Simpson's Rule

Midpoint Approx: \( f\left(\frac{a+b}{2}\right) \cdot (b-a) \equiv M \)

Taylor: \( f(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2} + O((x-a)^3) \)

True Area: \( \int_a^b f(x) dx = f(a)(b-a) + f'(a)\frac{(b-a)^2}{2} + f''(a)\frac{(b-a)^3}{6} + O((b-a)^4) \)

\( \Rightarrow \) now \( f\left(\frac{a+b}{2}\right) = f(a) + f'(a)\frac{b-a}{2} + f''(a)\frac{(b-a)^3}{8} + O((b-a)^4) \)

So True Area - midpoint approx is
\( f''(a)\frac{(b-a)^3}{6} - f''(a)\frac{(b-a)^3}{8} + O((b-a)^4) \)
\( = f''(a)\frac{(b-a)^3}{24} + O((b-a)^4) \)

Trapezoid Approx: \( (f(a) + f(b)) \cdot \frac{b-a}{2} \equiv T \)

\( \Rightarrow \) algebra: True Area - trapezoid approx is
\( -\frac{1}{12}(b-a)^3 f''(a) + O((b-a)^4) \)

Weighted Average: \( \frac{2M + T}{3} \)
\( \Rightarrow \) has error \( O((b-a)^4) \)

Yields: Simpson's rule:
\[ \int_a^b f(x) dx = \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] \frac{b-a}{6} + O((b-a)^4) \]

\( \Rightarrow \) note: error actually \( O((b-a)^5) \) b/c of symmetry of evaluations
Chapter 8: Numerical Methods

Section 8.1: The Euler or Tangent Line Method

1. Review tangent line

Application: Newton's Method

Consider first-order ODE: \( y' = f(t, y) \) with \( y(t_0) = y_0 \)

a) Assume \( f, f_y \) cont on rectangle \( R(t_0, y_0) \)

b) Thm 2.4.2: \( \exists! \) soln in nbhd to

\[ \text{Euler/Tangent line} \quad y_{n+1} = y_n + f(t_n, y_n) \cdot (t_{n+1} - t_n) \]

or \( y_{n+1} = y_n + f(t_n, y_n) \cdot h \) if stepsize constant

Idea: \( f(t, y) \) is slope, tells us how \( y \) is changing

**Errors:** \( \Phi(t) \) soln, \( E_n = \Phi(t_n) - y_n \): global truncation error

- at each step make errors, input data at each step only approximates
- if assume \( y_n \) exactly right, error is going one step forward to \( y_n \) is local truncation error.
- also have round-off error from computers
- Lorenz equations and birth of chaos
- story of restatements...

We'll assume no round-off and analyze just local truncation error

Assume \( y = \Phi(t) \) soln @ cont second derivative (true if \( f, f_t, f_y \) cont)

- \( \Phi'(t) = f(t, \Phi(t)) \)
- \( \Phi''(t) = f_{tt}(t, \Phi(t)) + f_y(t, \Phi(t)) \Phi'(t) \) chain rule, \( \Phi'(t) = f(t, \Phi(t)) \)

Taylor series: \( \Phi(t_n + h) = \Phi(t_n) + \Phi'(t_n)h + \frac{1}{2} \Phi''(t_n)h^2 \)

Tangent line: \( y_{n+1} = y_n + f(t_n, y_n)h \)

Subtract: \( \Phi(t_{n+1}) - y_{n+1} = (\Phi(t_n) - y_n) + h \left[ f(t_n, \Phi(t_n)) - f(t_n, y_n) \right] + \frac{1}{2} \Phi''(t_n)h^2 \)

- local error: \( |\Phi(t_{n+1}) - y_{n+1}| \leq \max|\Phi''| \cdot \frac{h^2}{2} \)

- Key fact: error is like \( h^2 \)