MATH 209: WHEN B DRIVES A OUT OF BUSINESS

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Abstract. We propose and analyze a model for how long it takes business B to drive business A out of the market, given that these are the only two firms in the industry. We analyze the functional form of the critical time in terms of A's initial market share, and discuss reasonable features of a model and what functional form the critical time should take.

1. Statement of problem

Consider the following problem. Two businesses, A and B, compete for market share. Assuming they are the only two businesses in the industry, between the two of them they have the entire market. Letting their respective shares at time t be $M_A(t)$ and $M_B(t)$, we have $M_A(t)+M_B(t)$. Assuming $M_A(0) < M_B(0)$, how long will it take for B to drive A from the market? Or, in other words, if t_{critical} is how long until A is driven from the market, find t_{critical}.

2. Conjectured Formula

We write down a reasonable guess for t_{critical}. We know it is a function of $M_A(0)$ and $M_B(0)$; however, as $M_A(t)+M_B(t) = 1$, we have $M_B(t) = 1 - M_A(t)$ or $M_B(0) = 1 - M_A(0)$. Thus t_{critical} should just be a function of $M_A(0)$, but which function? While there are infinitely many functions, a little thought suggests a reasonable conjecture.

In trying to figure out t_{critical}, we have that it is a function of just $M_A(0)$, so let us denote it by $t_{\text{critical}}(M_A(0))$. There are two situations where we know the answer. If $M_A(0) = 0$ then $t_{\text{critical}}(0) = 0$, as A has no market share! There is one other case we can easily determine, namely $M_A(0) = 1/2$. In this case, $t_{\text{critical}}(1/2) = \infty$, as A and B have equal market share and thus neither will drive the other from the market.

We are thus looking for a function of $M_A(0)$ that vanishes when $M_A(0) = 0$ and is ∞ when $M_A(0) = 1/2$. The simplest function that

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fits this is

$$\frac{cM_A(0)}{1/2 - M_A(0)} \tag{2.1}$$

for any constant c. We wrote the denominator as $1/2 - M_A(0)$ as whenever $M_A(0) < 1/2$ then it takes a finite positive amount of time for A to be driven from the market.

Of course, this is not the only possible function satisfying these properties. Let $g(x)$ be any nice function with no constant term. Then

$$g\left(\frac{M_A(0)}{1/2 - M_A(0)}\right) \tag{2.2}$$

is a reasonable guess for the amount of time A stays in business when $M_A(0) < 1/2$; our first function is the simplest case, namely $g(x) = cx$.

3. Reasonable model

We propose the following model:

$$\begin{aligned} dM_A/dt &= M_A(t) - M_B(t) \\ dM_B/dt &= M_B(t) - M_A(t). \end{aligned} \tag{3.1}$$

While there are other models, this is the simplest model that satisfies several reasonable conditions. The first is we note that adding the two equations gives $d(M_A + M_B)/dt = 0$ or $M_A(t) + M_B(t) = c$, a constant. This is the conservation of market share. We can re-write these equations by noting that $M_B(t) = 1 - M_A(t)$ and find

$$\begin{aligned} dM_A/dt &= 2M_A(t) - 1 \\ dM_B/dt &= 2M_B(t) - 1. \end{aligned} \tag{3.2}$$

Other observations supporting the reasonableness of this model is that B's market share is increasing at a greater rate when it has a greater share, and if $M_A(t) = M_B(t) = 1/2$ then there is no change (ie, it is an equilibrium).

4. Solving the model

We must solve $dM_A/dt = 2M_A(t) - 1$. There are several ways. The first is to use integrating factors; the second to use our theory of solving systems of linear equations. The simplest is probably to note that it is a separable equation. We find

$$-\frac{dM_A}{1/2 - M_A} = dt, \tag{4.1}$$

or

$$\ln\left(\frac{1}{2} - M_A(t)\right) \;=\; t + C. \tag{4.2}$$

The initial condition implies that $C = \ln\left(\frac{1}{2} - M_A(0)\right)$, and thus our function $M_A(t)$ satisfies

$$\ln\left(\frac{1}{2} - M_A(t)\right) \;=\; t + \ln\left(\frac{1}{2} - M_A(0)\right), \tag{4.3}$$

or by using our log-laws we find

$$\ln\left(\frac{1/2 - M_A(t)}{1/2 - M_A(0)}\right) \;=\; t. \tag{4.4}$$

We are interested in $t_{\text{critical}}(M_A(0))$, i.e., the time at which A is driven from the market. At that time A's market share is zero, and thus this time satisfies

$$t_{\text{critical}}(M_A(0)) \;=\; \ln\left(\frac{1/2}{1/2 - M_A(0)}\right). \tag{4.5}$$

We can obtain a more useful expression for the above by using our common trick of adding zero. We have

$$\begin{aligned} t_{\text{critical}}(M_A(0)) &= \ln\left(\frac{1/2}{1/2 - M_A(0)}\right) \\ &= \ln\left(\frac{1/2 - M_A(0) + M_A(0)}{1/2 - M_A(0)}\right) \\ &= \ln\left(1 + \frac{M_A(0)}{1/2 - M_A(0)}\right). \end{aligned} \tag{4.6}$$

Using $\log(1+u) = u + O(u^2)$ (where $O(u^2)$ means an error of size u^2), we see

$$t_{\text{critical}}(M_A(0)) \;=\; \frac{M_A(0)}{1/2 - M_A(0)} + O\left(\left(\frac{M_A(0)}{1/2 - M_A(0)}\right)^2\right) \tag{4.7}$$

Thus, to first order, we do recover the formula

$$t_{\text{critical}}(M_A(0)) \;\approx\; \frac{M_A(0)}{1/2 - M_A(0)}. \tag{4.8}$$

If instead we are looking for a function $g(x)$ that is zero at $x = 0$, we find $\ln\left(1 + \frac{M_A(0)}{1/2 - M_A(0)}\right)$.

5. Comments and generalizations

We see that our simple model leads to a reasonable form (which is natural to conjecture) for the amount of time A stays in business. An interesting exercise would be to consider other similar models and see what they lead to, for example,

$$\begin{aligned} dM_A/dt &= \alpha\left(M_A(t) - M_B(t)\right)^{\beta} \\ dM_B/dt &= \alpha\left(M_B(t) - M_A(t)\right)^{\beta}. \end{aligned} \tag{5.1}$$