MATH 209: TAKE AWAYS

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ABSTRACT. Below we summarize some items to take away from the class. In particular, what are one time tricks and methods, and what are general techniques to solve a variety of problems.

1. SPECIFIC TECHNIQUES

Obviously, in a 'service' class such as Differential Equations, one of the major objects is clearly to learn how to solve a variety of problems. There are often two stages to this: (1) coming up with a tractable mathematical model that captures many of the key features; (2) solving the model.

Much of the class has been about the second point, namely solving the model. To do this we've developed the theory of **Ordinary Differential Equations** (i.e., differential equations where everything is a function of just a single variable) for a wide class of problems, including:

(1) **First order linear differential equations:** $y'(x) + p(x)y(x) = g(x)$. The key idea here is integrating factors.

(2) **Separable equations:** $M(x) + N(y)dy/dx = 0$**.** Here the main point is that we can write it so that all the y-dependence is on one side and all the x-dependence on the other, so we have reduced the problem to integrating functions of one variable. While in general we won't be able to find closed form solutions, we can numerically approximate these integrals.

(3) **Exact equations:** $M(x, y) + N(x, y)dy/dx = 0$ **with** $\partial M/\partial y = \partial N/\partial x$**.** Here we find a function $\psi(x, y)$ such that $d\psi/dx = M(x, y) + N(x, y)dy/dx$, and thus the solution is $\psi(x, y) = c$ for some c.

(4) **Second order linear constant coefficient homogenous equations:** $y''(x)+ay(x)+by(x) = 0$. We solve by guessing $y(x) = \exp(rx)$, or modifying our guess if there is a repeated root of the characteristic polynomial $r^2 + ar + b = 0$. Note the theory generalizes to n^{th} order constant coefficient differential equations.
$\diamond$ A similar analysis solves linear, constant coefficient difference equations: $a_{n+1} = c_1 a_n + c_2 a_{n-1} + c_3 a_{n-2} + \cdots + c_k a_{n-k+1}$.

A natural question to ask is how many systems can be modeled by these types of equations. Fortunately, there are numerous, ranging from the Solow growth equations (separable equation) to why double-plus-one is a bad strategy at roulette (difference equation).

Date: May 27, 2009.

2. General Solution Techniques

Below we describe three techniques for solving differential equations. These techniques are singled out from the methods above as the ideas behind them can be generalized to work on a variety of problems.

(1) **Picard's iteration method.** This is an incredible method. Namely, to solve a first order differential equation (with initial conditions $y(0) = 0$) we guess the solution is $y(x) = 0$, and then use a completely specified method to determine a sequence of guesses which, under weak assumptions about f (where $y'(x) = f(x, y(x))$ converges to the solution to the problem. These iterative methods are very important in mathematics and its applications. In particular, they are useful in proving fixed point theorems (these arise very frequently in economics).

(2) **Method of Undermined Coefficients:** $ay'' + by' + cy = g(t)$ with $g(t) = e^{\alpha t} P_n(t)$ (with $P_n(t)$ a polynomial of degree n in t) or $g(t) = e^{\alpha t}\cos(\beta t)$ or $g(t) = e^{\alpha t}\sin(\beta t)$; see also the method of variation of parameters. What is nice about this method is that we start with a solution to a related problem, and then modify it (in the case of variation of parameters) or then bring in the solution to a related problem (in the method of undetermined coefficients).

(3) **Series expansions:** $p(x)y''(x) + q(x)y'(x) + r(x)y(x) = 0$ with $p(x_0) \neq 0$ and guessing $y(x) = \sum_{n=0}^{\infty} a_n (X - x_0)^n$. This is but one of many expansions to guess for a solution. Other popular approaches include doing a Fourier Series expansion for the guess ($y(x) = \sum_{n=-\infty}^{\infty} (a_n \sin(2\pi n x) + b_n \cos(2\pi n x))$); the exact nature of the guess depends on the symmetry of the problem. The difficulty in practice is that in general one does not get a nice closed form answer in the end, but rather an infinite series expansion where we have some (but *not* complete) knowledge of the coefficients of the expansion. While in principle this method is straightforward to use, the algebra can quickly become overwhelming.

3. General Techniques

When confronted with a theorem to prove or a problem to solve, it is often not clear how to begin. Over time, one accumulates experiences from solving similar problems. Here are some very powerful techniques and a quick description of where you may have seen them.

(1) **Adding zero / multiplying by one:** The difficult part of these methods is figuring out how to 'do nothing' in an intelligent way. The first example you might remember is proving the product rule from calculus. Let $A(x) = f(x)g(x)$. Then

$$\begin{aligned} A'(x) &= \lim_{h\to 0} \frac{A(x+h)-A(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h)g(x+h)-f(x)g(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h)g(x+h)\mathbf{-f(x)g(x+h) + f(x)g(x+h)} - f(x)g(x)}{h} \\ &= \lim_{h\to 0} \left[\frac{f(x+h)g(x+h)-f(x)g(x+h)}{h} + \frac{f(x)g(x+h)-f(x)g(x)}{h}\right] \\ &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} g(x+h) + \lim_{h\to 0} f(x)\frac{g(x+h)-g(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \lim_{h\to 0} g(x+h) + f(x)\lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\ &= f'(x)g(x)+f(x)g'(x). \end{aligned} \tag{3.1}$$

In our class, we've seen this a few times. One was in finding the series expansion to $xy''(x) + y'(x) + xy(x) = 0$ about $x_0 = 1$, where we wrote x as $x - 1 + 1$.

