, ds = \iint_{S_2} (\text{curl } \mathbf{F}) \cdot \mathbf{n}_2 \, dS \quad (7.8)$$

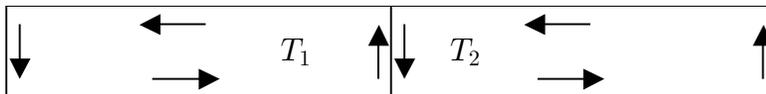
where \mathbf{n}_1 and \mathbf{n}_2 are normal unit vectors to S_1 and S_2 , respectively. If we now define \mathbf{n} to be \mathbf{n}_1 on S_1 and \mathbf{n}_2 on S_2 , we see that

$$\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n} \, dS = \iint_{S_1} (\text{curl } \mathbf{F}) \cdot \mathbf{n}_1 \, dS + \iint_{S_2} (\text{curl } \mathbf{F}) \cdot \mathbf{n}_2 \, dS. \quad (7.9)$$

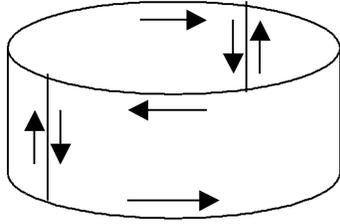
Combining eqs. (7.6) through (7.9), we see that eq. (7.4) holds for the composite surface S .

This kind of argument may be extended to any kind of surface that may be construed as the union of simpler surfaces knit together along part of their edges. The only requirement for this argument to go through is that it be possible for “counterclockwise” to be defined for each subsurface in such a way that the direction of integration along any arc that is a shared edge will be opposite. For example, surfaces with “holes” can be treated by introducing “cross-cuts.” A picture is worth a thousand words here, and I advise the reader to consult almost any text on advanced calculus.

We can knit together surfaces in more complicated ways. Consider the two rectangles T_1 and T_2 in uv -space shown below.



Let Γ_1 and Γ_2 denote the boundaries of T_1 and T_2 , respectively, and let $T \equiv T_1 \cup T_2$. Suppose that the image $\mathbf{r}(T)$ in \mathbb{R}^3 is the *cylinder* shown below where (in effect) the long strip T has been bent around until the image of the left edge of T_1 has been brought into coincidence with the image of the right edge of T_2 . The images $\mathbf{r}(\Gamma_1)$ and $\mathbf{r}(\Gamma_2)$ coincide on two arcs: the image $\mathbf{r}(\Gamma_1 \cap \Gamma_2)$ of the short vertical line where T_1 and T_2 join, and the common image under \mathbf{r} of the left edge of T_1 and the right edge of T_2 . An argument similar to that given above shows that Stokes' equation applies to this cylinder, where the total circulation is the sum of the line integrals taken over the upper and lower rims of the cylinder, and in the directions indicated in the diagram.



The argument works because the line integrals along arcs that are common to two regions are always in *opposite* directions, so they cancel. In summary, the sum of the line integrals over the two component surfaces is just equal to the line integral over the *exterior* edge (or edges) of the amalgamated surface because the contributions to line integrals over interior (and therefore shared) arcs just sum to zero.

Now suppose that the mapping \mathbf{r} gives, in effect, the strip T a half twist before the image of the left edge of T_1 and the image of the right edge of T_2 are brought into coincidence. The image $\mathbf{r}(T)$ is called a Möbius band. Stokes' equation fails to hold in this case because the direction of integration of the two line integrals is necessarily in the *same* direction along some arc that is common to $\mathbf{r}(T_1)$ and $\mathbf{r}(T_2)$. The Möbius band is an example of a *nonorientable* surface.

8. Some concluding remarks.

Remark 1. Green's theorem, Stokes' theorem, and the divergence theorem are all extensions of the second fundamental theorem of calculus. Each of these theorems states that the integral of some function over a “region” of \mathbb{R}^3 is equal to the integral of a related function over the boundary of that region. For Green's theorem and Stokes' theorem, the region is a surface and the boundary is a closed curve. For the divergence theorem, the region is a mathematical solid and the boundary is a closed surface.

Remark 2. The divergence (eq. (3.1)) and curl (eq. (3.2)) were defined for a vector field \mathbf{F} that's defined on a subset of \mathbb{R}^3 . That's adequate for electromagnetism, the subject for which these tools were essentially invented. However, the dot product is naturally extended to \mathbb{R}^n (see eq. (1.1)), and it's natural to extend the definition of divergence to \mathbb{R}^n . If \mathbf{F} is a vector field defined on a subset of \mathbb{R}^n , say

$$\mathbf{F}(\mathbf{r}) \equiv (F_1(\mathbf{r}), F_2(\mathbf{r}), \dots, F_n(\mathbf{r})) \quad \text{where} \quad \mathbf{r} \equiv (x_1, x_2, \dots, x_n)$$

then

$$\operatorname{div} \mathbf{F}(\mathbf{r}) \equiv \nabla \cdot \mathbf{F}(\mathbf{r}) \equiv \sum_{i=1}^n \frac{\partial F_i}{\partial x_i} \quad (8.1)$$

where all the partial derivatives are evaluated at \mathbf{r} . This plays an important role in (for example) the kinetic theory of gases.

Similarly, the gradient is naturally defined on \mathbb{R}^n , and plays an important role in many fields, including economics.

It follows that the “Laplacian” operator ∇^2 , defined as the divergence of a gradient, is naturally defined on \mathbb{R}^n : for any scalar field $\varphi(x_1, x_2, \dots, x_n)$,

$$\nabla^2 \varphi \equiv \operatorname{div}(\operatorname{grad} \varphi) = \sum_{i=1}^n \frac{\partial^2 \varphi}{\partial x_i^2}. \quad (8.2)$$

On the other hand, both the cross product (eq. (1.3)) and the curl are meaningful constructs only in \mathbb{R}^3 , so far as I can tell.

Remark 3: Some commuting and some non-commuting operators. The Laplacian operator ∇^2 is defined as the *divergence* of a *gradient*. If $\mathbf{F} \equiv (F_x, F_y, F_z)$ is a vector field in \mathbb{R}^3 , the Laplacian of \mathbf{F} is defined “component-wise”:

$$\nabla^2 \mathbf{F} \equiv (\nabla^2 F_x, \nabla^2 F_y, \nabla^2 F_z). \quad (8.3)$$

For any vector field \mathbf{F} in \mathbb{R}^3 , the *gradient* of the *divergence* of \mathbf{F}

$$\operatorname{grad}(\operatorname{div} \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F})$$

is a meaningful vector field. It occurs, for example, in the formula for the curl of a curl:

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}. \quad (8.4)$$

The unwary student might naively assume that $\operatorname{grad}(\operatorname{div}())$ is equal to $\operatorname{div}(\operatorname{grad}())$. This would be a gross error! Among other differences, $\operatorname{grad}(\operatorname{div}())$ is a vector, whereas $\operatorname{div}(\operatorname{grad}())$ is a scalar. In words, the operators “grad” and “div” *do not* commute.

