PROBLEMS ON DIFFERENCE EQUATIONS

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ABSTRACT. Below we give some exercises on linear difference equations with constant coefficients. These problems are taken from [MT-B].

1. EXERCISES

Exercise 1.1 (Recurrence Relations). *Let $\alpha_0, \dots, \alpha_{k-1}$ be fixed integers and consider the recurrence relation of order k*

$$x_{n+k} \ = \ \alpha_{k-1}x_{n+k-1} + \alpha_{k-2}x_{n+k-2} + \cdots + \alpha_1 x_{n+1} + \alpha_0 x_n. \tag{1.1}$$

Show once k values of x_m are specified, all values of x_n are determined. Let

$$f(r) \ = \ r^k - \alpha_{k-1}r^{k-1} - \cdots - \alpha_0; \tag{1.2}$$

we call this the characteristic polynomial of the recurrence relation. Show if $f(\rho) = 0$ then $x_n = c\rho^n$ satisfies the recurrence relation for any $c \in \mathbb{C}$.

Exercise 1.2. *Notation as in the previous problem, if $f(r)$ has k distinct roots $r_1, \dots, r_k$, show that any solution of the recurrence equation can be represented as*

$$x_n \ = \ c_1 r_1^n + \cdots + c_k r_k^n \tag{1.3}$$

for some $c_i \in \mathbb{C}$. The Initial Value Problem is when k values of x_n are specified; using linear algebra, this determines the values of $c_1, \dots, c_k$. Investigate the cases where the characteristic polynomial has repeated roots. For more on recursive relations, see [GKP], *§7.3.*

Exercise 1.3. *Solve the Fibonacci recurrence relation $F_{n+2} = F_{n+1} + F_n$, given $F_0 = F_1 = 1$. Show F_n grows exponentially, i.e., F_n is of size r^n for some $r > 1$. What is r? Let $r_n = \frac{F_{n+1}}{F_n}$. Show that the even terms r_{2m} are increasing and the odd terms r_{2m+1} are decreasing. Investigate $\lim_{n\to\infty} r_n$ for the Fibonacci numbers. Show r_n converges to the golden mean, $\frac{1+\sqrt{5}}{2}$. See* [PS2] *for a continued fraction involving Fibonacci numbers.*

Exercise 1.4 (Binet's Formula). *For F_n as in the previous exercise, prove*

$$F_{n-1} \ = \ \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]. \tag{1.4}$$

This formula should be surprising at first: F_n is an integer, but the expression on the right involves irrational numbers and division by 2.

Exercise 1.5. *Notation as in the previous problem, more generally for which positive integers m is*

$$\frac{1}{\sqrt{m}}\left[\left(\frac{1+\sqrt{m}}{2}\right)^n - \left(\frac{1-\sqrt{m}}{2}\right)^n\right] \tag{1.5}$$

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an integer for any positive integer n?

Exercise(h) 1.6 (Zeckendorf's Theorem)**.** Consider the set of distinct Fibonacci numbers: $\{1, 2, 3, 5, 8, 13, \dots\}$. Show every positive integer can be written uniquely as a sum of distinct Fibonacci numbers where we do not allow two consecutive Fibonacci numbers to occur in the decomposition. Equivalently, for any n there are choices of $\epsilon_i(n) \in \{0, 1\}$ such that

$$n \;=\; \sum_{i=2}^{\ell(n)} \epsilon_i(n) F_i, \quad \epsilon_i(n)\epsilon_{i+1}(n) = 0 \text{ for } i \in \{2, \dots, \ell(n) - 1\}. \tag{1.6}$$

Does a similar result hold for all recurrence relations? If not, can you find another recurrence relation where such a result holds?

Exercise(hr) 1.7. Assume all the roots of the characteristic polynomial are distinct, and let λ_1 be the largest root in absolute value. Show for almost all initial conditions that the coefficient of λ_1 is non-zero.

Exercise(hr) 1.8. Consider 100 tosses of a fair coin. What is the probability that at least three consecutive tosses are heads? What about at least five consecutive tosses? More generally, for a fixed k what can you say about the probability of getting at least k consecutive heads in N tosses as $N \to \infty$?

References

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[PS2] A. van der Poorten and J. Shallit, *A specialised continued fraction*, Canad. J. Math. **45** (1993), no. 5, 1067–1079.

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