(2) **Numerical accuracy:** Frequently we can write down solutions to differential equations, but to actually compute with these quantities can be difficult. For example, the method of integrating factors solves many first order linear differential equations, but involves two integrals which typically can only be approximated numerically. Thus it is very important to have an idea of how accurate an expansion is. On a related note, we frequently need to do computations quickly. Thus while there are straightforward 'naive' approaches to do a calculation, these can take too long in practice to be useful. Frequently an intelligent re-arrangement of the algebra can lead to an incredible time savings. Examples range from telescoping series (which is actually one of the key ingredients in the proof of the fundamental theorem of calculus) to Horner's algorithm (to evaluate polynomials rapidly) to Strassen's and others algorithms (to compute products of matrices). Another example, from linear algebra, is diagonalizing matrices. We talked about using matrices to model discrete time systems (Markov models); this requires us to be able to compute A^n for large n. If A is diagonalizable (say $S^{-1}AS = \Lambda$ so $A = S\Lambda S^{-1}$) then $A^2 = S\Lambda S^{-1}S\Lambda S^{-1} = S\Lambda^2 S^{-1}$, and more generally we see $A^n = S\Lambda^n S^{-1}$ (and Λ^n is easily computed as Λ is a diagonal matrix).

(3) **Linear combinations:** One of the most important techniques we've seen is the power of linear combinations of solutions or approaches. The first instance was in seeing how to find additional solutions to difference or differential equations with constant coefficients whose

characteristic polynomial has repeated roots. For example, consider $a_{n+2}-10a_{n+1}+25a_n = 0$. The characteristic polynomial arises from guessing $a_n = r^n$. We find $r^2-10r+25 = 0$, or $(r-5)^2 = 0$. Thus we do not have two distinct roots. Consider instead a related polynomial which has two distinct roots, say r_1 and r_2. The two solutions are r_1^n and r_2^n. While we may take the two independent solutions as $c_1r_1^n$ and $c_2r_2^n$, we could also take the solutions to be $b_1r_1^n + b_2r_2^n$ and $b_1'r_1^n + b_2'r_2^n$ for 'most' b_1, b_1', b_2, b_2'. There are particularly good choices to take: $(r_1^n + r_2^n)/(r_1 + r_2)$ and $(r_2^n - r_1^n)/(r_2 - r_1)$. These are not the most 'obvious' combinations of the two solutions r_1^n and r_2^n, but there is an advantage. The first converges to r_1^n as $r_2 \to r_1$; the second, however, is interesting. As $r_2 \to r_1$ we get $0/0$, and thus need to use L'Hopital's rule or the definition of the derivative and find

$$\lim_{r_2\to r_1} \frac{r_2^n - r_1^n}{r_2 - r_1} \ = \ \lim_{h\to 0} \frac{(r_1+h)^n - r_1^n}{h} \ = \ nr_1^{n-1}; \tag{3.2}$$

and, of course, since r_1 is a constant we see we may take nr_1^n and not nr_1^{n-1} as the second solution. Thus we find the second solution by taking an appropriate linear combination.

Another instance of this is in the proof of Simpson's rule. We are trying to approximate the area under $y = f(x)$ from a to b. We have the Median method, using $M = f\left(\frac{a+b}{2}\right) \cdot (b-a)$; we have the Trapezoid method, using $T = \frac{f(a)+f(b)}{2} \cdot (b-a)$. Each of these methods has an error of size $(b-a)^3$ (the first has an error $-\frac{1}{24}f''(a)(b-a)^3 + O((b-a)^4)$ and the second has an error of size $\frac{1}{12}f''(a)(b-a)^3 + O((b-a)^4)$; however, by taking the combination $\frac{2}{3}M + \frac{1}{3}T$, we get an error of size $O((b-a)^5)$ (while one would expect an error of size $O((b-a)^4)$, there is some extra cancelation). This is phenomenal; somehow by taking the 'right' combination of two methods that have errors of size $(b-a)^3$ we get something with an error of size $(b-a)^5$.

There are many other situations where the appropriate linear combinations lead to better results than one might initially expect. Another example is in economics or finance. Imagine there are two **independent** stocks (or mutual funds, investments, ...), denoted X_1 and X_2 with expected mean returns $\mu_1 = \mu_2 = \mu$ and variances $\sigma_1^2 = \sigma_2^2 = \sigma^2$. Consider the allocation of our funds among the two stocks, where ω percent of our money goes to X_1 or $1-\omega$ to X_2. Thus our new combined investment is $X = \omega X_1 + (1-\omega)X_2$. We have the expected value of X is $\omega\mu + (1-\omega)\mu = \mu$ (ie, the new combined investment has the same expected mean return); however, the variance of X is $\omega^2\sigma_1^2 + (1-\omega)^2\sigma_2^2 = \left(\omega^2 + (1-\omega)^2\right)\sigma^2$ (this is because, for independent random variables, $\mathrm{Var}(aX_1 + bX_2) = a^2\mathrm{Var}(X_1) + b^2\mathrm{Var}(X_2)$). Note that $\omega^2 + (1-\omega)^2 \le 1$; in fact, its minimum value arises when $\omega = 1/2$, in which it is $1/2$. Thus the new combined investment has the same expected return but a smaller variance (ie, less uncertainty).

All of these instances are examples of taking appropriate weighted combinations of the quantities of interest; these are just a few of numerous instances of the power of this method.

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