On the other hand, consider the two operators “curl” and “Laplacian.” For any vector field \mathbf{F} defined on a subset of \mathbb{R}^3 , the following formula

$$\nabla \times (\nabla^2 \mathbf{F}) = \nabla^2(\nabla \times \mathbf{F}) \quad (8.5)$$

is true. In words, “curl” and “Laplacian” *do* commute. Feynman passes eq. (8.5) off with the casual remark “[s]ince the Laplacian is a scalar operator, the order of the Laplacian and curl operations can be interchanged.” I don’t buy this; so far as I can see, eq. (8.5) needs a proof. The work is grungy but straightforward, and it all works out in the end.

Remark 4: spherical coordinates. The gradient, curl, and divergence were defined in terms of derivatives with respect to x , y , and z , the coordinates of a point relative to the standard coordinate system of \mathbb{R}^3 . For example, if $\mathbf{F} \equiv (F_x, F_y, F_z)$ is a vector field in \mathbb{R}^3 , then

$$\operatorname{div} \mathbf{F} \equiv \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}.$$

We gave geometric interpretations of the gradient, curl, and divergence that showed that these operations have physical meanings that are independent of the coordinate system used to locate points in \mathbb{R}^3 . For some problems, it's useful to express grad, div, and curl in terms of derivatives relative to alternative coordinate systems. In particular, one can find such expressions relative to cylindrical and spherical coordinates. A full discussion may be found in Schey. Here I'll just state the formulas for $\nabla^2 f$ in spherical coordinates. I need this formula for the discussion of “spherical waves” in the appendix.

A point \mathbf{r} in \mathbb{R}^3 is located in spherical coordinates by a triple of numbers (r, ϕ, θ) where $r \geq 0$ is a distance, and ϕ and θ are angles. Specifically, $r = \|\mathbf{r}\|$ is the distance from the origin to \mathbf{r} , θ is the angle between \mathbf{r} and the z -axis, and ϕ is the angle between the x -axis and the projection of \mathbf{r} onto the xy -plane. The angle θ corresponds to “latitude” in geography, except that θ is measured from the north pole rather than the equator. The angle ϕ corresponds to “longitude,” with the x -axis essentially in the role of “prime meridian,” except that ϕ is only measured in an “eastward” direction (i.e., counterclockwise as seen from the North Pole.) With spherical coordinates, a scalar field is expressed as a function of r , ϕ , and θ .

With these conventions, the following may be shown. Let $f(r, \phi, \theta)$ denote a scalar field. Then

$$\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2}. \quad (8.6)$$

Although this formula is impressively complicated, in a problem with spherical symmetry it quickly reduces to a much simpler expression.

References

My primary sources for this document are as follows.

Apostol, Tom. *Calculus*, Vol. II (2nd ed.). Xerox College Publishing, 1969.

Feynman, Richard, Robert Leighton, and Matthew Sands. *The Feynman Lectures on Physics: The Electromagnetic Field* (Vol. II). Addison-Wesley, 1964.

Schey, Harry Moritz. *div, grad, curl, and all that: an informal text on vector calculus* (2nd ed.). W. W. Norton & Company, 1992.

With regard to mathematics, Apostol is the most rigorous of these authors in his presentation, Feynman is the most relaxed, and Schey is intermediate in rigor. Apostol is always careful to state precisely the conditions under which a given result is true, and always gives rigorous proofs. While this is all good in a mathematical reference book, it makes his text rather difficult reading for undergraduates. In contrast, Feynman is quite cavalier in his statement of theorems. Schey is careful in his statements of theorems, but his proofs are generally heuristic and informal instead of being rigorous. This document is closest in rigor to Schey.

With regard to physics, my primary source was Feynman (of course), but I found useful material in the other two books. I also extracted some physics from the following text.

Stauffer, Dietrich, and H. Eugene Stanley. *From Newton to Mandelbrot: A Primer in Theoretical Physics*. Springer-Verlag, 1991

Appendix: Supplementary Material.

1. Linearity of grad, div, and curl. The gradient, divergence, and curl are all *linear* operators. That is, if a and b are scalars, φ and ψ are scalar fields on \mathbb{R}^3 , and \mathbf{F} and \mathbf{G} are vector fields on \mathbb{R}^3 , then

$$\nabla(a\varphi + b\psi) = a\nabla\varphi + b\nabla\psi, \quad (\text{A1.1})$$

$$\nabla \cdot (a\mathbf{F} + b\mathbf{G}) = a(\nabla \cdot \mathbf{F}) + b(\nabla \cdot \mathbf{G}), \quad (\text{A1.2})$$

$$\nabla \times (a\mathbf{F} + b\mathbf{G}) = a(\nabla \times \mathbf{F}) + b(\nabla \times \mathbf{G}). \quad (\text{A1.3})$$

2. Product Differentiation Formulas. The following identities are all generalizations of the rule in elementary calculus for differentiating the product of two functions. Let φ and ψ be scalar fields on \mathbb{R}^3 , and let \mathbf{F} be a vector field on \mathbb{R}^3 . Then

$$\nabla(\varphi\psi) = \varphi\nabla\psi + \psi\nabla\varphi, \quad (\text{A2.1})$$

$$\nabla \cdot (\varphi\mathbf{F}) = \varphi(\nabla \cdot \mathbf{F}) + \mathbf{F} \cdot \nabla\varphi, \quad (\text{A2.2})$$

$$\nabla \times (\varphi\mathbf{F}) = \varphi(\nabla \times \mathbf{F}) + \nabla\varphi \times \mathbf{F}. \quad (\text{A2.3})$$

3. “Irrotational” and “Solenoidal” Vector Fields. If \mathbf{F} is a vector field in \mathbb{R}^3 and $\text{curl } \mathbf{F} = \mathbf{0}$ on some set $D \subseteq \mathbb{R}^3$, then \mathbf{F} is said to be *irrotational*. We know that if $\mathbf{F} = \nabla\varphi$ for some scalar field φ with continuous second-order mixed partial derivatives, then \mathbf{F} is irrotational. Conversely, it's known that if \mathbf{F} is irrotational at all points in an open convex set D , then there exists a scalar field φ defined on D such that $\mathbf{F} = \nabla\varphi$.

If \mathbf{F} is a vector field in \mathbb{R}^3 and $\text{div } \mathbf{F} = 0$ on some set $D \subseteq \mathbb{R}^3$, then \mathbf{F} is said to be *solenoidal*. We know that if all the mixed partial derivatives of a vector field \mathbf{G} are continuous, then $\mathbf{F} \equiv \text{curl } \mathbf{G}$ is solenoidal. Conversely, if \mathbf{F} is solenoidal everywhere in some open interval D , then there exists a vector field \mathbf{G} defined on D such that $\mathbf{F} = \text{curl } \mathbf{G}$.

