

# A STRICTLY STATIONARY, $N$ -TUPLEWISE INDEPENDENT COUNTEREXAMPLE TO THE CENTRAL LIMIT THEOREM

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**Abstract.** For an arbitrary integer  $N \geq 2$ , this paper gives a construction of a strictly stationary,  $N$ -tuplewise independent sequence of (nondegenerate) bounded random variables such that the Central Limit Theorem fails to hold. The sequence is in part an adaptation of a nonstationary example with similar properties constructed by one of the authors (ARP) in a paper published in 1998.

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## 1. Introduction and main result

Suppose  $X := (X_k, k \in \mathbf{Z})$  is a sequence of random variables on a probability space  $(\Omega, \mathcal{F}, P)$ . This sequence  $X$  is said to be “strictly stationary” if for all choices of integers  $j$  and  $\ell$  and nonnegative integer  $m$ , the random vectors  $(X_j, X_{j+1}, \dots, X_{j+m})$  and  $(X_\ell, X_{\ell+1}, \dots, X_{\ell+m})$  have the same distribution. For a given integer  $N \geq 2$ , the sequence  $X$  (stationary or not) is said to satisfy “ $N$ -tuplewise independence” if for every choice of  $N$  distinct integers  $k(1), k(2), \dots, k(N)$ , the random variables  $X_{k(1)}, X_{k(2)}, \dots, X_{k(N)}$  are independent. For  $N = 2$  (resp.  $N = 3$ ), the word “ $N$ -tuplewise” is also expressed as “pairwise” (resp. “triplewise”).

Etemadi [5] proved a strong law of large numbers for sequences of pairwise independent, identically distributed random variables with finite absolute first moment. Janson [7] showed with several classes of counterexamples that for strictly stationary sequences of pairwise independent, nondegenerate, square-integrable random variables, the Central Limit Theorem (henceforth abbreviated CLT) need not hold. Subsequently, Bradley [2, Theorem 1] constructed another such counterexample, a 3-state one that has the additional property of satisfying the absolute regularity (weak Bernoulli) condition. For an arbitrary fixed integer  $N \geq 3$ , Pruss [9] constructed a (not strictly stationary) sequence of bounded, nondegenerate,  $N$ -tuplewise independent, identically distributed random variables for which the CLT fails to hold. In that paper, Pruss left open the question whether, for any integer  $N \geq 3$ , a strictly stationary counterexample exists. For  $N = 3$ , Bradley [3, Theorem 1] answered that question by showing that the counterexample in [2, Theorem 1] alluded to above is in fact triplewise independent.

In a similar spirit, for an arbitrary integer  $N \geq 2$ , Flaminio [6] constructed a strictly stationary, finite-state,  $N$ -tuplewise independent random sequence  $X := (X_k, k \in \mathbf{Z})$  which also has zero entropy and is mixing (in the ergodic-theoretic sense). That paper explicitly left open the question of whether those examples satisfy the CLT.

In this paper, we shall answer affirmatively the question in [9], by constructing for an arbitrary fixed integer  $N \geq 2$  a strictly stationary,  $N$ -tuplewise independent sequence of bounded, nondegenerate random variables such that the CLT fails to hold. The construction will be in part an adaptation of the (not strictly stationary) counterexample in [9].

Before the main result is stated, a few notations are needed:

The Borel  $\sigma$ -field on the real number line  $\mathbf{R}$  will be denoted  $\mathcal{R}$ .

Convergence in distribution will be denoted by  $\Rightarrow$ .

The set of positive integers will be denoted by  $\mathbf{N}$ . For a given sequence  $X := (X_k, k \in \mathbf{Z})$  of random variables, the partial sums will be denoted for  $n \in \mathbf{N}$  by

$$S_n := S(X, n) := X_1 + X_2 + \dots + X_n. \quad (1.1)$$

Here is our main result:

**Theorem 1.1.** *Suppose  $N$  is an integer such that  $N \geq 2$ . Then there exists a strictly stationary sequence  $X := (X_k, k \in \mathbf{Z})$  of random variables such that the following statements hold:*

(A) The random variable  $X_0$  is uniformly distributed on the interval  $[-3^{1/2}, 3^{1/2}]$  (and hence  $EX_0 = 0$  and  $EX_0^2 = 1$ ).

(B) For every choice of  $N$  distinct integers  $k(1), k(2), \dots, k(N)$ , the random variables  $X_{k(1)}, X_{k(2)}, \dots, X_{k(N)}$  are independent.

(C) The random variables  $|X_k|$ ,  $k \in \mathbf{Z}$  are independent (and identically distributed).

(D) For every infinite set  $Q \subset \mathbf{N}$ , there exist an infinite set  $T \subset Q$  and a nondegenerate, non-normal probability measure  $\mu$  on  $(\mathbf{R}, \mathcal{R})$  such that  $S_n/n^{1/2} \Rightarrow \mu$  as  $n \rightarrow \infty$ ,  $n \in T$ .

Here are some comments on Theorem 1.1:

By property (A), the “natural normalization” to consider for the central limit question for the partial sums of this sequence is  $S_n/n^{1/2}$ .

Property (B) is of course  $N$ -tuplewise independence.

By property (D) and the Theorem of Types (see e.g. [1, p. 193, Theorem 14.2]), there do not exist constants  $a_n > 0$  and  $b_n \in \mathbf{R}$  for  $n \in \mathbf{N}$  such that  $a_n S_n + b_n \Rightarrow N(0, 1)$ , even along a subsequence of the positive integers.

Also by property (D) and an elementary argument, there do not exist constants  $b_n \in \mathbf{R}$  for  $n \in \mathbf{N}$  such that  $n^{-1/2} S_n + b_n \rightarrow 0$  in probability, even along a subsequence of the positive integers.

In property (D), the probability measure  $\mu$  may depend on the set  $Q$ .

**Remark 1.2.** With essentially the same construction, one obtains an analog of Theorem 1.1 with property (A) replaced by the following one: (A') The random variable  $X_0$  has the  $N(0, 1)$  distribution.

We did not investigate the ergodic-theoretic properties of the sequence  $X$  in Theorem 1.1, nor did we try to ascertain the particular class of probability measures  $\mu$  that can arise in statement (D) there. Bradley [4] gives a construction of a (nondegenerate, two-state) strictly stationary, 5-tuplewise independent random sequence which fails to satisfy the CLT (instead, it satisfies property (D) in Theorem 1.1), and also has the extra properties of being “causal” (for an appropriate use of that term) and therefore “Bernoulli” (i.e. isomorphic to a Bernoulli shift) and also having a trivial double tail  $\sigma$ -field. There does not seem to be an obvious way to build those extra properties into the construction here for Theorem 1.1.

The proof of Theorem 1.1 will be given in sections 2 and 3. It will be rather intricate and will require a fair amount of notation. Refer to properties (A) and (C) in Theorem 1.1. The construction will basically involve taking a sequence of independent, identically distributed random variables uniformly distributed on the interval  $[-3^{1/2}, 3^{1/2}]$ , and changing the signs of the variables in such a way as to introduce a dependence that preserves  $N$ -tuplewise independence but is incompatible with the CLT.

In essence it involves the conversion of the (not strictly stationary) counterexample in [9] to one that is strictly stationary. The main ideas for that conversion were outlined in an e-mail message by one of the authors (Pruss [10]) to the other author (RCB), and are developed in section 3 here. Section 2 gives some vital “preliminary” information; much of it is taken or adapted from [9], but will be given again here in detail because

of considerable extra complications in our context. The use (in both sections 2 and 3) of higher order moments to establish property (D) in Theorem 1.1, is adapted from the analogous use of 6th moments for the same purpose in (an earlier, 2006 version of) the preprint [4]. Underlying all this is the repeated creation of “big” collections of  $N$ -tuplewise independent random variables from “smaller” ones; such procedures are well known in the theory of error-correcting codes (see e.g. [8]).

## 2. Part 1 of proof of Theorem 1.1

Sections 2 and 3 together will give the proof of Theorem 1.1. Both sections will be divided into several “steps,” including a “definition,” some “lemmas,” etc. Throughout this proof, the setting is a probability space  $(\Omega, \mathcal{F}, P)$ , “enlarged” as necessary to accommodate all random variables defined in this proof.

**Step 2.1.** Let  $L$  be an arbitrary fixed integer such that

$$L \text{ is even and } L \geq 6. \tag{2.1}$$

To prove Theorem 1.1, it suffices to construct a strictly stationary,  $(L - 1)$ -tuplewise independent random sequence  $X := (X_k, k \in \mathbf{Z})$  that also satisfies properties (A), (C), and (D) in Theorem 1.1. That will be the goal of sections 2 and 3.

**Step 2.2.** The following notations and conventions will be used:

(a) Refer to (2.1). For  $n \in \{0, 1, 2, \dots\}$ , when the term  $L^n$  appears in a subscript or exponent, it will be written as  $L \uparrow n$  for typographical convenience.

(b) Suppose  $n \in \mathbf{N}$ . A vector  $x \in \mathbf{R}^n$  will often be represented as  $x := (x_0, x_1, \dots, x_{n-1})$  (instead of  $(x_1, x_2, \dots, x_n)$ ), for “bookkeeping” convenience. For a given  $x := (x_0, x_1, \dots, x_{n-1}) \in \mathbf{R}^n$ , define the two real numbers

$$\text{sum } x := \sum_{i=0}^{n-1} x_i \quad \text{and} \quad \text{prod } x := \prod_{i=0}^{n-1} x_i. \tag{2.2}$$

Notations of the form  $f((x_0, x_1, \dots, x_{n-1}))$  will be written simply as  $f(x_0, x_1, \dots, x_{n-1})$ .

(c) Sometimes the coordinates of a vector will be permuted. If  $n \in \mathbf{N}$ ,  $x := (x_0, x_1, \dots, x_{n-1}) \in \mathbf{R}^n$ , and  $\sigma$  is a permutation of the set  $\{0, 1, \dots, n - 1\}$ , then define the vector  $x_\sigma \in \mathbf{R}^n$  by  $x_\sigma := ((x_\sigma)_0, (x_\sigma)_1, \dots, (x_\sigma)_{n-1}) := (x_{\sigma(0)}, x_{\sigma(1)}, \dots, x_{\sigma(n-1)})$ .

(d) If  $(x_k, k \in \mathbf{Z}) \in \mathbf{R}^{\mathbf{Z}}$ , and  $a \leq b$  are integers, then the vector  $(x_a, x_{a+1}, \dots, x_b)$  will also be denoted  $(x_k : a \leq k \leq b)$ .

(e) A “measure” on the space  $\mathbf{R}$  or  $\mathbf{R}^n$  ( $n \in \mathbf{N}$ ) or  $\mathbf{R}^{\mathbf{Z}}$  will always mean a measure on the Borel  $\sigma$ -field on that space.

(f) If  $\eta$  is a random variable/vector/sequence, then the distribution of  $\eta$  on the appropriate space (such as  $\mathbf{R}$ ,  $\mathbf{R}^n$ , or  $\mathbf{R}^{\mathbf{Z}}$ ) will be denoted  $\mathcal{L}(\eta)$ . If also  $F$  is an event such that  $P(F) > 0$ , then  $\mathcal{L}(\eta | F)$  will denote the conditional distribution of  $\eta$ , given  $F$ .

(g) For a given  $n \in \mathbf{N}$ , an “ $\mathbf{R}^n$ -valued random vector” is simply a random vector with  $n$  coordinates. Of course, if  $n, m \in \mathbf{N}$ ,  $\eta_1$  and  $\eta_2$  are  $\mathbf{R}^n$ -valued random vectors,  $\mathcal{L}(\eta_1) = \mathcal{L}(\eta_2)$  (on  $\mathbf{R}^n$ ), and  $f : \mathbf{R}^n \rightarrow \mathbf{R}^m$  is a Borel function, then  $\mathcal{L}(f(\eta_1)) = \mathcal{L}(f(\eta_2))$  (on  $\mathbf{R}^m$ ).