Suppose that \mathbf{F} is a continuously differentiable vector field defined on an open interval D in \mathbb{R}^3 . It's known that every such vector field may be written in the form $\mathbf{F} = \mathbf{C} + \mathbf{G}$ where \mathbf{C} is solenoidal and \mathbf{G} is irrotational [Apostol, p. 452]. As \mathbf{C} is solenoidal, it follows that $\mathbf{C} = \text{curl } \mathbf{H}$ for some vector field \mathbf{H} . Similarly, as \mathbf{G} is irrotational, it follows that $\mathbf{G} = \nabla\varphi$ for some scalar field φ . Hence, we can write

$$\mathbf{F} = \text{curl } \mathbf{H} + \nabla\varphi. \quad (\text{A3.1})$$

To find \mathbf{H} and φ given \mathbf{F} , we take the curl and divergence of each side of eq. (A3.1) and

make use of the linearity of curl and divergence. This yields the following partial differential equations for φ and \mathbf{H} :

$$\nabla^2 \varphi = \operatorname{div} \mathbf{F}$$

and

$$\operatorname{curl}(\operatorname{curl} \mathbf{H}) = \nabla(\nabla \cdot \mathbf{H}) - \nabla^2 \mathbf{H} = \operatorname{curl} \mathbf{F}.$$

- 4. “Central” Vector Fields.** We previously defined “central” force fields. We now wish to extend this vocabulary to general vector fields. A vector field \mathbf{F} defined on \mathbb{R}^n is said to be *central* if it can be written in the form $\mathbf{F}(\mathbf{r}) = g(r)\mathbf{r}$ where $r \equiv \|\mathbf{r}\|$. The purpose of this section is to record some of the properties of central vector fields.

We previously showed that every central force field is conservative; that is, if \mathbf{F} is a central force field defined on some set $D \subseteq \mathbb{R}^3$, then $\mathbf{F} = -\nabla U$ for some potential function U . This result clearly isn't restricted just to force fields: if \mathbf{F} is a central vector field defined on $D \subseteq \mathbb{R}^n$, then $\mathbf{F} = \nabla \varphi$ for some scalar field φ defined on D . As a corollary, we see that every central vector field is irrotational.

We next want to find $\nabla g(\|\mathbf{r}\|)$ for an arbitrary function $g(\cdot)$. Let $\mathbf{r} = (x_1, x_2, \dots, x_n)$ and let $r = \|\mathbf{r}\|$. Recall eq. (2.2):

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r} \quad \text{for } i = 1, \dots, n. \quad (2.2)$$

Hence

$$\frac{\partial g}{\partial x_i} = g'(r) \frac{\partial r}{\partial x_i} = g'(r) \frac{x_i}{r}.$$

It follows that

$$\nabla g(r) = \frac{g'(r)}{r} \mathbf{r}. \quad (\text{A4.1})$$

We may use this result to find the divergence of a central vector field. Suppose that $\mathbf{F} = g(r)\mathbf{r}$ is a central vector field in \mathbb{R}^3 . From eq. (A2.2), Example 1 of Section 3, and eq. (A4.1), we find

$$\operatorname{div} \mathbf{F} = g(r)(\operatorname{div} \mathbf{r}) + \mathbf{r} \cdot \nabla g(r) = 3g(r) + rg'(r). \quad (\text{A4.2})$$

In particular, consider the central vector field given by $\mathbf{F} = r^L \mathbf{r}$ for some constant L . Then

$$\operatorname{div} \mathbf{F} = 3r^L + r(Lr^{L-1}) = (3 + L)r^L. \quad (\text{A4.3})$$

If $L \geq 0$ this formula holds for all \mathbf{r} . If $L < 0$ this formula holds for all $\mathbf{r} \neq \mathbf{0}$. In the

interesting case that $L = -3$, we have

$$\operatorname{div} \mathbf{F} = 0 \quad \text{unless} \quad \mathbf{r} = \mathbf{0}. \quad (\text{A4.4})$$

Hence, the vector field $\mathbf{F} = r^{-3}\mathbf{r}$ is irrotational everywhere in \mathbb{R}^3 , and is solenoidal everywhere except at $\mathbf{r} = \mathbf{0}$. We'll use this result in the next section.

5. Inverse Square Laws. If the magnitude of a central force field \mathbf{F} at a point $\mathbf{r} \in \mathbb{R}^3$ is inversely proportional to $\|\mathbf{r}\|^2$ (i.e., inversely proportional to the square of the distance from the origin), then the force field is said to obey an “inverse square law.” Let $F = \|\mathbf{F}\|$ and let $r = \|\mathbf{r}\|$. The best known examples of inverse square laws are Newton's law of gravitation

$$F = \frac{GMm}{r^2}$$

(which gives the gravitational force between a point mass of M and a point mass of m) and Coulomb's law

$$F = \frac{1}{4\pi\epsilon_0} \frac{|qq_0|}{r^2}$$

which gives the magnitude of the electrical force acting between stationary charges q and q_0 . (G and ϵ_0 are constants.) To give the *direction* as well as the *magnitude* of these forces, let $\mathbf{e}_r \equiv \|\mathbf{r}\|^{-1}\mathbf{r}$ be a unit vector in the direction of \mathbf{r} . Then Newton's law may be written

$$\mathbf{F} = -\frac{GMm}{r^2}\mathbf{e}_r = -\frac{GMm}{r^3}\mathbf{r}$$

(where the point mass M is at the origin, the point mass m is at \mathbf{r} , and \mathbf{F} denotes the force acting on the point mass m), and Coulomb's law may be written

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0} \frac{qq_0}{r^2}\mathbf{e}_r = \frac{1}{4\pi\epsilon_0} \frac{qq_0}{r^3}\mathbf{r}$$

(where the charge q_0 is at the origin, the charge q is at \mathbf{r} , and \mathbf{F} denotes the force acting on the charge q).

A force that obeys an inverse square law can be written

$$\mathbf{F} = \pm \frac{F_1}{r^2}\mathbf{e}_r = \pm \frac{F_1}{r^3}\mathbf{r} \quad (\text{A5.1})$$

where F_1 denotes the magnitude of the force at unit distance. The plus sign is used if the force is repulsive, and the negative sign is used if the force is attractive. Note that

$$U(\mathbf{r}) = \pm \frac{F_1}{r} \quad (\text{A5.2})$$

is a potential function for the force given by eq. (A5.1); i.e., $\mathbf{F} = -\nabla U$. Also, note that

eq. (A4.4) applies if \mathbf{F} obeys an inverse square law. I'll only consider repulsive forces in the remainder of this section, but the same results apply to attractive forces with suitable modifications of language.

Let \mathbf{F} satisfy eq. (A5.1) with a plus sign. Let's compute the flux of \mathbf{F} out of the surface of a sphere of radius r centered at the origin. Let $\mathbf{n} \equiv \|\mathbf{r}\|^{-1}\mathbf{r} = \mathbf{e}_r$ denote the unit normal at any point on the surface of the sphere. Hence,

$$\mathbf{F} \cdot \mathbf{n} = \frac{F_1}{r^2} \mathbf{n} \cdot \mathbf{n} = \frac{F_1}{r^2}.$$

The total flux out of this surface is therefore

$$\int_S \frac{F_1}{r^2} dS = \frac{F_1}{r^2} \int_S dS = \frac{F_1}{r^2} \cdot 4\pi r^2 = 4\pi F_1. \quad (\text{A5.3})$$

Note that this flux is *independent* of the radius r . That is, the flux of \mathbf{F} through the surface of *any* sphere centered at the origin equals $4\pi F_1$. Combining this result with the observation that $\text{div } \mathbf{F} = 0$ except at the origin yields the following theorem.

Theorem. Suppose that \mathbf{F} obeys the inverse square law of eq. (A5.1) (with a plus sign). Let V denote a solid that includes the origin \mathbf{O} as an interior point. Then the flux of \mathbf{F} through the surface of V equals $4\pi F_1$.

Proof. As \mathbf{O} is an interior point of V , we can find $\epsilon > 0$ such that a sphere with radius ϵ centered at \mathbf{O} will be entirely contained within V . Let $V(\epsilon)$ denote this sphere, and let $S(\epsilon)$ denote the surface of $V(\epsilon)$. Let S denote the surface of V and define $V' \equiv V - V(\epsilon)$. You may think of V' as being V with a bubble removed. The surface of V' equals $S \cup S(\epsilon)$. We'll say that S is the *exterior* surface of V' and $S(\epsilon)$ is the *interior* surface of V' . As $\text{div } \mathbf{F} = 0$ throughout V' , it follows from the divergence theorem that the total flux of \mathbf{F} out of V' must equal zero. Therefore, the total flux of \mathbf{F} into V' through the interior surface $S(\epsilon)$ must equal the total flux of \mathbf{F} out of V' through the exterior surface S . But the flux of \mathbf{F} into V' through the interior surface $S(\epsilon)$ is just equal to $4\pi F_1$, the flux of \mathbf{F} out of $V(\epsilon)$.

6. Maxwell's Equations. In the following 4 equations, t denotes time, \mathbf{E} , \mathbf{B} , and \mathbf{J} are vector fields in \mathbb{R}^3 , ρ is a scalar field in \mathbb{R}^3 , and c and ϵ_0 are constants. In somewhat more detail,

$$\begin{aligned} \mathbf{E} &= \text{the electric field,} \\ \mathbf{B} &= \text{the magnetic field,} \\ \mathbf{J} &= \text{current density, and} \\ \rho &= \text{charge density.} \end{aligned}$$

Maxwell's equations in *differential form* are as follows:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \quad (\text{A6.1})$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (\text{A6.2})$$

$$c^2(\nabla \times \mathbf{B}) = \frac{\partial \mathbf{E}}{\partial t} + \frac{\mathbf{J}}{\epsilon_0}, \quad (\text{A6.3})$$

$$\nabla \cdot \mathbf{B} = 0. \quad (\text{A6.4})$$

If we apply the divergence theorem to the first and last of these equations, and apply Stokes' theorem to the second and third, we obtain Maxwell's equations in *integrated form*:

- (1) The flux of \mathbf{E} through a closed surface S equals the total charge contained within S divided by ϵ_0 . (Both this statement and eq. (A6.1) are known as “Gauss' law.”)
- (2) The clockwise circulation of \mathbf{E} around a closed loop C is equal to the rate of change of the flux of \mathbf{B} through any surface that caps C . (Both this statement and eq. (A6.2) are known as “Faraday's law.”)
- (3) c^2 times the counterclockwise circulation of \mathbf{B} around any closed loop C equals the rate of change of the flux of \mathbf{E} through any surface S that caps C , plus the total flux of electric current through S divided by ϵ_0 .
- (4) The flux of \mathbf{B} through any closed surface is zero.

7. Electrostatics and Magnetostatics. If the charge density ρ and the current density \mathbf{J} in Maxwell's equations do not depend on time, then the two time derivatives equal zero, and Maxwell's equations reduce to two pairs of equations:

Electrostatics:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \quad (\text{A7.1})$$

$$\nabla \times \mathbf{E} = \mathbf{0}. \quad (\text{A7.2})$$

Magnetostatics:

$$c^2(\nabla \times \mathbf{B}) = \frac{\mathbf{J}}{\epsilon_0}, \quad (\text{A7.3})$$

$$\nabla \cdot \mathbf{B} = 0. \quad (\text{A7.4})$$

Equation (A7.3) is known as “Ampere's law.”

In this static situation, the electric field \mathbf{E} appears in only the first two equations and the magnetic field \mathbf{B} appears in only the second two equations. Hence, if charges and currents are static, then electricity and magnetism are distinct and separate phenomena.

Notice that in electrostatics, the electric field \mathbf{E} is irrotational. Hence, there exists a scalar field Φ such that

$$\mathbf{E} = -\nabla\Phi. \quad (\text{A7.5})$$

The scalar field Φ is called the *electrostatic potential*. By substituting eq. (A7.5) into eq. (A7.1), we see that Φ satisfies *Poisson's equation*:

$$\nabla^2\Phi = -\frac{\rho}{\epsilon_0}. \quad (\text{A7.6})$$

The specialization of Poisson's equation obtained when $\rho = 0$, i.e.,

$$\nabla^2\Phi = 0, \quad (\text{A7.7})$$

is called *Laplace's equation*.

Before turning to the subject of magnetostatics, let's examine the electrostatic potential in a little more detail. We may write Coulomb's law as

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0} \frac{qq_1}{r^2} \mathbf{e}_r = \frac{1}{4\pi\epsilon_0} \frac{qq_1}{r^3} \mathbf{r}$$

where the charge q_1 is at the origin, the charge q is at \mathbf{r} , and \mathbf{F} denotes the force acting on the charge q . Hence $\mathbf{F} = q\mathbf{E}$ where the electric field \mathbf{E} at \mathbf{r} produced by a charge q_1 is at the origin is

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{q_1}{r^2} \mathbf{e}_r = \frac{1}{4\pi\epsilon_0} \frac{q_1}{r^3} \mathbf{r}.$$

Note that $\mathbf{E} = -\nabla\Phi$ where

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{q_1}{r}.$$

Generalizing, the electric field at \mathbf{r} produced by a point charge q_1 at \mathbf{r}_1 is given by

$$\mathbf{E}_1 = \frac{1}{4\pi\epsilon_0} \frac{q_1}{\|\mathbf{r} - \mathbf{r}_1\|^3} (\mathbf{r} - \mathbf{r}_1)$$

and $\mathbf{E}_1 = -\nabla\Phi_1$ where

$$\Phi_1(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{q_1}{\|\mathbf{r} - \mathbf{r}_1\|}.$$

Now suppose we have m point charges q_1, q_2, \dots, q_m at points $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_m$. By the

principle of superposition, the electric field \mathbf{E} at any point \mathbf{r} is the vector sum of the electric fields produced by the individual point charges. That is,