(h) Several notations will be defined here. Suppose  $Y := (Y_i, i \in I)$  is a family of random variables, where  $I$  is a nonempty (possibly infinite) index set. The  $\sigma$ -field of events generated by this family will be denoted  $\sigma(Y)$  or  $\sigma(Y_i, i \in I)$ . To avoid any confusion, this family  $Y$  is said to satisfy “ $(L - 1)$ -tuplewise independence” if either (i)  $\text{card } I = 1$ , or (ii)  $\text{card } I \geq 2$  and for every set  $S \subset I$  such that  $2 \leq \text{card } S \leq L - 1$ , the random variables  $Y_i, i \in S$  are independent. (The point there is to formally include the case  $\text{card } I = 1$  in that terminology.) Here and below, “card” means cardinality. In the case of a random vector  $Y := (Y_0, Y_1, \dots, Y_{n-1})$ , where  $n \in \mathbf{N}$ , the phrase “ $Y$  satisfies  $(L - 1)$ -tuplewise independence” simply means that the family of its coordinates  $(Y_i, i \in \{0, 1, \dots, n - 1\})$  satisfies  $(L - 1)$ -tuplewise independence.

**Step 2.3.** Here a few useful functions on  $\mathbf{R}^n$  will be defined.

**Step 2.3(A).** For each  $n \in \mathbf{N}$ , define the function  $\varphi_n : \mathbf{R}^n \rightarrow \mathbf{R}^n$  as follows: For  $x \in \mathbf{R}^n$  (see (2.2)),

$$\varphi_n(x) := \begin{cases} x & \text{if } \text{sum } x > 0 \\ 0_n & \text{if } \text{sum } x = 0 \\ -x & \text{if } \text{sum } x < 0 \end{cases} \quad (2.3)$$

where  $0_n$  denotes the origin in  $\mathbf{R}^n$ .

Thus for example  $\varphi_3(5, -7, 4) = \varphi_3(-5, 7, -4) = (5, -7, 4)$ .

*Remark 1.* For each  $n \in \mathbf{N}$  and each  $x \in \mathbf{R}^n$ ,  $\text{sum } \varphi_n(x) = |\text{sum } x|$ .

*Remark 2.* Suppose  $n \in \mathbf{N}$ ,  $x := (x_0, x_1, \dots, x_{n-1}) \in \mathbf{R}^n$ ,  $y := \varphi_n(x) := (y_0, y_1, \dots, y_{n-1})$ , and  $\sigma$  is a permutation of the set  $\{0, 1, \dots, n - 1\}$ . Recall from step 2.2(c) the notations  $x_\sigma := (x_{\sigma(0)}, x_{\sigma(1)}, \dots, x_{\sigma(n-1)})$  and  $y_\sigma := (y_{\sigma(0)}, y_{\sigma(1)}, \dots, y_{\sigma(n-1)})$ . Then  $y_\sigma = \varphi_n(x_\sigma)$ . This holds by a careful trivial argument, using the fact that  $\text{sum } x_\sigma = \text{sum } x$ . Thus for  $0 \leq i \leq n - 1$  (see step 2.2(c) again),  $(\varphi_n(x_\sigma))_i = (y_\sigma)_i = y_{\sigma(i)} = (\varphi_n(x))_{\sigma(i)}$ .

*Remark 3.* If  $n, x$ , and  $y$  are as in Remark 2 above (with  $y = \varphi_n(x)$ ), and also  $\text{sum } x \neq 0$ , then  $|x_i| = |y_i|$  for each  $i \in \{0, 1, \dots, n - 1\}$ .

*Remark 4.* Suppose  $n \in \mathbf{N}$ ,  $Y$  is an  $\mathbf{R}^n$ -valued random vector (recall step 2.2(g)) such that  $\mathcal{L}(-Y) = \mathcal{L}(Y)$  and  $\mathcal{L}(Y)$  is absolutely continuous with respect to Lebesgue measure on  $\mathbf{R}^n$ , and  $V$  is a random variable independent of  $Y$  such that  $P(V = -1) = P(V = 1) = 1/2$ . Then  $P(\text{sum } Y = 0) = 0$ ; and for any Borel set  $B \subset \{x \in \mathbf{R}^n : \text{sum } x > 0\}$ ,

$$P(Y \in B) = P(-Y \in B) = (1/2) \cdot P(\varphi_n(Y) \in B);$$

and hence by a trivial argument,  $\mathcal{L}(V\varphi_n(Y)) = \mathcal{L}(Y)$ .

**Step 2.3(B).** Refer to (2.1). Occasionally, “big” vectors will be created by the splicing together of  $L$  “smaller” ones. Suppose  $n$  is a positive integer, and for each  $\ell \in \{0, 1, \dots, L - 1\}$ ,  $x^{(\ell)} := (x_0^{(\ell)}, x_1^{(\ell)}, \dots, x_{n-1}^{(\ell)}) \in \mathbf{R}^n$ . Then the notation

$$\left\langle x^{(0)} \mid x^{(1)} \mid x^{(2)} \mid \dots \mid x^{(L-1)} \right\rangle \quad (2.4)$$

means the vector  $y := (y_0, y_1, y_2, \dots, y_{Ln-1}) \in \mathbf{R}^{Ln}$  such that for each  $\ell \in \{0, 1, \dots, L - 1\}$ ,  $(y_{\ell n}, y_{\ell n+1}, \dots, y_{\ell n+n-1}) = x^{(\ell)}$ .

**Step 2.3(C).** For each  $n \in \mathbf{N}$  and each  $j \in \{0, 1, \dots, L-1\}$ , define the function  $\psi_{n,j} : \mathbf{R}^{Ln} \rightarrow \mathbf{R}^{Ln}$  as follows: Suppose  $y \in \mathbf{R}^{Ln}$ . Represent  $y := \langle x^{(0)} | x^{(1)} | \dots | x^{(L-1)} \rangle$  as in (2.4), where for each  $\ell \in \{0, 1, \dots, L-1\}$ ,  $x^{(\ell)} \in \mathbf{R}^n$ . Then (see (2.2)) define

$$\psi_{n,j}(y) := \begin{cases} y & \text{if } \prod_{\ell=0}^{L-1} (\text{sum } x^{(\ell)}) \leq 0 \\ \langle x^{(0)} | \dots | x^{(j-1)} | -x^{(j)} | x^{(j+1)} | \dots | x^{(L-1)} \rangle & \text{if } \prod_{\ell=0}^{L-1} (\text{sum } x^{(\ell)}) > 0. \end{cases} \quad (2.5)$$

That is, if  $\prod_{\ell=0}^{L-1} (\text{sum } x^{(\ell)}) > 0$ , then to define  $\psi_{n,j}(y)$ , one just multiplies by  $-1$  the coordinates of  $x^{(j)}$  (the  $(j+1)^{\text{st}}$  “ $n$ -coordinate piece” of  $y$ ); all other coordinates of  $y$  are left alone.

*Preview.* We shall show later on that if for each  $\ell \in \{0, 1, \dots, L-1\}$ ,  $Y^{(\ell)}$  is an  $\mathbf{R}^n$ -valued random vector satisfying  $(L-1)$ -tuplewise independence (see (2.1)) and certain other conditions, and these random vectors  $Y^{(\ell)}$ ,  $\ell \in \{0, 1, \dots, L-1\}$  are independent of each other, and one defines the  $\mathbf{R}^{Ln}$ -valued random vector  $Y := \langle Y^{(0)} | Y^{(1)} | \dots | Y^{(L-1)} \rangle$ , then for any  $j \in \{0, 1, \dots, L-1\}$ , the  $\mathbf{R}^{Ln}$ -valued random vector  $\psi_{n,j}(Y)$  satisfies  $(L-1)$ -tuplewise independence. Iterative use of this construction will generate the final sequence  $X := (X_k, k \in \mathbf{Z})$  for Theorem 1.1.

**Step 2.4.** Here several special probability measures on  $\mathbf{R}$  or  $\mathbf{R}^n$  or  $\mathbf{R}^{\mathbf{Z}}$  will be defined.

**Step 2.4(A).** The notation  $\lambda_{\text{unps3}}$  will refer to the uniform distribution on the interval  $[-3^{1/2}, 3^{1/2}]$ , regarded as a probability measure on  $\mathbf{R}$ . (The subscript “unps3” stands for **u**niform on the interval from **n**egative to **p**ositive **s**quare root of **3**.) For any positive integer  $m$ , the  $m$ -fold product measure  $\lambda_{\text{unps3}} \times \lambda_{\text{unps3}} \times \dots \times \lambda_{\text{unps3}}$  on  $\mathbf{R}^m$  will be denoted  $\lambda_{\text{unps3}}^{[m]}$ .

*Remark.* If  $U$  is a random variable such that  $\mathcal{L}(U) = \lambda_{\text{unps3}}$  (see step 2.2(f)), then of course  $EU^n = 0$  for odd  $n \in \mathbf{N}$  and  $EU^n = (n+1)^{-1} \cdot 3^{n/2}$  for even  $n \in \mathbf{N}$ . Hence, if also  $Z$  is a  $N(0, 1)$  random variable, then (see e.g. [1, p. 275, eq. (21.7)]) for every  $n \in \mathbf{N}$ ,  $EZ^n \geq EU^n \geq 0$  (with equality for odd  $n$  and with  $EZ^2 = EU^2 = 1$ ).

**Step 2.4(B).** Suppose  $n \in \mathbf{N}$ , and  $\mu$  is a probability measure on  $\mathbf{R}^n$ . Then the notation  $\mathcal{D}(n, \mu)$  means the distribution on  $\mathbf{R}^{\mathbf{Z}}$  of a random sequence  $Y := (Y_k, k \in \mathbf{Z})$  such that (i) for each  $u \in \mathbf{Z}$ , the random vector  $\zeta^{(u)} := (Y_{nu}, Y_{nu+1}, Y_{nu+2}, \dots, Y_{nu+n-1})$  satisfies  $\mathcal{L}(\zeta^{(u)}) = \mu$ , and (ii) these random vectors  $\zeta^{(u)}$ ,  $u \in \mathbf{Z}$  are independent of each other.

**Step 2.4(C).** Define (see (2.1) and (2.2)) the set  $\Upsilon := \{x \in \{-1, 1\}^L : \text{prod } x = -1\}$ . Then  $\text{card } \Upsilon = 2^{L-1}$ . Let  $\nu$  denote the uniform distribution on  $\Upsilon$  — that is, the probability measure  $\nu$  on  $\mathbf{R}^L$  such that  $\nu(\{x\}) = 1/2^{L-1}$  for each  $x \in \Upsilon$ .

**Step 2.4(D).** Suppose  $n \in \mathbf{N}$ , and  $\mu$  is a probability measure on  $\mathbf{R}^n$ . Then define the probability measure  $\theta(\mu)$  on  $\mathbf{R}^{Ln}$  by  $\theta(\mu) := \mathcal{L}(Y)$  where the random vector  $Y := (Y_0, Y_1, \dots, Y_{Ln-1})$  is as defined below:

Let  $W^{(0)}, W^{(1)}, \dots, W^{(L-1)}$  each be an  $\mathbf{R}^n$ -valued random vector, such that  $\mathcal{L}(W^{(\ell)}) = \mu$  for each  $\ell \in \{0, 1, \dots, L-1\}$ , and such that these random vectors  $W^{(\ell)}$ ,  $\ell \in \{0, 1, \dots, L-1\}$  are independent of each other. Let  $V := (V_0, V_1, \dots, V_{L-1})$  be a  $\Upsilon$ -valued random vector such that  $\mathcal{L}(V) = \nu$  (see step 2.4(C) above), with  $V$  being independent of the family

$(W^{(\ell)}, \ell \in \{0, 1, \dots, L-1\})$ . Referring to step 2.3(A)(B), define the  $\mathbf{R}^{Ln}$ -valued random vector  $Y$  by

$$Y := \left\langle V_0 \varphi_n(W^{(0)}) \mid V_1 \varphi_n(W^{(1)}) \mid V_2 \varphi_n(W^{(2)}) \mid \cdots \mid V_{L-1} \varphi_n(W^{(L-1)}) \right\rangle. \quad (2.6)$$

**Remark 2.5.** Suppose  $V := (V_0, V_1, \dots, V_{L-1})$  is a random vector such that  $\mathcal{L}(V) = \nu$  (see (2.1) and step 2.4(C)). Then by elementary arguments, the following statements hold:

- (i) For each  $k \in \{0, 1, \dots, L-1\}$ ,  $P(V_k = -1) = P(V_k = 1) = 1/2$ .
- (ii)  $V$  satisfies  $(L-1)$ -tuplewise independence.
- (iii)  $\mathcal{L}(-V) = \nu$ .
- (iv) For every permutation  $\sigma$  on the set  $\{0, 1, \dots, L-1\}$ , the random vector  $V_\sigma := (V_{\sigma(0)}, V_{\sigma(1)}, \dots, V_{\sigma(L-1)})$  (see step 2.2(c)) satisfies  $\mathcal{L}(V_\sigma) = \nu$ .
- (v)  $\text{prod } V = -1$  a.s. (see (2.2)). Hence for any  $j \in \{0, 1, \dots, L-1\}$ ,  $V_j = -\prod_{\ell \in \{0, 1, \dots, L-1\} - \{j\}} V_\ell$  a.s.

**Definition 2.6.** Suppose  $n \in \mathbf{N}$ , and  $\mu$  is a probability measure on  $\mathbf{R}^n$ . Then  $\mu$  is said to satisfy ‘‘Condition  $\mathcal{H}(n)$ ’’ if the following seven statements (a)–(g) hold, where  $W := (W_0, W_1, \dots, W_{n-1})$  is a random vector such that  $\mathcal{L}(W) = \mu$ :

- (a)  $\mu$  is absolutely continuous with respect to Lebesgue measure on  $\mathbf{R}^n$ .
- (b) For each  $k \in \{0, 1, \dots, n-1\}$ ,  $\mathcal{L}(W_k) = \lambda_{\text{unps3}}$  (see step 2.4(A)).
- (c)  $W$  satisfies  $(L-1)$ -tuplewise independence.
- (d) The random variables  $|W_0|, |W_1|, \dots, |W_{n-1}|$  are independent (if  $n \geq 2$ ).
- (e)  $\mathcal{L}(-W) = \mu$ .
- (f) For any  $j \in \{0, 1, \dots, n-1\}$ , there exists a permutation  $\sigma$  on the set  $\{0, 1, \dots, n-1\}$  such that (i)  $\sigma(0) = j$  and (ii)  $\mathcal{L}(W_\sigma) = \mu$  (where  $W_\sigma := (W_{\sigma(0)}, W_{\sigma(1)}, \dots, W_{\sigma(n-1)})$ ) — see step 2.2(c).
- (g) Either (i)  $n < L$ , or (ii)  $n \geq L$ , and for every set  $S \subset \{0, 1, \dots, n-1\}$  such that  $\text{card } S = L$ , one has that  $E(\prod_{i \in S} W_i) \leq 0$ .

**Lemma 2.7.** Suppose  $n \in \mathbf{N}$ , and  $W := (W_0, W_1, \dots, W_{n-1})$  is a random vector whose distribution  $\mathcal{L}(W)$  on  $\mathbf{R}^n$  satisfies Condition  $\mathcal{H}(n)$ . Referring to (2.3), define the random vector  $Y := \varphi_n(W) := (Y_0, Y_1, \dots, Y_{n-1})$ . Then the following three statements hold (see (2.2)):

- (i)  $E(\text{sum } Y) = E|\text{sum } W| \geq (1/2)n^{1/2}$ .
- (ii)  $\mathcal{L}(Y_0) = \mathcal{L}(Y_1) = \cdots = \mathcal{L}(Y_{n-1})$ .
- (iii) For any  $k \in \{0, 1, \dots, n-1\}$ ,  $EY_k = (1/n) \cdot E|\text{sum } W|$ .

**Proof.** Let us first prove (i). Trivially  $\text{sum } Y = |\text{sum } W|$  by (2.3). We just need to prove the (‘‘latter’’) inequality in (i).

Refer to (2.1) and Definition 2.6(b)(c). By simple calculations, including the argument in [1, p. 85, proof of Theorem 6.1] (which uses only 4-tuplewise independence), one has that for each  $k \in \{0, 1, \dots, n-1\}$ ,  $\mathcal{L}(W_k) = \lambda_{\text{unps3}}$ ,  $EW_k = 0$ ,  $EW_k^2 = 1$ , and  $EW_k^4 = 9/5$ , and hence  $E(\text{sum } W)^2 = n$  and

$$E(\text{sum } W)^4 = n \cdot EW_0^4 + 3n(n-1) \cdot (EW_0^2)^2 < 3n^2.$$

Hence by Hölder's inequality,

$$\begin{aligned}
n = E(\text{sum } W)^2 &\leq \left\| |\text{sum } W|^{2/3} \right\|_{3/2} \cdot \left\| |\text{sum } W|^{4/3} \right\|_3 \\
&= [E|\text{sum } W|]^{2/3} \cdot [E(\text{sum } W)^4]^{1/3} \\
&\leq [E|\text{sum } W|]^{2/3} \cdot (3n^2)^{1/3}.
\end{aligned}$$

Hence  $n^{3/2} \leq E|\text{sum } W| \cdot (3n^2)^{1/2}$ . Hence the (“latter”) inequality in (i) holds.

Now let us prove (ii). Suppose  $j \in \{0, 1, \dots, n-1\}$ . Referring to Definition 2.6(f), let  $\sigma$  be a permutation of the set  $\{0, 1, \dots, n-1\}$  such that  $\sigma(0) = j$  and  $\mathcal{L}(W_\sigma) = \mathcal{L}(W)$ . Then  $\mathcal{L}(\varphi_n(W_\sigma)) = \mathcal{L}(\varphi_n(W)) = \mathcal{L}(Y)$ . Also,  $Y_\sigma = \varphi_n(W_\sigma)$  by Remark 2 in step 2.3(A), and hence  $\mathcal{L}(Y_\sigma) = \mathcal{L}(Y)$ . Hence  $\mathcal{L}(Y_j) = \mathcal{L}(Y_{\sigma(0)}) = \mathcal{L}(Y_0)$ . Since  $j \in \{0, 1, \dots, n-1\}$  was arbitrary, (ii) follows.

Statement (iii) follows trivially from statements (i) and (ii).

**Lemma 2.8.** *Suppose  $n \in \mathbf{N}$ , and  $\mu$  is a probability measure on  $\mathbf{R}^n$  that satisfies Condition  $\mathcal{H}(n)$ . Then the following statements hold:*

- (A) *The probability measure  $\theta(\mu)$  on  $\mathbf{R}^{Ln}$  (see step 2.4(D)) satisfies Condition  $\mathcal{H}(Ln)$ .*
- (B) *Suppose  $W$  is an  $\mathbf{R}^{Ln}$ -valued random vector of the form*

$$W := (W_0, W_1, \dots, W_{Ln-1}) := \left\langle \zeta^{(0)} \mid \zeta^{(1)} \mid \zeta^{(2)} \mid \dots \mid \zeta^{(L-1)} \right\rangle \quad (2.7)$$

(see (2.4)) where (i) for each  $\ell \in \{0, 1, \dots, L-1\}$ ,  $\zeta^{(\ell)}$  is an  $\mathbf{R}^n$ -valued random vector such that  $\mathcal{L}(\zeta^{(\ell)}) = \mu$ , and (ii) these random vectors  $\zeta^{(\ell)}$ ,  $\ell \in \{0, 1, \dots, L-1\}$  are independent of each other. Then (see (2.5)) for every  $j \in \{0, 1, \dots, L-1\}$ ,  $\mathcal{L}(\psi_{n,j}(W)) = \theta(\mu)$ .

(C) *Suppose the  $\mathbf{R}^{Ln}$ -valued random vector  $W$  is as in statement (B) (satisfying all conditions there), and  $Y := (Y_0, Y_1, \dots, Y_{Ln-1})$  is an  $\mathbf{R}^{Ln}$ -valued random vector such that  $\mathcal{L}(Y) = \theta(\mu)$ . Suppose  $S$  is a nonempty subset of  $\{0, 1, \dots, Ln-1\}$ . Then the following statements (i), (ii), (iii) hold:*

- (i) *For each integer  $M \in \{1, 2, \dots, L-1\}$ ,  $E(\sum_{k \in S} Y_k)^M = E(\sum_{k \in S} W_k)^M$ . (In the case where  $M$  is odd, both sides of that equality are 0.)*
- (ii)  *$E(\sum_{k \in S} Y_k)^L \leq E(\sum_{k \in S} W_k)^L$ .*
- (iii) *For the case where  $S = \{0, 1, \dots, Ln-1\}$  itself, one has (see (2.2))*

$$E(\text{sum } W)^L - E(\text{sum } Y)^L \geq 2^{-L} \cdot L! \cdot n^{L/2}. \quad (2.8)$$

Statements (A) and (B) and their proofs are taken or adapted from [9]. Their proofs will be included here because of extra complications in our context.

**Proof.** The proofs of statements (B), (A), and (C) will be given in that order.

**Proof of statement (B).** Before the random vector  $W$  in statement (B) is brought into the picture, some preliminary work is needed.

Let  $j \in \{0, 1, \dots, L-1\}$  be arbitrary but fixed. (See the role of  $j$  in statement (B).)



Let  $V'_0, V'_1, \dots, V'_{L-1}$  be independent, identically distributed  $\{-1, 1\}$ -valued random variables such that  $P(V'_0 = -1) = P(V'_0 = 1) = 1/2$ . Define the random vector  $V^* := (V_0^*, V_1^*, \dots, V_{L-1}^*)$  as follows: For each  $\ell \in \{0, 1, \dots, L-1\} - \{j\}$ ,  $V_\ell^* := V'_\ell$ ; and

$$V_j^* := - \prod_{\ell \in \{0, 1, \dots, L-1\} - \{j\}} V_\ell^*. \quad (2.9)$$

Now the random variables  $V_\ell^*$ ,  $\ell \in \{0, 1, \dots, L-1\} - \{j\}$  are independent and identically distributed,  $P(V_\ell^* = -1) = P(V_\ell^* = 1) = 1/2$  for each  $\ell \in \{0, 1, \dots, L-1\} - \{j\}$ , and (2.9) holds. These conditions together uniquely determine  $\mathcal{L}(V^*)$ . Also, the random vector  $V$  (with  $\mathcal{L}(V) = \nu$ ) in Remark 2.5 satisfies the analogs of those conditions (see Remark 2.5(i)(ii)(v)). It follows that

$$\mathcal{L}(V^*) = \nu. \quad (2.10)$$

Next, let  $\xi^{(0)}, \xi^{(1)}, \dots, \xi^{(L-1)}$  be independent  $\mathbf{R}^n$ -valued random vectors such that (i)  $\mathcal{L}(\xi^{(\ell)}) = \mu$  for each  $\ell \in \{0, 1, \dots, L-1\}$  and (ii)  $\sigma(\xi^{(0)}, \xi^{(1)}, \dots, \xi^{(L-1)})$  is independent of  $\sigma(V')$  (and hence independent of  $\sigma(V^*)$ ) — see step 2.2(h). Define the  $\mathbf{R}^{Ln}$ -valued random vectors  $W'$  and  $W^*$  by (see (2.3) and (2.4))

$$\begin{aligned} W' &:= (W'_0, W'_1, \dots, W'_{L-1}) \\ &:= \left\langle V'_0 \varphi_n(\xi^{(0)}) \mid V'_1 \varphi_n(\xi^{(1)}) \mid \dots \mid V'_{L-1} \varphi_n(\xi^{(L-1)}) \right\rangle \end{aligned} \quad (2.11)$$

and

$$\begin{aligned} W^* &:= (W_0^*, W_1^*, \dots, W_{L-1}^*) \\ &:= \left\langle V_0^* \varphi_n(\xi^{(0)}) \mid V_1^* \varphi_n(\xi^{(1)}) \mid \dots \mid V_{L-1}^* \varphi_n(\xi^{(L-1)}) \right\rangle. \end{aligned} \quad (2.12)$$

By (2.10), (2.12), and step 2.4(D),

$$\mathcal{L}(W^*) = \theta(\mu). \quad (2.13)$$

Since (by hypothesis)  $\mu$  satisfies Condition  $\mathcal{H}(n)$ , one has by Definition 2.6(a)(e) and Remark 4 in step 2.3(A) that  $\forall \ell \in \{0, 1, \dots, L-1\}$ ,

$$\mathcal{L}(V'_\ell \varphi_n(\xi^{(\ell)})) = \mathcal{L}(\xi^{(\ell)}) = \mu. \quad (2.14)$$