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^m \frac{q_i}{\|\mathbf{r} - \mathbf{r}_i\|^3} (\mathbf{r} - \mathbf{r}_i). \quad (\text{A7.8})$$

Now let Φ_i denote the potential function associated with the point charge q_i at \mathbf{r}_i ,

$$\Phi_i(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{q_i}{\|\mathbf{r} - \mathbf{r}_i\|} \quad \text{for } i = 1, \dots, m$$

and define

$$\Phi(\mathbf{r}) \equiv \sum_{i=1}^m \Phi_i(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^m \frac{q_i}{\|\mathbf{r} - \mathbf{r}_i\|}. \quad (\text{A7.9})$$

By the linearity of the gradient,

$$-\nabla\Phi = \sum_{i=1}^m (-\nabla\Phi_i) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^m \frac{q_i}{\|\mathbf{r} - \mathbf{r}_i\|^3} (\mathbf{r} - \mathbf{r}_i) = \mathbf{E}(\mathbf{r}).$$

In summary, the principle of superposition applies to potential functions as well as to force and electric fields.

We may extend these results from point charges to a continuous distribution of charge over \mathbb{R}^3 . This yields an electric field

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \iiint \frac{\rho(\mathbf{r}')}{\|\mathbf{r} - \mathbf{r}'\|^3} (\mathbf{r} - \mathbf{r}') dV \quad (\text{A7.10})$$

and an associated potential function

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \iiint \frac{\rho(\mathbf{r}')}{\|\mathbf{r} - \mathbf{r}'\|} dV \quad (\text{A7.11})$$

where these integrals are over all \mathbf{r}' in \mathbb{R}^3 . In principle, eq. (A7.11) provides an explicit solution to eq. (A7.6). Whether this is a *practical* method of finding the electrostatic potential depends on the particular situation under consideration. We have, then, two methods to find \mathbf{E} : we can evaluate the explicit integral given by (A7.10), or we can find Φ , either by the explicit integral (A7.11) or by solving eq. (A7.6), and then find $-\nabla\Phi$. In general, it is somewhat easier to evaluate (A7.11) than (A7.10), for two reasons. First, the integral of (A7.10) is actually 3 integrals, one for each of the components of \mathbf{E} . Second, the integrand of (A7.10) involves $\|\mathbf{r} - \mathbf{r}'\|$ raised to the third power, and this usually makes the integrand of (A7.10) more complicated than the integrand of (A7.11).

We now turn to the subject of magnetostatics. From eq. (A6.4), we see that the magnetic field is solenoidal. The physical meaning of this is often stated as “there are no magnetic monopoles.” As \mathbf{B} is solenoidal, it follows that there exists a vector field \mathbf{A} such that

$$\mathbf{B} = \nabla \times \mathbf{A} = \text{curl } \mathbf{A}. \quad (\text{A7.12})$$

The vector field \mathbf{A} is called the *vector potential*. In magnetostatics, we may combine eqs. (A7.12) and (A7.3) to see that \mathbf{A} satisfies

$$\text{curl}(\text{curl } \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J} \quad (\text{A7.13})$$

where

$$\mu_0 \equiv \frac{1}{c^2 \epsilon_0}. \quad (\text{A7.14})$$

The vector potential \mathbf{A} is not uniquely determined by eq. (A7.12). Let \mathbf{A} be a vector field that satisfies eq. (A7.12), let ψ be *any* scalar field defined on \mathbb{R}^3 , and let $\mathbf{A}' \equiv \mathbf{A} + \nabla\psi$. Because the curl of a gradient is always $\mathbf{0}$, it follows that

$$\nabla \times \mathbf{A}' = \nabla \times (\mathbf{A} + \nabla\psi) = \nabla \times \mathbf{A} = \mathbf{B}.$$

In short, we have a considerable amount of freedom in how the vector potential \mathbf{A} is chosen. In particular, it's possible (and convenient) to impose the restriction

$$\nabla \cdot \mathbf{A} = \text{div } \mathbf{A} = 0.$$

With this restriction, eq. (A7.13) simplifies to

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}. \quad (\text{A7.15})$$

Hence, the vector potential \mathbf{A} in magnetostatics may be found by solving a vector version of Poisson's equation. That is, eq. (A7.15) is really *three* equations: one for each of the components of \mathbf{A} . By comparing eqs. (A7.6), (A7.11), and (A7.15), we see that an explicit solution of (A7.15) is given by

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \iiint_V \frac{\mathbf{J}(\mathbf{r}')}{\|\mathbf{r} - \mathbf{r}'\|} dV. \quad (\text{A7.16})$$

8. Conservation of Charge (and Other “Stuff”). Let V be a mathematical solid in \mathbb{R}^3 with a boundary surface S , and let $\rho(x, y, z, t)$ denote “charge density” at any point (x, y, z) in V at time t . The total amount of “charge” inside V at time t is therefore given by

$$Q(t) \equiv \iiint_V \rho(x, y, z, t) dV.$$

Hence, the rate of change of Q is given by

$$Q'(t) = \iiint_V \frac{\partial \rho}{\partial t} dV. \quad (\text{A8.1})$$

(The operation of differentiating under the integral sign is justified if $\partial\rho/\partial t$ is continuous.)

On the other hand, the only way the amount of charge in V can change is if there is a current across the border of V . If we let \mathbf{J} denote current density as before, it follows that

$$Q'(t) = - \int \int_S \mathbf{J} \cdot \mathbf{n} dS \quad (\text{A8.2})$$

where \mathbf{n} is a outward unit normal. (The integral on the right hand side of eq. (A8.2) is the flux of current across S , and the negative sign is motivated by the observation that a *positive* flux of current across S implies a *decrease* in charge inside V .) Applying the divergence theorem to the right hand side of eq. (A8.2), we find

$$Q'(t) = - \int \int \int_V \nabla \cdot \mathbf{J} dV. \quad (\text{A8.3})$$

Combining eqs. (A8.1) and (A8.3), we find that

$$\int \int \int_V \frac{\partial \rho}{\partial t} dV = - \int \int \int_V \nabla \cdot \mathbf{J} dV. \quad (\text{A8.4})$$

But as the solid V is quite arbitrary, it follows that

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J}$$

at all points where ρ is defined. This equation is usually written

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0. \quad (\text{A8.5})$$

Equation (A8.5) is called a *continuity equation* as it expresses a conservation law: in this case, the conservation of charge. However, it applies in any situation where there is some kind of “stuff” that is conserved where fields ρ and \mathbf{J} may be defined that quantify the density of stuff at a point and the movement of stuff through space. For example, this analysis applies to the study of heat.