Also, the random vectors  $V'_\ell \varphi_n(\xi^{(\ell)})$ ,  $\ell \in \{0, 1, \dots, L-1\}$  are independent (by the properties above). Hence by (2.11) and (2.7) and the (other) assumptions in statement (B),

$$\mathcal{L}(W') = \mathcal{L}(W). \quad (2.15)$$

Our next task is to compare the random vectors  $W'$  and  $W^*$ . For that purpose, observe that by (2.14) and Definition 2.6(a) (and the hypothesis of Lemma 2.8),

$$P \left( \prod_{\ell=0}^{L-1} \text{sum} \left( V'_\ell \varphi_n(\xi^{(\ell)}) \right) = 0 \right) = 0,$$

and for all  $\omega \in \Omega$  not in that event,

$$V_j^*(\omega)\varphi_n(\xi^{(j)}(\omega)) = \begin{cases} -V_j'(\omega)\varphi_n(\xi^{(j)}(\omega)) & \text{if } \prod_{\ell=0}^{L-1} \text{sum}(V_\ell'(\omega)\varphi_n(\xi^{(\ell)}(\omega))) > 0 \\ V_j'(\omega)\varphi_n(\xi^{(j)}(\omega)) & \text{if } \prod_{\ell=0}^{L-1} \text{sum}(V_\ell'(\omega)\varphi_n(\xi^{(\ell)}(\omega))) < 0. \end{cases} \quad (2.16)$$

To verify (2.16), consider first the case where  $\prod_{\ell=0}^{L-1} \text{sum}(V_\ell'(\omega)\varphi_n(\xi^{(\ell)}(\omega))) > 0$ . This can be rewritten as  $\prod_{\ell=0}^{L-1} [V_\ell'(\omega) \cdot \text{sum} \varphi_n(\xi^{(\ell)}(\omega))] > 0$ , or  $[\prod_{\ell=0}^{L-1} V_\ell'(\omega)] \cdot [\prod_{\ell=0}^{L-1} \text{sum} \varphi_n(\xi^{(\ell)}(\omega))] > 0$ . Since  $\text{sum} \varphi_n(\xi^{(\ell)}(\omega)) > 0$  for each  $\ell \in \{0, 1, \dots, L-1\}$ , this forces  $\prod_{\ell=0}^{L-1} V_\ell'(\omega) = 1$ . That in turn implies  $V_j'(\omega) = \prod_{\ell \neq j} V_\ell'(\omega)$ , and hence by (2.9) and its entire sentence,

$$V_j^*(\omega) = - \prod_{\ell \neq j} V_\ell^*(\omega) = - \prod_{\ell \neq j} V_\ell'(\omega) = -V_j'(\omega),$$

and hence (2.16) holds. In the other case, where  $\prod_{\ell=0}^{L-1} \text{sum}(V_\ell'(\omega)\varphi_n(\xi^{(\ell)}(\omega))) < 0$ , eq. (2.16) holds by a similar argument.

Now of course  $V_\ell^*\varphi_n(\xi^{(\ell)}) = V_\ell'\varphi_n(\xi^{(\ell)})$  for each  $\ell \in \{0, 1, \dots, L-1\} - \{j\}$ . Hence by (2.5), (2.11), (2.12), and (2.16) and its entire sentence,

$$\psi_{n,j}(W') = W^* \text{ a.s.} \quad (2.17)$$

Now by (2.15), (2.17), and (2.13),  $\mathcal{L}(\psi_{n,j}(W)) = \mathcal{L}(\psi_{n,j}(W')) = \mathcal{L}(W^*) = \theta(\mu)$ . Since  $j \in \{0, 1, \dots, L-1\}$  was arbitrary, statement(B) in Lemma 2.8 holds.

**Proof of statement (A).** This will be a continuation of the argument above for statement (B). (The integer  $j$  in that argument will not play a role here; as a formality, one can take, say,  $j = 0$ .) Using (2.12) and (2.13), we shall verify, one by one, the analogs of conditions (a)–(g) in Definition 2.6 for the probability measure  $\theta(\mu)$  on  $\mathbf{R}^{Ln}$ .

By (2.14) and the paragraph after (2.10),  $\mathcal{L}(\xi^{(0)}, \xi^{(1)}, \dots, \xi^{(L-1)}) = \mu \times \mu \times \dots \times \mu$  (the  $L$ -fold product measure on  $(\mathbf{R}^n)^L$ ), which (by Definition 2.6(a) for  $\mu$  and a standard measure-theoretic argument) is absolutely continuous with respect to Lebesgue measure on  $(\mathbf{R}^n)^L$ . Hence by (2.3), Definition 2.6(e) for  $\mu$ , and an elementary argument,  $\mathcal{L}(\varphi_n(\xi^{(0)}), \varphi_n(\xi^{(1)}), \dots, \varphi_n(\xi^{(L-1)}))$  is absolutely continuous. Hence so is  $\mathcal{L}(W^*)$ , by (2.12) and an elementary argument. Hence by (2.13), the analog of condition (a) in Definition 2.6 holds for  $\theta(\mu)$ .

Next, by (2.14), (2.10), Remark 2.5(i)(ii), and the paragraph after (2.10), one has that

$$\forall \ell \in \{0, 1, \dots, L-1\}, \quad \mathcal{L}(V_\ell^*\varphi_n(\xi^{(\ell)})) = \mathcal{L}(V_\ell'\varphi_n(\xi^{(\ell)})) = \mu, \quad (2.18)$$

and also that every  $L-1$  of the random vectors  $V_\ell^*\varphi_n(\xi^{(\ell)})$ ,  $\ell \in \{0, 1, \dots, L-1\}$  are independent. Hence by (2.12), Definition 2.6(b)(c) for  $\mu$ , and a simple argument, one has that  $\mathcal{L}(W_k^*) = \lambda_{\text{unps3}}$  for each  $k \in \{0, 1, \dots, Ln-1\}$ , and that the random vector  $W^*$  satisfies  $(L-1)$ -tuplewise independence. Hence by (2.13), the analogs of conditions (b) and (c) in Definition 2.6 hold for  $\theta(\mu)$ .

Next, by (2.7) and its entire sentence, together with Definition 2.6(d) for  $\mu$ , the random variables  $|W_i|$ ,  $i \in \{0, 1, \dots, Ln-1\}$  are independent. Hence by (2.15), the same holds

for the random variables  $|W'_i|$ ,  $0 \leq i \leq Ln - 1$ . Now by (2.11), (2.12), and the entire paragraph containing (2.9),  $|W_i^*| = |W'_i|$  for each  $i \in \{0, 1, \dots, Ln - 1\}$ ; and hence the random variables  $|W_i^*|$ ,  $0 \leq i \leq Ln - 1$  are independent. Hence by (2.12) and (2.13), the analog of condition (d) in Definition 2.6 holds for  $\theta(\mu)$ .

Next, by (2.10), Remark 2.5(iii), and the paragraph after (2.10),

$$\begin{aligned} \mathcal{L}(V^*, \xi^{(0)}, \xi^{(1)}, \dots, \xi^{(L-1)}) &= \nu \times \mu \times \mu \times \dots \times \mu \\ &= \mathcal{L}(-V^*, \xi^{(0)}, \xi^{(1)}, \dots, \xi^{(L-1)}). \end{aligned} \quad (2.19)$$

Also, by (2.12),  $W^* = f(V^*, \xi^{(0)}, \xi^{(1)}, \dots, \xi^{(L-1)})$  for an obvious Borel function  $f : \mathbf{R}^L \times (\mathbf{R}^n)^L \rightarrow \mathbf{R}^{nL}$ , and  $-W^* = f(-V^*, \xi^{(0)}, \xi^{(1)}, \dots, \xi^{(L-1)})$ . Hence  $\mathcal{L}(-W^*) = \mathcal{L}(W^*)$  by (2.19). Hence by (2.13), the analog of condition (e) in Definition 2.6 holds for  $\theta(\mu)$ .

*Proof of Definition 2.6(f) for  $\theta(\mu)$ .* Refer to the notations in step 2.2(c), and refer to Remark 2 in step 2.3(A).

Let  $J \in \{0, 1, \dots, Ln - 1\}$  be arbitrary but fixed. Refer to (2.12) and (2.13). To prove the analog of property (f) in Definition 2.6 for  $\theta(\mu)$ , it suffices to show that there exists a permutation  $\tau$  of the set  $\{0, 1, \dots, Ln - 1\}$  such that

$$\tau(0) = J \text{ and } \mathcal{L}(W_\tau^*) = \mathcal{L}(W^*). \quad (2.20)$$

Let  $\ell' \in \{0, 1, \dots, L - 1\}$  and  $j' \in \{0, 1, \dots, n - 1\}$  be such that

$$J = \ell'n + j'. \quad (2.21)$$

Let  $\alpha$  be a permutation of the set  $\{0, 1, \dots, L - 1\}$  such that  $\alpha(0) = \ell'$ . Then by (2.19), (2.10), and Remark 2.5(iv),

$$\mathcal{L}\left(V_\alpha^*, \xi^{\alpha(0)}, \xi^{\alpha(1)}, \dots, \xi^{\alpha(L-1)}\right) = \nu \times \mu \times \mu \times \dots \times \mu. \quad (2.22)$$

Applying Definition 2.6(f) for  $\mu$  (see (2.14)), let  $\beta$  be a permutation of the set  $\{0, 1, \dots, n - 1\}$  such that  $\beta(0) = j'$  and (say)  $\mathcal{L}(\xi_\beta^{\alpha(0)}) = \mu$ . Then by (2.22) and a trivial argument,

$$\mathcal{L}\left(V_\alpha^*, \xi_\beta^{\alpha(0)}, \xi^{\alpha(1)}, \xi^{\alpha(2)}, \dots, \xi^{\alpha(L-1)}\right) = \nu \times \mu \times \mu \times \dots \times \mu. \quad (2.23)$$

Define the  $\mathbf{R}^{Ln}$ -valued random vectors  $W^{**}$  and  $W^{***}$  by

$$\begin{aligned} W^{**} &:= (W_0^{**}, W_1^{**}, \dots, W_{Ln-1}^{**}) \\ &:= \left\langle (V_\alpha^*)_0 \varphi_n(\xi^{\alpha(0)}) \mid (V_\alpha^*)_1 \varphi_n(\xi^{\alpha(1)}) \mid \dots \mid (V_\alpha^*)_{L-1} \varphi_n(\xi^{\alpha(L-1)}) \right\rangle \end{aligned} \quad (2.24)$$

and

$$\begin{aligned} W^{***} &:= (W_0^{***}, W_1^{***}, \dots, W_{Ln-1}^{***}) \\ &:= \left\langle (V_\alpha^*)_0 \varphi_n(\xi_\beta^{\alpha(0)}) \mid (V_\alpha^*)_1 \varphi_n(\xi^{\alpha(1)}) \mid (V_\alpha^*)_2 \varphi_n(\xi^{\alpha(2)}) \mid \right. \\ &\quad \left. \dots \mid (V_\alpha^*)_{L-1} \varphi_n(\xi^{\alpha(L-1)}) \right\rangle. \end{aligned} \quad (2.25)$$

By (2.19) (its first equality), (2.22), (2.23), (2.24), (2.25), (2.12), and (2.13),

$$\mathcal{L}(W^{***}) = \mathcal{L}(W^{**}) = \mathcal{L}(W^*) = \theta(\mu). \quad (2.26)$$

By (2.21) and the sentence after it, together with (2.24) and (2.12), the random variables  $W_k^{**}$ ,  $0 \leq k \leq Ln - 1$  in (2.24) are (to put this informally) simply a permutation of the random variables  $W_k^*$ ,  $0 \leq k \leq Ln - 1$  in (2.12), such that  $W_{j'}^{**} = W_j^*$ . Similarly, by (2.24), (2.25), Remark 2 in step 2.3(A), and the sentence after (2.22), the random variables  $W_k^{***}$ ,  $0 \leq k \leq Ln - 1$  in (2.25) are simply a permutation of the random variables  $W_k^{**}$ ,  $0 \leq k \leq Ln - 1$  in (2.24), such that  $W_0^{***} = W_{j'}^{**}$ . Hence the random variables  $W_k^{***}$ ,  $0 \leq k \leq Ln - 1$  in (2.25) are simply a permutation of the random variables  $W_k^*$ ,  $0 \leq k \leq Ln - 1$  in (2.12), such that  $W_0^{***} = W_j^*$ . Thus for the resulting permutation  $\tau$  of the indices  $\{0, 1, \dots, Ln - 1\}$ , one has that  $W^{***} = W_\tau^*$  and by (2.26), equation (2.20) holds. That completes the argument for Definition 2.6(f) for  $\theta(\mu)$ .

*Proof of Definition 2.6(g) for  $\theta(\mu)$ .* Let  $S$  be an arbitrary fixed subset of  $\{0, 1, \dots, Ln - 1\}$  such that  $\text{card } S = L$ . Refer to (2.12) and (2.13). Our task is to show that  $E(\prod_{k \in S} W_k^*) \leq 0$ .

For each  $\ell \in \{0, 1, \dots, L - 1\}$ , define the set

$$\Phi(\ell) := \{\ell n, \ell n + 1, \dots, \ell n + n - 1\}. \quad (2.27)$$

These sets form a partition of the set  $\{0, 1, \dots, Ln - 1\}$ . The argument here will be divided into three cases according to how many indices  $\ell \in \{0, 1, \dots, L - 1\}$  are such that the set  $S \cap \Phi(\ell)$  is nonempty. First recall from (2.12) that for each  $\ell \in \{0, 1, \dots, L - 1\}$ ,

$$(W_{\ell n}^*, W_{\ell n + 1}^*, \dots, W_{\ell n + n - 1}^*) = V_\ell^* \varphi_n(\xi^{(\ell)}). \quad (2.28)$$

*Case 1.*  $S \subset \Phi(\ell)$  for some  $\ell \in \{0, 1, \dots, L - 1\}$ . Then by (2.28) and (2.27) (for that  $\ell$ ), (2.18), and Definition 2.6(g) for  $\mu$ ,  $E(\prod_{k \in S} W_k^*) \leq 0$ .

*Case 2.* The set  $T := \{\ell \in \{0, 1, \dots, L - 1\} : S \cap \Phi(\ell) \neq \emptyset\}$  satisfies  $2 \leq \text{card } T \leq L - 1$ . This forces  $\text{card}(S \cap \Phi(\ell)) \leq L - 1$  for each  $\ell \in T$ . For each  $\ell \in T$ , by (2.28), (2.27), (2.18), and Definition 2.6(b)(c) for  $\mu$ ,  $E(\prod_{k \in S \cap \Phi(\ell)} W_k^*) = \prod_{k \in S \cap \Phi(\ell)} E W_k^* = 0$ . Also, by (2.28), (2.27), and the phrase immediately after (2.18), the  $\sigma$ -fields  $\sigma(W_k^*, k \in S \cap \Phi(\ell))$ ,  $\ell \in T$  are independent. It follows that  $E(\prod_{k \in S} W_k^*) = 0$ .

*Case 3.* For each  $\ell \in \{0, 1, \dots, L - 1\}$ , the set  $\Phi(\ell)$  contains exactly one element of  $S$ . (Recall that  $\text{card } S = L$ .) For each  $\ell \in \{0, 1, \dots, L - 1\}$ , let  $k(\ell)$  denote the element of  $S \cap \Phi(\ell)$ . By (2.12), for each  $\ell \in \{0, 1, \dots, L - 1\}$ ,  $W_{k(\ell)}^* = V_\ell^* T_\ell$  where  $T_\ell$  is one of the coordinates of the random vector  $\varphi_n(\xi^{(\ell)})$ . For each  $\ell \in \{0, 1, \dots, L - 1\}$ ,  $E T_\ell > 0$  by (2.14) and Lemma 2.7(i)(iii). Hence by (2.10), the paragraph after (2.10), and Remark 2.5(v),

$$\begin{aligned} E \left( \prod_{k \in S} W_k^* \right) &= E \left( \prod_{\ell=0}^{L-1} W_{k(\ell)}^* \right) = E \left( \prod_{\ell=0}^{L-1} V_\ell^* \right) \cdot \prod_{\ell=0}^{L-1} E T_\ell \\ &= -1 \cdot \prod_{\ell=0}^{L-1} E T_\ell < 0. \end{aligned}$$

That completes the argument for Case 3, for Definition 2.6(g) for  $\theta(\mu)$ , and for statement (A).

**Proof of statement (C).** In this argument, for any  $\mathbf{R}^{Ln}$ -valued random vector  $\eta := (\eta_0, \eta_1, \dots, \eta_{Ln-1})$  and any nonempty set  $Q \subset \{0, 1, \dots, Ln-1\}$ , the notation  $(\eta_k, k \in Q)$  will sometimes be used to denote the random vector  $(\eta_{k(1)}, \eta_{k(2)}, \dots, \eta_{k(m)})$  where  $m = \text{card } Q$  and  $k(1), k(2), \dots, k(m)$  are in strictly increasing order the elements of  $Q$ .

We shall use appropriate arguments from the proofs of (A) and (B).

For any nonempty set  $Q \subset \{0, 1, \dots, Ln-1\}$  such that  $\text{card } Q \leq L-1$ , one has (see step 2.4(A)) that

$$\mathcal{L}(W_k, k \in Q) = \lambda_{\text{unps3}}^{[m]} = \mathcal{L}(Y_k, k \in Q). \quad (2.29)$$

The first equality holds by the entire sentence containing (2.7), together with Definition 2.6(b)(c) for  $\mu$  and a simple argument. The second equality holds by Definition 2.6(b)(c) for  $\theta(\mu)$  (recall statement (A), proved above).

Refer to the set  $S$  in statement (C). To prove item (i) in statement (C), note that if  $1 \leq M \leq L-1$ , then for any  $M$ -tuple  $(k(1), k(2), \dots, k(M)) \in S^M$  (the  $k(i)$ 's need not be distinct),  $E(\prod_{i=1}^M W_{k(i)}) = (\prod_{i=1}^M Y_{k(i)})$  by (2.29) and its entire sentence. Adding up over all  $M$ -tuples  $\in S^M$ , one obtains  $E(\sum_{k \in S} W_k)^M = E(\sum_{k \in S} Y_k)^M$ , the equality in (i). The last sentence in (i) follows from Definition 2.6(e) for  $\theta(\mu)$  (applied to  $Y$ ).

*Proof of statement (C)(ii).* Let  $(k(1), k(2), \dots, k(L))$  be an arbitrary fixed  $L$ -tuple  $\in S^L$ . (The  $k(i)$ 's need not be distinct.) In order to prove statement (C)(ii), it suffices to prove that

$$E \left( \prod_{i=1}^L Y_{k(i)} \right) \leq E \left( \prod_{i=1}^L W_{k(i)} \right). \quad (2.30)$$

(For then by adding up both sides over all  $L$ -tuples  $\in S^L$ , one would obtain statement (C)(ii).)

If two or more of the integers  $k(i)$  are equal, then (2.30) holds with equality by (2.29) and its entire sentence. Therefore, we now assume that the integers  $k(i)$  are distinct.

Now refer to the sets  $\Phi(\ell)$ ,  $\ell \in \{0, 1, \dots, L-1\}$  in (2.27). Suppose first that  $\{k(1), k(2), \dots, k(L)\} \subset \Phi(\ell')$  for some  $\ell' \in \{0, 1, \dots, L-1\}$ . By (2.27) and (2.7) and its entire sentence,

$$\mathcal{L}(W_k, k \in \Phi(\ell')) = \mathcal{L}(\zeta^{(\ell')}) = \mu. \quad (2.31)$$

Also, since  $\mathcal{L}(Y) = \theta(\mu) = \mathcal{L}(W^*)$  by (2.13) and the assumption on  $Y$ , one has by (2.27), (2.12), and (2.18),

$$\mathcal{L}(Y_k, k \in \Phi(\ell')) = \mathcal{L}(W_k^*, k \in \Phi(\ell')) = \mathcal{L}(V_{\ell'}^* \varphi_n(\zeta^{(\ell')})) = \mu. \quad (2.32)$$

Hence  $\mathcal{L}(Y_{k(1)}, Y_{k(2)}, \dots, Y_{k(L)}) = \mathcal{L}(W_{k(1)}, W_{k(2)}, \dots, W_{k(L)})$ , and (2.30) holds with equality.

Now consider the remaining case where  $\{k(1), k(2), \dots, k(L)\} \cap \Phi(\ell)$  is nonempty for at least two indices  $\ell \in \{0, 1, \dots, L-1\}$ . Then none of the sets  $\Phi(\ell)$ ,  $\ell \in \{0, 1, \dots, L-1\}$  can contain more than  $L-1$  of the integers  $k(i)$ . Now (2.31) holds for any  $\ell' \in \{0, 1, \dots, L-1\}$ ;

and hence by Definition 2.6(b)(c) for  $\mu$ , one has that for any  $\ell \in \{0, 1, \dots, L-1\}$  such that  $\{k(1), \dots, k(L)\} \cap \Phi(\ell)$  is nonempty,  $E \prod W_{k(i)} = 0$  where the product is taken over all  $i \in \{0, 1, \dots, L\}$  such that  $k(i) \in \Phi(\ell)$ . Also, by (2.27) and (2.7) and its entire sentence, the  $\sigma$ -fields  $\sigma(W_k, k \in \Phi(\ell))$ ,  $\ell \in \{0, 1, \dots, L-1\}$  are independent. It follows that  $E(\prod_{i=1}^L W_{k(i)}) = 0$ . Since (by hypothesis)  $\mathcal{L}(Y) = \theta(\mu)$ , one has by Definition 2.6(g) for  $\theta(\mu)$  that  $E(\prod_{i=1}^L Y_{k(i)}) \leq 0$ . Thus (2.30) holds. That completes the proof of statement (C)(ii).

*Proof of statement (C)(iii).* Refer again to (2.27). For each  $\ell \in \{0, 1, \dots, L-1\}$ , define the random variables

$$\eta'_\ell := \sum_{k \in \Phi(\ell)} W_k \quad \text{and} \quad \eta^*_\ell := \sum_{k \in \Phi(\ell)} Y_k. \quad (2.33)$$

Then

$$\text{sum } W = \sum_{\ell=0}^{L-1} \eta'_\ell \quad \text{and} \quad \text{sum } Y = \sum_{\ell=0}^{L-1} \eta^*_\ell. \quad (2.34)$$

By (2.27) and (2.7) and its entire sentence (note that  $\eta'_\ell = \text{sum } \zeta^{(\ell)}$  for each  $\ell \in \{0, 1, \dots, L-1\}$ ), the random variables  $\eta'_\ell$ ,  $\ell \in \{0, 1, \dots, L-1\}$  are independent. Recall again that  $\mathcal{L}(W^*) = \theta(\mu) = \mathcal{L}(Y)$  by (2.13) and the assumption on  $Y$ . By (2.33), (2.27), (2.12), and the phrase immediately after (2.18), every  $L-1$  of the random variables  $\eta^*_\ell$ ,  $\ell \in \{0, 1, \dots, L-1\}$  are independent. Hence by (2.31), (2.32), and (2.33), if  $j(1), j(2), \dots, j(L-1)$  are distinct elements of  $\{0, 1, \dots, L-1\}$ , then

$$\mathcal{L}\left(\eta'_{j(1)}, \eta'_{j(2)}, \dots, \eta'_{j(L-1)}\right) = \mathcal{L}\left(\eta^*_{j(1)}, \eta^*_{j(2)}, \dots, \eta^*_{j(L-1)}\right).$$

Hence, if  $j(1), j(2), \dots, j(L)$  are each an element of  $\{0, 1, \dots, L-1\}$  and two or more of the  $j(i)$ 's are equal, then  $E(\prod_{i=1}^L \eta'_{j(i)}) = E(\prod_{i=1}^L \eta^*_{j(i)})$ . Hence by (2.34) and a simple calculation,

$$E(\text{sum } W)^L - E(\text{sum } Y)^L = L! \cdot \left[ E\left(\prod_{\ell=0}^{L-1} \eta'_\ell\right) - E\left(\prod_{\ell=0}^{L-1} \eta^*_\ell\right) \right]. \quad (2.35)$$

Now by (2.7) and its entire sentence, and Definition 2.6(b) for  $\mu$ , one has that  $EW_k = 0$  for each  $k \in \{0, 1, \dots, Ln-1\}$ ; and hence by (2.33),  $E\eta'_\ell = 0$  for each  $\ell \in \{0, 1, \dots, L-1\}$ . Hence by the sentence after (2.34),  $E(\prod_{\ell=0}^{L-1} \eta'_\ell) = 0$ . Hence by (2.35),

$$E(\text{sum } W)^L - E(\text{sum } Y)^L = -L! \cdot E\left(\sum_{\ell=0}^{L-1} \eta^*_\ell\right). \quad (2.36)$$

Now recall again the equality  $\mathcal{L}(W^*) = \mathcal{L}(Y)$  in the second sentence after (2.34). By (2.33), (2.27), and (2.12),

$$E\left(\prod_{\ell=0}^{L-1} \eta^*_\ell\right) = E\left[\prod_{\ell=0}^{L-1} \text{sum}\left(V_\ell^* \varphi_n(\xi^{(\ell)})\right)\right] = E\left[\prod_{\ell=0}^{L-1} \left(V_\ell^* \cdot \text{sum}\left(\varphi_n(\xi^{(\ell)})\right)\right)\right].$$

Hence by (2.10), Remark 2.5(v), and the sentence after (2.10),

$$\begin{aligned} E \left( \prod_{\ell=0}^{L-1} \eta_{\ell}^* \right) &= E \left[ \prod_{\ell=0}^{L-1} V_{\ell}^* \right] \cdot \prod_{\ell=0}^{L-1} E \left[ \text{sum } \varphi_n(\xi^{(\ell)}) \right] \\ &= -1 \cdot \left[ E \left( \text{sum } \varphi_n(\xi^{(0)}) \right) \right]^L. \end{aligned} \quad (2.37)$$

Now by the sentence after (2.10), together with Lemma 2.7(i),  $E(\text{sum } \varphi_n(\xi^{(0)})) \geq (1/2)n^{1/2}$ . Hence by (2.37),  $E(\prod_{\ell=0}^{L-1} \eta_{\ell}^*) \leq -2^{-L}n^{L/2}$ . Hence by (2.36), eq. (2.8) holds. That completes the proof of statement C(iii), and of Lemma 2.8.