The derivation of eq. (A8.5) given above was intended to motivate its' interpretation as an expression of the conservation of some “stuff.” It is also possible to derive eq. (A8.5) directly from Maxwell's equations. By taking the divergence of both sides of eq. (A6.3) we obtain

$$\nabla \cdot \frac{\partial \mathbf{E}}{\partial t} + \frac{1}{\epsilon_0} \nabla \cdot \mathbf{J} = 0 \quad (\text{A8.6})$$

as the divergence of a curl is always zero. Now,

$$\nabla \cdot \frac{\partial \mathbf{E}}{\partial t} = \frac{\partial}{\partial t} (\nabla \cdot \mathbf{E})$$

as we may exchange the order of time and space derivatives. But $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$ from eq. (A6.1), so

$$\nabla \cdot \frac{\partial \mathbf{E}}{\partial t} = \frac{1}{\epsilon_0} \frac{\partial \rho}{\partial t}.$$

Substituting this relation into eq. (A8.6) yields eq. (A8.5). The point of this demonstration is to show that the conservation of charge is a consequence of Maxwell's equations.

9. Waves. We next want to take up the topic of electromagnetic radiation. This requires a brief review of the physics of waves.

The Wave Equation. Based on physical consideration, a wave propagating at speed c along the x -axis may be modeled by the hyperbolic partial differential equation

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2}. \quad (\text{A9.1})$$

This is the (one-dimensional) *wave equation*. It's easy to show that any function of the form $\psi = f(x - ct)$ is a solution. This represents a wave propagating to the right. Another solution is $\psi = f(x + ct)$, which represents a wave propagating to the left. Equation (A9.1) is *linear*. This implies that if ψ_1 and ψ_2 are two solutions to the wave equation, then any linear combination of ψ_1 and ψ_2 is also a solution.

Sinusoidal waves and fundamental wave vocabulary. The sinusoidal waves are solutions of (A9.1) of fundamental importance. These solutions can be written in the form

$$\psi(x, t) = A \cos(Kx - \omega t) \quad (\text{A9.2})$$

where $c = \omega/K$. The three coefficients A , K , and ω are named and interpreted as follows. A is the *amplitude* of the wave and measures its vertical size relative to the x axis. K is called the “wave number” and specifies how the wave varies with space. If the unit of space is the “meter,” then K specifies the number of *radians per meter*. ω is the “angular frequency” and specifies how the wave varies with time. If the unit of time is the “second,” then ω specifies the number of *radians per second*. The combined expression $Kx - \omega t$ is called the “phase” of the wave.

We can relate K and ω to properties of waves that may be more familiar to the reader. Suppose we look at a snapshot of the wave taken at a particular moment (so t is fixed). The *wavelength* λ of the wave is the distance (in meters) between peaks. This is the change in x required to change the phase by 2π , so

$$\lambda = \frac{2\pi}{K}.$$

Now fix x and consider how ψ varies with time. The *period* t_0 is the amount of time required for the phase to change by 2π , so

$$t_0 = \frac{2\pi}{\omega}.$$

Putting these two equations together, the speed of the wave (in meters per second) is given by

$$c = \frac{\lambda}{t_0} = \frac{2\pi/K}{2\pi/\omega} = \frac{\omega}{K}$$

as stated above. Let ν denote the frequency of the wave measured in *cycles* per second. As there are 2π radians per cycle, it follows that

$$\nu = \frac{\omega}{2\pi} = \frac{1}{t_0},$$

which makes sense, as t_0 is the number of seconds per cycle.

The wave equation in space. As noted above, eq. (A9.1) is the wave equation for a wave propagating along the x -axis. The equation for a wave propagating at speed c in \mathbb{R}^3 is

$$\nabla^2 \psi = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2}. \quad (\text{A9.3})$$

Some authors write eq. (A9.3) as

$$\square \psi = 0$$

where the “wave operator” \square (also called the d'Alembert operator or “quabla”) is defined as

$$\square \equiv \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}. \quad (\text{A9.4})$$

Equation (A9.3) is linear, so any linear combination of solutions is also a solution.

Sinusoidal “plane” waves. The reader may confirm that one solution of eq. (A9.3) is given by

$$\psi(\mathbf{r}, t) = A \cos(\mathbf{K} \cdot \mathbf{r} - \omega t) \quad (\text{A9.5})$$

where

$$\begin{aligned} \mathbf{r} &= (x, y, z), \\ \mathbf{K} &= (K_x, K_y, K_z), \\ c &= \omega / \|\mathbf{K}\|. \end{aligned}$$

The wave number K in eq. (A9.2) has been replaced by a “wave vector” \mathbf{K} whose components give the number of radians per meter in the directions of the three coordinate axes. If \mathbf{r}_1 and \mathbf{r}_2 are two points in \mathbb{R}^3 such that $\mathbf{r}_1 - \mathbf{r}_2$ is perpendicular to \mathbf{K} , then \mathbf{r}_1 and \mathbf{r}_2 are on the same “wave front”; i.e., for any time t ,

$$\mathbf{K} \cdot \mathbf{r}_1 - \omega t = \mathbf{K} \cdot \mathbf{r}_2 - \omega t.$$

It follows that the wave specified by eq. (A9.4) propagates in the direction of \mathbf{K} , the wave fronts of eq. (A9.4) are planes perpendicular to \mathbf{K} (which is why we call this solution “plane waves”), and $\|\mathbf{K}\|$ gives the number of radians per meter in the direction of propagation. If c does not depend on ω (which is the case with light, for example), then it's convenient to write $\|\mathbf{K}\| = \omega/c$.

Spherical waves. Although plane waves are mathematically and conceptually agreeable, they are physically problematic: it's difficult to imagine a mechanism that can generate a plane wave that is not physically infinite in some way. Therefore, we now consider solutions of eq. (A9.5) whose wave fronts consist of spheres expanding at speed c away from the origin. Specifically, consider

$$\psi = \frac{1}{r} f(r - ct). \quad (\text{A9.6})$$

We wish to show that eq. (A9.6) satisfies eq. (A9.3). It's convenient to use spherical coordinates for this problem. Because ψ has no dependence on θ or ϕ , the equation for $\nabla^2 \psi$ becomes

$$\nabla^2 \psi = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right).$$

The reader may use this formula to confirm that (A9.6) satisfies the wave equation. Notice that the amplitude of these waves are inversely proportional to r .