**Lemma 2.9.** *Suppose  $n$  is a positive integer, and  $\mu$  is a probability measure on  $\mathbf{R}^n$  that satisfies Condition  $\mathcal{H}(n)$  (see Definition 2.6). Suppose  $W := (W_k, k \in \mathbf{Z})$  and  $Y := (Y_k, k \in \mathbf{Z})$  are random sequences such that (see step 2.4(B)(D))  $\mathcal{L}(W) = \mathcal{D}(n, \mu)$  and  $\mathcal{L}(Y) = \mathcal{D}(Ln, \theta(\mu))$ . Then the following statements hold:*

(A) *The random variables  $W_k, k \in \mathbf{Z}$  satisfy  $(L-1)$ -tuplewise independence as well as (for all  $k \in \mathbf{Z}$ )  $\mathcal{L}(W_k) = \lambda_{\text{unps3}}$  (see step 2.4(A)); and the same holds for the random variables  $Y_k, k \in \mathbf{Z}$ .*

(B) *For any nonempty finite set  $S \subset \mathbf{Z}$ , one has that  $E(\sum_{k \in S} Y_k)^L \leq E(\sum_{k \in S} W_k)^L$ .*

(C) *If  $j \in \mathbf{Z}$ , and  $S$  is a finite subset of  $\mathbf{Z}$  such that  $\{j, j+1, j+2, \dots, j+2Ln-1\} \subset S$ , then*

$$E \left( \sum_{k \in S} W_k \right)^L - E \left( \sum_{k \in S} Y_k \right)^L \geq 2^{-L} \cdot L! \cdot n^{L/2}. \quad (2.38)$$

**Proof.** Statement (A) holds by Lemma 2.8(A), Definition 2.6(b)(c) (for both  $\mu$  and  $\theta(\mu)$ ), step 2.4(B), and a trivial extra argument.

Statements (B) and (C) will be proved together. For that purpose, define for each  $u \in \mathbf{Z}$  the set

$$\Lambda(u) := \{Lnu, Lnu+1, Lnu+2, \dots, Lnu+Ln-1\}. \quad (2.39)$$

In this argument, we shall follow the convention in the first paragraph of the proof of statement (C) in Lemma 2.8. By step 2.4(B) and a trivial argument, the random vectors  $(W_k, k \in \Lambda(u)), u \in \mathbf{Z}$  are independent of each other; and the random vectors  $(Y_k, k \in \Lambda(u)), u \in \mathbf{Z}$  are independent of each other.

Now suppose  $S$  is a nonempty finite subset of  $\mathbf{Z}$ . Let  $Q$  denote the (nonempty, finite) set of all integers  $u$  such that the set  $S \cap \Lambda(u)$  is nonempty. For each  $q \in Q$ , define the random variables

$$T'_q := \sum_{k \in S \cap \Lambda(q)} W_k \quad \text{and} \quad T^*_q := \sum_{k \in S \cap \Lambda(q)} Y_k. \quad (2.40)$$

Then

$$\sum_{k \in S} W_k = \sum_{q \in Q} T'_q \quad \text{and} \quad \sum_{k \in S} Y_k = \sum_{q \in Q} T^*_q. \quad (2.41)$$

Now for each  $q \in \mathbf{Z}$ , by step 2.4(B), the random vectors  $(W_{Lmq}, W_{Lmq+1}, \dots, W_{Lmq+Ln-1})$  and  $(Y_{Lmq}, Y_{Lmq+1}, \dots, Y_{Lmq+Ln-1})$  satisfy the hypothesis of Lemma 2.8(C). Hence for each

$q \in Q$  and each  $m \in \{1, 2, \dots, L-1\}$ , by (2.40) and Lemma 2.8(C)(i),  $E(T'_q)^m = E(T_q^*)^m$ . Also, the random variables  $T'_q$ ,  $q \in Q$  are independent, and the random variables  $T_q^*$ ,  $q \in Q$  are independent, by (2.40), (2.39), and the two sentences right after (2.39). Hence, if a given  $L$ -tuple  $(q(1), q(2), \dots, q(L)) \in Q^L$  is such that at least two of the entries  $q(i)$  are different from each other, then by a simple argument,  $E(\prod_{i=1}^L T'_{q(i)}) = E(\prod_{i=1}^L T_{q(i)}^*)$ . Hence by (2.41) and a simple calculation,

$$E \left[ \sum_{k \in S} W_k \right]^L - E \left[ \sum_{k \in S} Y_k \right]^L = \sum_{q \in Q} [E(T'_q)^L - E(T_q^*)^L]. \quad (2.42)$$

Now by (2.40), the sentence after (2.41), and Lemma 2.8(C)(ii),  $E(T'_q)^L \geq E(T_q^*)^L$  for every  $q \in Q$ . Hence by (2.42), statement (B) (in Lemma 2.9) holds.

To continue this argument in order to prove statement (C), note that under the hypothesis of (C), there exists  $p \in Q$  such that (see (2.39) again)  $\Lambda(p) \subset S$ . By (2.40), the sentence after (2.41), and Lemma 2.8(C)(iii),  $E(T'_p)^L - E(T_p^*)^L \geq 2^{-L} \cdot L! \cdot n^{L/2}$ . Hence by (2.42) and the sentence right after it, equation (2.38) holds. That completes the proof of statement (C), and of Lemma 2.9.

### 3. Part 2 of proof of Theorem 1.1

This section is a direct continuation of section 2. As in section 2, the argument here in section 3 will be divided into several “steps,” including some “lemmas.”

**Step 3.1.** Refer again to the even integer  $L \geq 6$  fixed in step 2.1. Refer to steps 2.2(a) and 2.4(A)(D). Recursively define as follows, for each  $n \in \{0, 1, 2, \dots\}$ , the probability measure  $\mu_n$  on  $\mathbf{R}^{L \uparrow n}$ : (i)  $\mu_0 := \lambda_{\text{unps3}}$ , and (ii) for each  $n \geq 0$ ,  $\mu_{n+1} := \theta(\mu_n)$ .

*Remark.* For each  $n \in \{0, 1, 2, \dots\}$ , the probability measure  $\mu_n$  satisfies Condition  $\mathcal{H}(L^n)$ . (For  $n = 0$ , that holds by Definition 2.6 and a trivial argument; and then for  $n \geq 1$ , it holds by Lemma 2.8(A) and induction.)

For the next lemma, and for the rest of section 3 here, let  $Z$  be a  $N(0, 1)$  random variable.

**Lemma 3.2.** *Suppose  $h$  and  $m$  are positive integers such that  $m \leq h$ . Suppose  $Y := (Y_k, k \in \mathbf{Z})$  is a random sequence such that  $\mathcal{L}(Y) = \mathcal{D}(L^h, \mu_h)$  (see step 2.4(B)). Suppose  $S$  is a finite set  $\subset \mathbf{Z}$  such that for some integer  $j$ , one has that  $\{j, j+1, j+2, \dots, j+2L^m-1\} \subset S$ . Then*

$$E \left[ \sum_{k \in S} Y_k \right]^L \leq E \left[ (\text{card } S)^{1/2} Z \right]^L - 2^{-L} \cdot L! \cdot L^{(m-1)L/2}. \quad (3.1)$$

**Proof.** For each  $n \in \{0, 1, 2, \dots\}$ , let  $W^{(n)} := (W_k^{(n)}, k \in \mathbf{Z})$  be a random sequence such that  $\mathcal{L}(W^{(n)}) = \mathcal{D}(L^n, \mu_n)$ . Then by Lemma 2.9(B),

$$\forall n \geq 1, \quad E \left[ \sum_{k \in S} W_k^{(n)} \right]^L \leq E \left[ \sum_{k \in S} W_k^{(n-1)} \right]^L. \quad (3.2)$$



Also by Lemma 2.9(C) (with  $n = L^{m-1}$ ) and the assumptions on  $S$ ,

$$E \left[ \sum_{k \in S} W_k^{(m)} \right]^L \leq E \left[ \sum_{k \in S} W_k^{(m-1)} \right]^L - 2^{-L} \cdot L! \cdot L^{(m-1)L/2}.$$

Combining that with (3.2) and trivial induction, one has that

$$\forall n \geq m, \quad E \left[ \sum_{k \in S} W_k^{(n)} \right]^L \leq E \left[ \sum_{k \in S} W_k^{(0)} \right]^L - 2^{-L} \cdot L! \cdot L^{(m-1)L/2}.$$

Since  $h \geq m$  (by hypothesis) and  $\mathcal{L}(Y) = \mathcal{L}(W^{(h)})$ , it follows that

$$E \left[ \sum_{k \in S} Y_k \right]^L \leq E \left[ \sum_{k \in S} W_k^{(0)} \right]^L - 2^{-L} \cdot L! \cdot L^{(m-1)L/2}.$$

Hence, to complete the proof of (3.1), it suffices to prove that

$$E \left[ \sum_{k \in S} W_k^{(0)} \right]^L \leq E \left[ (\text{card } S)^{1/2} Z \right]^L. \quad (3.3)$$

Now (see steps 3.1 and 2.4(A) again), the random variables  $W_k^{(0)}$ ,  $k \in \mathbf{Z}$  are independent and identically distributed, with  $\mathcal{L}(W_0^{(0)}) = \lambda_{\text{unps3}}$ . Let  $Z_k$ ,  $k \in \mathbf{Z}$  be independent  $N(0, 1)$  random variables. If  $p$  is a positive integer,  $k(1), k(2), \dots, k(p)$  are distinct integers, and  $a(1), a(2), \dots, a(p)$  are (not necessarily distinct) positive integers, then by the Remark in step 2.4(A),

$$E \left[ \prod_{i=1}^p \left( W_{k(i)}^{(0)} \right)^{a(i)} \right] = \prod_{i=1}^p E \left( W_{k(i)}^{(0)} \right)^{a(i)} \leq \prod_{i=1}^p E Z_{k(i)}^{a(i)} = E \left[ \prod_{i=1}^p Z_{k(i)}^{a(i)} \right].$$

Thus for any  $L$ -tuple  $(k(1), k(2), \dots, k(L)) \in \mathbf{Z}^L$  (the  $k(\ell)$ 's need not be distinct),  $E[\prod_{\ell=1}^L W_{k(\ell)}^{(0)}] \leq E[\prod_{\ell=1}^L Z_{k(\ell)}]$ . Adding up both sides over all  $L$ -tuples  $\in S^L$ , one obtains  $E[\sum_{k \in S} W_k^{(0)}]^L \leq E[\sum_{k \in S} Z_k]^L$ . Since  $\mathcal{L}(\sum_{k \in S} Z_k) = N(0, \text{card } S) = \mathcal{L}((\text{card } S)^{1/2} Z)$ , eq. (3.3) follows. That completes the proof of (3.1) and of Lemma 3.2.

**Step 3.3.** Let  $\kappa_1, \kappa_2, \kappa_3, \dots$  be a sequence of independent, identically distributed random variables, taking their values in the set  $\{0, 1, 2, \dots, L-1\}$  and uniformly distributed on that set.

Define the random variables  $J(n)$ ,  $n \in \{0, 1, 2, \dots\}$  as follows:

$$J(0) := 0 \text{ (constant)}, \text{ and for each } n \in \mathbf{N}, J(n) := \sum_{u=1}^n L^{u-1} \kappa_u. \quad (3.4)$$

*Remarks.* Recall that in the base- $L$  number system, for a given  $n \geq 1$  and a given  $\beta \in \{0, 1, 2, \dots, L^n - 1\}$ , there is a unique representation  $\beta = \sum_{u=1}^n L^{u-1} \alpha_u$  such that  $\alpha_1, \alpha_2, \dots, \alpha_n \in \{0, 1, \dots, L-1\}$ . Also, by (3.4),

$$\forall n \in \mathbf{N}, J(n) = L^{n-1} \kappa_n + J(n-1). \quad (3.5)$$

From those facts and trivial arguments, one has the following observations:

*Remark 1.* For each  $n \in \{0, 1, 2, \dots\}$ , the random variable  $J(n)$  takes its values in the set  $\{0, 1, 2, \dots, L^n - 1\}$  and is uniformly distributed on that set.

*Remark 2.* For each  $n \in \mathbf{N}$  (and each  $\omega \in \Omega$ ),  $J(n) \geq J(n-1)$  and in fact

$$\begin{aligned} -J(n) &\leq -J(n-1) \leq 0 \\ &\leq -J(n-1) + L^{n-1} - 1 \leq -J(n) + L^n - 1. \end{aligned} \quad (3.6)$$

(For the last inequality, apply (3.5) and the fact  $L^{n-1} \kappa_n \leq L^{n-1}(L-1) = L^n - L^{n-1}$ .)

*Remark 3.* For each  $n \in \mathbf{N}$  and each  $\omega \in \Omega$  such that  $1 \leq \kappa_n(\omega) \leq L-2$ , one has (again by (3.5)) that

$$\begin{aligned} -J(n)(\omega) &\leq -L^{n-1} - J(n-1)(\omega) \quad \text{and} \\ (-J(n-1)(\omega) + L^{n-1} - 1) + L^{n-1} &\leq (-J(n)(\omega) + L^n - 1). \end{aligned} \quad (3.7)$$

*Remark 4.* For each  $n \in \mathbf{N}$  and each  $j \in \{0, 1, 2, \dots, L^n - 1\}$ , there exists a unique choice of integers  $j_0 \in \{0, 1, 2, \dots, L^{n-1} - 1\}$  and  $k_0 \in \{0, 1, \dots, L-1\}$  such that

$$\{J(n) = j\} = \{J(n-1) = j_0\} \cap \{\kappa_n = k_0\}. \quad (3.8)$$

*Remark 5.* For each  $n \in \mathbf{N}$ , the two random variables  $J(n-1)$  and  $\kappa_n$  are independent.

**Step 3.4.** For each  $n \in \{0, 1, 2, \dots\}$ , we shall construct a sequence  $X^{(n)} := (X_k^{(n)}, k \in \mathbf{Z})$  of random variables. The definition will be recursive, and is as follows:

(A) To start off, let  $X^{(0)} := (X_k^{(0)}, k \in \mathbf{Z})$  be a sequence of independent, identically distributed random variables such that (i)  $\mathcal{L}(X_0^{(0)}) = \lambda_{\text{unps3}}$  (again see step 2.4(A)) and (ii) this sequence  $X^{(0)}$  is independent of the sequence  $(\kappa_1, \kappa_2, \kappa_3, \dots)$  in step 3.3.

(B) Now suppose  $n \geq 1$ , and the random sequence  $X^{(n-1)} := (X_k^{(n-1)}, k \in \mathbf{Z})$  is already defined. Define the random sequence  $X^{(n)} := (X_k^{(n)}, k \in \mathbf{Z})$  as follows:

For a given  $\omega \in \Omega$ , in the notations in step 2.2(a)(d) and step 2.3(C),

$$\begin{aligned} &(X_k^{(n)}(\omega) : -J(n)(\omega) \leq k \leq -J(n)(\omega) + L^n - 1) \\ &:= \begin{cases} \psi_{L \uparrow(n-1), 1}(X_k^{(n-1)}(\omega) : -J(n)(\omega) \leq k \leq -J(n)(\omega) + L^n - 1) \\ \quad \text{if } J(n)(\omega) = J(n-1)(\omega) \\ \psi_{L \uparrow(n-1), 0}(X_k^{(n-1)}(\omega) : -J(n)(\omega) \leq k \leq -J(n)(\omega) + L^n - 1) \\ \quad \text{if } J(n)(\omega) > J(n-1)(\omega), \end{cases} \end{aligned} \quad (3.9)$$

and for every integer  $K \in \mathbf{Z} - \{0\}$ ,

$$\begin{aligned} & (X_k^{(n)}(\omega) : -J(n)(\omega) + KL^n \leq k \leq -J(n)(\omega) + KL^n + L^n - 1) \\ & := \psi_{L\uparrow(n-1),0}(X_k^{(n-1)}(\omega) : -J(n)(\omega) + KL^n \leq k \leq -J(n)(\omega) + KL^n + L^n - 1). \end{aligned} \quad (3.10)$$

That completes the recursive definition. By induction on  $n$ , one has (see step 2.2(h)) that

$$\forall n \in \mathbf{N}, \sigma(X^{(n)}) \subset \sigma(X^{(0)}) \vee \sigma(\kappa_1, \kappa_2, \dots, \kappa_n). \quad (3.11)$$

By step 3.3 and paragraph (A) (property (ii)) above, one now has that for all  $n \in \{0, 1, 2, \dots\}$ , the two  $\sigma$ -fields  $\sigma(J(n), X^{(n)})$  and  $\sigma(\kappa_{n+1})$  are independent.

**Lemma 3.5.** *For each  $n \in \{0, 1, 2, \dots\}$ , one has that*

$$\forall j \in \{0, 1, 2, \dots, L^n - 1\}, \quad \mathcal{L}\left(X_{-j+k}^{(n)}, k \in \mathbf{Z} \mid J(n) = j\right) = \mathcal{D}(L^n, \mu_n). \quad (3.12)$$

**Proof.** Eq. (3.12) holds for  $n = 0$  by (3.4) and step 3.4(A), step 3.1, and step 2.4(B).

Now for induction, suppose that  $n \geq 1$ , and that (3.12) holds with  $n$  replaced by  $n - 1$ . To complete the induction step and the proof, our task is to prove (3.12) for the given  $n$ .

Referring to (3.12) suppose  $j \in \{0, 1, 2, \dots, L^n - 1\}$ .

Referring to step 3.3 (its Remark 4), let  $j_0 \in \{0, 1, \dots, L^{n-1} - 1\}$  and  $k_0 \in \{0, 1, \dots, L - 1\}$  be such that (3.8) holds. By the sentence after (3.11), the  $\sigma$ -fields  $\sigma(J(n-1), X^{(n-1)})$  and  $\sigma(\kappa_n)$  are independent. Hence by (3.8) and a simple calculation,

$$\begin{aligned} \forall B \in \sigma(X^{(n-1)}), \quad P(B \mid J(n) = j) &= P(B \mid \{J(n-1) = j_0\} \cap \{\kappa_n = k_0\}) \\ &= P(B \mid J(n-1) = j_0). \end{aligned} \quad (3.13)$$

Now by (3.8) and (3.5),  $j = L^{n-1}k_0 + j_0$ . By the induction hypothesis,

$$\mathcal{L}\left(X_{-j(0)+k}^{(n-1)}, k \in \mathbf{Z} \mid J(n-1) = j_0\right) = \mathcal{D}(L^{n-1}, \mu_{n-1}) \quad (3.14)$$

(where  $j_0$  is also written as  $j(0)$  for typographical convenience). Also (see step 2.4(B)), the distribution  $\mathcal{D}(L^{n-1}, \mu_{n-1})$  on  $\mathbf{R}^{\mathbf{Z}}$  is invariant under a shift of the indices by (any multiple of)  $L^{n-1}$ . Hence by (3.14),

$$\mathcal{L}\left(X_{-j+k}^{(n-1)}, k \in \mathbf{Z} \mid J(n-1) = j_0\right) = \mathcal{D}(L^{n-1}, \mu_{n-1}).$$

Hence by (3.13),

$$\mathcal{L}\left(X_{-j+k}^{(n-1)}, k \in \mathbf{Z} \mid J(n) = j\right) = \mathcal{D}(L^{n-1}, \mu_{n-1}). \quad (3.15)$$

Referring to step 2.2(d), consider the following three classes of random vectors for  $K \in \mathbf{Z}$ :

$$\begin{aligned}\zeta'_K &:= \left( X_k^{(n-1)} : -j + KL^{n-1} \leq k \leq -j + KL^{n-1} + L^{n-1} - 1 \right), \\ \zeta''_K &:= \left( X_k^{(n-1)} : -j + KL^n \leq k \leq -j + KL^n + L^n - 1 \right), \\ \zeta'''_K &:= \left( X_k^{(n)} : -j + KL^n \leq k \leq -j + KL^n + L^n - 1 \right).\end{aligned}$$

Referring to (2.4), note that for each  $K \in \mathbf{Z}$ ,  $\zeta''_K = \langle \zeta'_{LK} \mid \zeta'_{LK+1} \mid \zeta'_{LK+2} \mid \cdots \mid \zeta'_{LK+L-1} \rangle$ . Also, by (3.9)–(3.10) (and the inclusion  $\{J(n) = j\} \subset \{J(n-1) = j_0\}$  from (3.8)), for each  $K \in \mathbf{Z}$ , either (if  $K \neq 0$ , or  $K = 0$  and  $j > j_0$ )  $\zeta'''_K(\omega) = \psi_{L\uparrow(n-1),0}(\zeta''_K(\omega))$  for all  $\omega \in \{J(n) = j\}$ , or (if  $K = 0$  and  $j = j_0$ )  $\zeta'''_K(\omega) = \psi_{L\uparrow(n-1),1}(\zeta''_K(\omega))$  for all  $\omega \in \{J(n) = j\}$ . Now by (3.15) and step 2.4(B), conditional on the event  $\{J(n) = j\}$ , the random vectors  $\zeta'_K$ ,  $K \in \mathbf{Z}$  are independent and have distribution  $\mu_{n-1}$ . Hence conditional on  $\{J(n) = j\}$ , the random vectors  $\zeta''_K$ ,  $K \in \mathbf{Z}$  are independent. Hence by step 3.1 and Lemma 2.8(B), conditional on  $\{J(n) = j\}$ , the random vectors  $\zeta'''_K$ ,  $K \in \mathbf{Z}$  are independent and have distribution  $\mu_n$ . Thus by step 2.4(B), eq. (3.12) holds for our given  $n$  and  $j$ . Since  $j \in \{0, 1, \dots, L^n - 1\}$  was arbitrary, that completes the induction step and the proof.