10. Electromagnetic Radiation. We now consider solutions of Maxwell's equations in “free space.” In a region of \mathbb{R}^3 where there is no charge and no current (so $\rho = 0$ and $\mathbf{J} = \mathbf{0}$), Maxwell's equations become

$$\nabla \cdot \mathbf{E} = 0, \quad (\text{A10.1})$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (\text{A10.2})$$

$$c^2(\nabla \times \mathbf{B}) = \frac{\partial \mathbf{E}}{\partial t}, \quad (\text{A10.3})$$

$$\nabla \cdot \mathbf{B} = 0. \quad (\text{A10.4})$$

The situation here is “dual” in some sense to the situation considered in electrostatics and magnetostatics, where we allowed (constant) charge density ρ and (steady) currents \mathbf{J} , but required that \mathbf{E} and \mathbf{B} not vary with time. The “trivial” solution of these equations is $\mathbf{E} = \mathbf{B} = \mathbf{0}$, but we're interested in the possibility of non-trivial solutions. To start, rewrite eq. (A10.2) as

$$\frac{\partial \mathbf{B}}{\partial t} = -(\nabla \times \mathbf{E}).$$

Now differentiate with respect to time. Under normal conditions, which we assume here, we can exchange the order of differentiation, so

$$\frac{\partial^2 \mathbf{B}}{\partial t^2} = -\left(\nabla \times \frac{\partial \mathbf{E}}{\partial t}\right).$$

Substituting eq. (A10.3), we obtain

$$\frac{\partial^2 \mathbf{B}}{\partial t^2} = -c^2[\nabla \times (\nabla \times \mathbf{B})].$$

From eqs. (8.4) and (A10.4),

$$\nabla \times (\nabla \times \mathbf{B}) = \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} = -\nabla^2 \mathbf{B},$$

so

$$\frac{\partial^2 \mathbf{B}}{\partial t^2} = c^2 \nabla^2 \mathbf{B}.$$

We'll rewrite this as

$$\nabla^2 \mathbf{B} = \frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} \tag{A10.5}$$

which we recognize as having the form of a “vector” wave equation. The reader may show that eq. (A10.5) is actually three equations, one for each component of \mathbf{B} :

$$\nabla^2 B_x = \frac{1}{c^2} \frac{\partial^2 B_x}{\partial t^2}, \quad \nabla^2 B_y = \frac{1}{c^2} \frac{\partial^2 B_y}{\partial t^2}, \quad \text{and} \quad \nabla^2 B_z = \frac{1}{c^2} \frac{\partial^2 B_z}{\partial t^2}.$$

An exactly parallel derivation starting with eq. (A10.3) and using eq. (A10.1) yields

$$\nabla^2 \mathbf{E} = \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{A10.6}$$

which actually means

$$\nabla^2 E_x = \frac{1}{c^2} \frac{\partial^2 E_x}{\partial t^2}, \quad \nabla^2 E_y = \frac{1}{c^2} \frac{\partial^2 E_y}{\partial t^2}, \quad \text{and} \quad \nabla^2 E_z = \frac{1}{c^2} \frac{\partial^2 E_z}{\partial t^2}.$$

In summary, Maxwell's equations in free space permit solutions for each component of \mathbf{E} and \mathbf{B} that have the form of waves traveling with speed c , the speed of light. These waves are *electromagnetic radiation*, the most familiar example being light itself.

We can say more about the nature of electromagnetic radiation. To begin, let's consider “plane wave” solutions for \mathbf{E} and \mathbf{B} . Without loss of generality, suppose that \mathbf{E} and \mathbf{B}

propagate in plane waves in the direction of the x -axis, so the wave fronts of \mathbf{E} and \mathbf{B} are perpendicular to the x -axis. This implies that \mathbf{E} and \mathbf{B} can have no dependence on y or z , so we can write

$$\mathbf{E} = \mathbf{E}(\mathbf{r}, t) = (E_x(x, t), E_y(x, t), E_z(x, t))$$

and

$$\mathbf{B} = \mathbf{B}(\mathbf{r}, t) = (B_x(x, t), B_y(x, t), B_z(x, t)).$$

Without going into the details (see Feynman, Chapter 20), eqs. (A10.1) - (A10.4) imply that $E_x = 0$ and $B_x = 0$, so we may write

$$\mathbf{E} = \mathbf{E}(\mathbf{r}, t) = (0, E_y(x, t), E_z(x, t))$$

and

$$\mathbf{B} = \mathbf{B}(\mathbf{r}, t) = (0, B_y(x, t), B_z(x, t)).$$

That is, all the variation in \mathbf{E} and \mathbf{B} is in a plane perpendicular to the direction of propagation of the waves.

To make further headway, let's consider a "trial solution" of the following form:

$$\mathbf{E} = (0, f(x - ct), 0) = f(x - ct)\mathbf{j}.$$

That is, \mathbf{E} is a wave traveling to the right and the \mathbf{k} component of \mathbf{E} is zero. It follows that

$$\frac{\partial \mathbf{B}}{\partial t} = -(\nabla \times \mathbf{E}) = - \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & f(x - ct) & 0 \end{bmatrix} = -f'(x - ct)\mathbf{k}.$$

Hence, the x and y components of \mathbf{B} are constant over time. As above, the only physically interesting solution of these equations is $B_x = B_y = 0$. Hence, \mathbf{B} is zero except in the direction of \mathbf{k} , and

$$\frac{\partial B_z}{\partial t} = -f'(x - ct).$$

Integrating, we obtain $B_z = c^{-1}f(x - ct)$ plus a constant of integration. On physical grounds again, it may be shown that the constant of integration is zero, so we conclude in this case that

$$B_z(x, t) = c^{-1}f(x - ct).$$

We may repeat this analysis under the assumption that the y component of \mathbf{E} is zero,

$$\mathbf{E} = (0, 0, f(x - ct)) = f(x - ct)\mathbf{k}.$$

We may also repeat both analyses under the trial solution of a wave traveling to the left:

$$(E_y, E_z) = (f(x + ct), 0) \quad \text{and} \quad (E_y, E_z) = (0, f(x + ct)).$$

Our results are summarized in the following table.

E_y	E_z	B_y	B_z
$f(x - ct)$	0	0	$c^{-1}f(x - ct)$
$f(x + ct)$	0	0	$-c^{-1}f(x + ct)$
0	$f(x - ct)$	$-c^{-1}f(x - ct)$	0
0	$f(x + ct)$	$c^{-1}f(x + ct)$	0

By the linearity of the wave equation, the general formula for \mathbf{E} as a plane wave moving along the x -axis is an arbitrary combination of the components given in the columns headed E_y and E_z , and the implied solution for \mathbf{B} is the same combination of the components given in the columns headed B_y and B_z . For example, if a wave propagating to the right is written

$$\mathbf{E} = \alpha_1 f_1(x - ct)\mathbf{j} + \alpha_2 f_2(x - ct)\mathbf{k},$$

(where α_1 and α_2 are arbitrary constants, and f_1 and f_2 are arbitrary functions), then

$$\mathbf{B} = \frac{1}{c} \left[\alpha_1 f_1(x - ct)\mathbf{k} - \alpha_2 f_2(x - ct)\mathbf{j} \right].$$

Similarly, if a wave propagating to the left is written

$$\mathbf{E} = \alpha_1 f_1(x + ct)\mathbf{j} + \alpha_2 f_2(x + ct)\mathbf{k},$$

then

$$\mathbf{B} = \frac{1}{c} \left[-\alpha_1 f_1(x + ct)\mathbf{k} + \alpha_2 f_2(x + ct)\mathbf{j} \right].$$

Note that \mathbf{E} and \mathbf{B} are perpendicular to one another in both cases.

An important class of solutions to these equations are the sinusoidal waves. To fix ideas, suppose \mathbf{E} is a wave propagating to the right along the x -axis and oscillating with angular frequency ω . Then we may write \mathbf{E} as

$$\mathbf{E} = A_y \cos(Kx - \omega t)\mathbf{j} + A_z \cos(Kx - \omega t - \alpha)\mathbf{k}$$

where $K = \omega/c$. The parameter α is a “phase shifter” that may vary from $-\pi$ to π . If $\alpha = 0$, then the two components of \mathbf{E} are “in phase” and the path of \mathbf{E} in the yz -plane is a straight line segment from (A_y, A_z) to $(-A_y, -A_z)$. If $\alpha = \pm\pi$, then the two components of \mathbf{E} are 180° out of phase, and the path of \mathbf{E} in the yz -plane is a straight line segment from $(A_y, -A_z)$ to $(-A_y, A_z)$. If $\alpha = \pm\frac{1}{2}\pi$, then the path of \mathbf{E} is an ellipse with semi-axes A_y and A_z . For fixed x , if $\alpha = \frac{1}{2}\pi$ the path is traversed in a clockwise direction as t increases, and if $\alpha = -\frac{1}{2}\pi$, the path is traversed in a counterclockwise direction as t increases. (These directions of traversal are reversed if t is fixed and x is allowed to

increase.) The phase shifter α and the two amplitudes A_y and A_z control the “polarization” of \mathbf{E} . In any case, the value of \mathbf{B} implied by this equation is

$$\mathbf{B} = \frac{1}{c} \left[-A_z \cos(Kx - \omega t - \alpha) \mathbf{j} + A_y \cos(Kx - \omega t) \mathbf{k} \right].$$

11. Solving Maxwell's equations. In section 7 of this appendix we solved Maxwell's equations for electrostatics and magnetostatics. We found that

$$\mathbf{E} = -\nabla\Phi$$

where the “electrostatic potential” Φ satisfies Poisson's equation

$$\nabla^2\Phi = -\frac{\rho}{\epsilon_0},$$

and

$$\mathbf{B} = \nabla \times \mathbf{A}$$

where the “vector potential” \mathbf{A} satisfies a vector version of Poisson's equation

$$\nabla^2\mathbf{A} = -\mu_0\mathbf{J}. \quad (\text{A11.1})$$

To get eq. (A11.1) we needed to impose a restriction on \mathbf{A} , namely

$$\nabla \cdot \mathbf{A} = 0. \quad (\text{A11.2})$$

We now show how this analysis may be extended to solve Maxwell's equations in general. For reference, here are Maxwell's equations.

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \quad (\text{A11.3})$$

$$\nabla \times \mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}, \quad (\text{A11.4})$$

$$c^2(\nabla \times \mathbf{B}) = \frac{\partial\mathbf{E}}{\partial t} + \frac{\mathbf{J}}{\epsilon_0}, \quad (\text{A11.5})$$

$$\nabla \cdot \mathbf{B} = 0. \quad (\text{A11.6})$$

As before, we begin with eq. (A11.6). As \mathbf{B} is solenoidal, it follows that we may write

$$\mathbf{B} = \nabla \times \mathbf{A} \quad (\text{A11.7})$$

for some vector field \mathbf{A} called the vector potential (as before).

Next, substitute eq. (A11.7) into eq. (A11.4). This yields

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{A}) = -\left(\nabla \times \frac{\partial \mathbf{A}}{\partial t}\right).$$

Hence,

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = \mathbf{0}.$$

In the language introduced above, $\mathbf{E} + \partial \mathbf{A}/\partial t$ is irrotational. Therefore, there exists a scalar field Φ called the *scalar potential* such that

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla \Phi.$$

We rewrite this as

$$\mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t}. \quad (\text{A11.8})$$

As before, there's some flexibility in our choice of \mathbf{A} . For given \mathbf{B} and \mathbf{E} , suppose \mathbf{A} and Φ satisfy eqs. (A11.7) and (A11.8). If we make the substitution

$$\mathbf{A}' = \mathbf{A} + \nabla \psi$$

for some scalar field ψ , then eq. (A11.7) will still be satisfied, but eq. (A11.8) will not. However, if we make the simultaneous substitution

$$\mathbf{A}' = \mathbf{A} + \nabla \psi \quad \text{and} \quad \Phi' = \Phi - \frac{\partial \psi}{\partial t}, \quad (\text{A11.9})$$

then both eq. (A11.7) and (A11.8) will be satisfied. The simultaneous transformation

$$(\mathbf{A}, \Phi) \rightarrow (\mathbf{A}', \Phi')$$

is called a *gauge transformation*.

Equations (A11.7) and (A11.8) express \mathbf{E} and \mathbf{B} in terms of vector potential \mathbf{A} and a scalar potential Φ . We now substitute eqs. (A11.7) and (A11.8) into eqs. (A11.3) and (A11.5) to obtain equations for \mathbf{A} and Φ in terms of the “sources” ρ and \mathbf{J} . This yields

$$\nabla^2 \Phi + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{\rho}{\epsilon_0}$$

and

$$\nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J} + \nabla \left(\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} \right)$$

where $\mu_0 \equiv 1/c^2 \epsilon_0$. To simplify the mathematics, we impose a gauge transformation (the “Lorentz gauge”) such that

$$\nabla \cdot \mathbf{A} = -\frac{1}{c^2} \frac{\partial \Phi}{\partial t}. \quad (\text{A11.10})$$

Substituting eq. (A11.10) into the preceding two equations, we obtain

$$\nabla^2 \Phi - \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} = \square \Phi = -\frac{\rho}{\epsilon_0} \quad (\text{A11.11})$$

and

$$\nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = \square \mathbf{A} = -\mu_0 \mathbf{J}. \quad (\text{A11.12})$$

Equation (A11.12) is actually three equations, one for each component of \mathbf{A} :

$$\square A_x = -\mu_0 J_x, \quad \square A_y = -\mu_0 J_y, \quad \text{and} \quad \square A_z = -\mu_0 J_z.$$

Recall that an equation of the form $\square \psi = 0$ is said to be a “wave equation.” Given this, you shouldn't be too surprised to learn that an equation of the form $\square \psi = \sigma$ is called a “wave equation with a source term.” A wave equation with a source term effectively combines a wave equation

$$\nabla^2 \psi - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = 0$$

and Poisson's equation

$$\nabla^2 \psi = \sigma.$$

To summarize, we've replaced Maxwell's four equations with the four equations

$$\begin{aligned} \mathbf{B} &= \nabla \times \mathbf{A} \\ \mathbf{E} &= -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t} \\ \square \Phi &= -\frac{\rho}{\epsilon_0}, \quad \text{and} \\ \square \mathbf{A} &= -\mu_0 \mathbf{J}. \end{aligned}$$

These four equations contain the same physical content as Maxwell's equations, and in many circumstances are easier to handle. I refer the reader to Feynman for the physical interpretation of \mathbf{A} and Φ .