**Lemma 3.6.** *For each  $n \in \mathbf{N}$ , the random sequence  $X^{(n)}$  is strictly stationary and satisfies  $\mathcal{L}(X_0^{(n)}) = \lambda_{\text{unps3}}$  (see step 2.4(A)) and  $(L-1)$ -tuplewise independence, and the random variables  $|X_k^{(n)}|$ ,  $k \in \mathbf{Z}$  are independent.*

**Proof.** Let  $n \in \mathbf{N}$  be arbitrary but fixed.

To prove strict stationarity, let  $\zeta := (\zeta_k, k \in \mathbf{Z})$  be a random sequence such that (see step 2.4(B))  $\mathcal{L}(\zeta) = \mathcal{D}(L^n, \mu_n)$ . For each  $i \in \mathbf{Z}$ , define the probability measure  $\eta_i$  on  $\mathbf{R}^{\mathbf{Z}}$  by  $\eta_i := \mathcal{L}(\zeta_{i+k}, k \in \mathbf{Z})$ . Then  $\eta_0 = \mathcal{D}(\mathcal{L}^n, \mu_n)$ , and from step 2.4(B) one has the “periodicity”  $\eta_{i+L\uparrow n} = \eta_i$  for  $i \in \mathbf{Z}$ . By Lemma 3.5, if  $j \in \{0, 1, \dots, L^n - 1\}$  then  $\mathcal{L}(X_{-j+k}^{(n)}, k \in \mathbf{Z} \mid J(n) = j) = \eta_0$ , and if also  $i \in \mathbf{Z}$  then  $\mathcal{L}(X_{-j+k+i}^{(n)}, k \in \mathbf{Z} \mid J(n) = j) = \eta_i$ . Hence by step 3.3 (its Remark 1) and a simple calculation, for any  $i \in \mathbf{Z}$ ,

$$\begin{aligned}\mathcal{L}\left(X_{i+k}^{(n)}, k \in \mathbf{Z}\right) &= L^{-n} \cdot \sum_{j=0}^{(L\uparrow n)-1} \mathcal{L}\left(X_{i+k}^{(n)}, k \in \mathbf{Z} \mid J(n) = j\right) \\ &= L^{-n} \cdot \sum_{j=0}^{(L\uparrow n)-1} \mathcal{L}\left(X_{-j+k+i+j}^{(n)}, k \in \mathbf{Z} \mid J(n) = j\right) \\ &= L^{-n} \cdot \sum_{j=0}^{(L\uparrow n)-1} \eta_{i+j} = L^{-n} \cdot \sum_{u=0}^{(L\uparrow n)-1} \eta_u\end{aligned}$$

where the last equality comes from the “periodicity” mentioned above. The last expression does not depend on  $i$ . Hence the sequence  $X^{(n)}$  is strictly stationary.

Next, by Lemma 3.5, step 3.1 (its Remark), Definition 2.6(b)(c)(d), and step 2.4(B)

(and where necessary a trivial extra argument), one has the following observations: First,

$$\begin{aligned}\mathcal{L}\left(X_0^{(n)}\right) &= L^{-n} \cdot \sum_{j=0}^{(L\uparrow n)-1} \mathcal{L}\left(X_0^{(n)} \mid J(n) = j\right) \\ &= L^{-n} \cdot \sum_{j=0}^{(L\uparrow n)-1} \lambda_{\text{unps}3} = \lambda_{\text{unps}3}.\end{aligned}$$

Second, if  $k(1), k(2), \dots, k(L-1)$  are any choice of  $L-1$  distinct integers, then (with a notation at the end of step 2.4(A))

$$\begin{aligned}\mathcal{L}\left(X_{k(1)}^{(n)}, X_{k(2)}^{(n)}, \dots, X_{k(L-1)}^{(n)}\right) &= L^{-n} \cdot \sum_{j=0}^{(L\uparrow n)-1} \mathcal{L}\left(X_{k(1)}^{(n)}, X_{k(2)}^{(n)}, \dots, X_{k(L-1)}^{(n)} \mid J(n) = j\right) \\ &= L^{-n} \cdot \sum_{j=0}^{(L\uparrow n)-1} \lambda_{\text{unps}3}^{[L-1]} = \lambda_{\text{unps}3}^{[L-1]};\end{aligned}$$

and hence the sequence  $X^{(n)}$  satisfies  $(L-1)$ -tuplewise independence. Third, by a similar argument, the random variables  $|X_k^{(n)}|$ ,  $k \in \mathbf{Z}$  are independent. The proof of Lemma 3.6 is complete.

**Lemma 3.7.** *Suppose  $h$  and  $m$  are positive integers such that  $m \leq h$ . Suppose  $p$  is a positive integer such that  $p \geq 2L^m$ . Then (recall the sentence preceding Lemma 3.2)*

$$E\left[\sum_{k=1}^p X_k^{(h)}\right]^L \leq E\left[p^{1/2}Z\right]^L - 2^{-L} \cdot L! \cdot L^{(m-1)L/2}. \quad (3.16)$$

**Proof.** By Lemma 3.5 and Lemma 3.2, for each  $j \in \{0, 1, \dots, L^h - 1\}$ ,

$$E\left(\left[\sum_{k=1}^p X_k^{(h)}\right]^L \mid J(n) = j\right) \leq [\text{RHS of (3.16)}].$$

Hence by step 3.3 (its Remark 1) and a trivial calculation, (3.16) holds.

**Lemma 3.8.** *There exists an event  $G$  with  $P(G) = 1$  such that the following holds: For every  $\omega \in G$  and every  $k \in \mathbf{Z}$ , there exists a positive integer  $m = m(\omega, k)$  such that*

$$\forall n \geq m, \quad X_k^{(n)}(\omega) = X_k^{(m)}(\omega). \quad (3.17)$$

**Proof.** First some preliminary calculations will be useful.

Suppose  $\omega \in \Omega$ ,  $n \geq 1$ , and  $J(n)(\omega) = J(n-1)(\omega)$ . Then by (3.9) and step 2.3(C),  $X_k^{(n)}(\omega) = X_k^{(n-1)}(\omega)$  will hold for all  $k \in \mathbf{Z}$  such that  $-J(n)(\omega) \leq k \leq -J(n)(\omega) + L^n - 1$

with the possible exception of the indices  $k$  such that  $-J(n)(\omega) + L^{n-1} \leq k \leq -J(n)(\omega) + 2L^{n-1} - 1$ . Thus (if  $J(n)(\omega) = J(n-1)(\omega)$ )  $X_k^{(n)}(\omega) = X_k^{(n-1)}(\omega)$  will hold for all  $k$  such that  $-J(n)(\omega) \leq k \leq -J(n)(\omega) + L^{n-1} - 1$ , that is, for all  $k$  such that

$$-J(n-1)(\omega) \leq k \leq -J(n-1)(\omega) + L^{n-1} - 1. \quad (3.18)$$

Now suppose  $\omega \in \Omega$ ,  $n \geq 1$ , and (instead)  $J(n)(\omega) > J(n-1)(\omega)$ . Then by (3.5),  $J(n)(\omega) \geq J(n-1)(\omega) + L^{n-1}$  and hence  $-J(n)(\omega) + L^{n-1} \leq -J(n-1)(\omega)$ ; and also by (3.9) and step 2.3(C),  $X_k^{(n)}(\omega) = X_k^{(n-1)}(\omega)$  will hold for (at least) all integers  $k$  such that  $-J(n)(\omega) + L^{n-1} \leq k \leq -J(n)(\omega) + L^n - 1$ , and in particular (see also the last inequality in (3.6)) for all integers  $k$  such that (3.18) holds.

By the preceding two paragraphs, one has that if  $\omega \in \Omega$  and  $n \geq 1$ , then  $X_k^{(n)}(\omega) = X_k^{(n-1)}(\omega)$  for all integers  $k$  such that (3.18) holds.

Hence by (3.6) and induction, one has that if  $\omega \in \Omega$ ,  $n \geq 1$ , and  $k$  satisfies (3.18), then  $X_k^{(n-1)}(\omega) = X_k^{(n)}(\omega) = X_k^{(n+1)}(\omega) = X_k^{(n+2)}(\omega) = \dots$ .

Now define the event  $G$  by

$$G := \{1 \leq \kappa_n \leq L - 2 \text{ for infinitely many } n \in \mathbf{N}\}.$$

By (2.1) and step 3.3,  $P(1 \leq \kappa_n \leq L - 2) \geq 2/3$  for each  $n \in \mathbf{N}$ , and hence by step 3.3 and the (second) Borel–Cantelli Lemma,  $P(G) = 1$ . Also, for each  $\omega \in G$ , [LHS of (3.18)]  $\rightarrow -\infty$  and [RHS of (3.18)]  $\rightarrow \infty$  as  $n \rightarrow \infty$  by (3.7). Hence for every  $\omega \in G$  and every  $k \in \mathbf{Z}$ , there exists  $n \in \mathbf{N}$  such that (3.18) holds. Hence by the observation in the preceding paragraph, Lemma 3.8 holds.

**Step 3.9.** Define the sequence  $X := (X_k, k \in \mathbf{Z})$  of random variables as follows: For each  $k \in \mathbf{Z}$ ,

$$X_k := \lim_{n \rightarrow \infty} X_k^{(n)}. \quad (3.19)$$

By Lemma 3.8, for each  $k \in \mathbf{Z}$ , this limit exists almost surely.

**Step 3.10.** The task now is to show that the sequence  $X$  defined in (3.19) has all of the properties stated in Theorem 1.1.

Suppose  $p$  is a positive integer and  $k(1), k(2), \dots, k(p)$  are distinct elements of  $\mathbf{Z}$ . By (3.19), for any  $p$ -tuple  $(t_1, t_2, \dots, t_p) \in \mathbf{R}^p$ , one has that  $\sum_{i=1}^p t_i X_{k(i)}^{(n)} \rightarrow \sum_{i=1}^p t_i X_{k(i)}$  a.s. as  $n \rightarrow \infty$ . Hence by [1, p. 330, Theorem 25.2, and p. 383, Theorem 29.4],

$$\mathcal{L} \left( X_{k(1)}^{(n)}, X_{k(2)}^{(n)}, \dots, X_{k(p)}^{(n)} \right) \Longrightarrow \mathcal{L} \left( X_{k(1)}, X_{k(2)}, \dots, X_{k(p)} \right) \text{ as } n \rightarrow \infty \quad (3.20)$$

(weak convergence on  $\mathbf{R}^p$ ). Recall the elementary fact that for weak convergence on  $\mathbf{R}^p$ , the limiting distribution is unique.

For a given  $j \in \mathbf{Z}$  and a given  $m \in \mathbf{N}$ , one has that

$$\mathcal{L} \left( X_{j+1}^{(n)}, X_{j+2}^{(n)}, \dots, X_{j+m}^{(n)} \right) = \mathcal{L} \left( X_1^{(n)}, X_2^{(n)}, \dots, X_m^{(n)} \right)$$

for each  $n \geq 1$  by Lemma 3.6, and hence by (3.20) one has that

$$\mathcal{L}(X_{j+1}, X_{j+2}, \dots, X_{j+m}) = \mathcal{L}(X_1, X_2, \dots, X_m).$$

Hence the sequence  $X$  is strictly stationary.

Since  $\mathcal{L}(X_0^{(n)}) = \lambda_{\text{unps}3}$  for each  $n \in \mathbf{N}$  by Lemma 3.6, it follows from (3.20) that  $\mathcal{L}(X_0) = \lambda_{\text{unps}3}$ . Thus property (A) in Theorem 1.1 holds.

In the case  $p = L - 1$ , the left side of (3.20) is  $\lambda_{\text{unps}3}^{[L-1]}$  for each  $n \in \mathbf{N}$  by Lemma 3.6 (see the end of step 2.4(A)), and hence the right side of (3.20) is also  $\lambda_{\text{unps}3}^{[L-1]}$ . Thus property (B) (here  $(L - 1)$ -tuplewise independence) in Theorem 1.1 holds (see step 2.1).

Property (C) in Theorem 1.1 holds by a similar argument involving Lemma 3.6 and eq. (3.20) with  $p$  arbitrarily large.

**Step 3.11.** The final task is to prove property (D) in Theorem 1.1. Recall the notations for partial sums in eq. (1.1).

By property (A) (proved above) in Theorem 1.1, and a routine calculation,  $EX_0 = 0$ ,  $EX_0^2 = 1$ , and  $EX_0^4 = 9/5$ . Recall eq. (2.1) and the property of  $(L - 1)$ -tuplewise independence (proved above) for the sequence  $X$ . One has that

$$\forall n \geq 1, \quad E\left(n^{-1/2}S(X, n)\right) = 0 \quad \text{and} \quad (3.21)$$

$$\forall n \geq 1, \quad E\left(n^{-1/2}S(X, n)\right)^2 = 1. \quad (3.22)$$

Further, by the well known, elementary calculation in [1, p. 85, proof of Theorem 6.1] (which requires only 4-tuplewise independence), one has that for each  $n \in \mathbf{N}$ ,

$$\begin{aligned} E(S(X, n))^4 &= n \cdot EX_0^4 + 3n(n-1)(EX_0^2)^2 \\ &= n \cdot (9/5) + (3n^2 - 3n) \cdot 1 \leq 3n^2. \end{aligned}$$

Hence

$$\forall n \geq 1, \quad E\left(n^{-1/2}S(X, n)\right)^4 \leq 3. \quad (3.23)$$

By (say) (3.22) and Chebyshev's inequality, the family of distributions of the normalized partial sums  $n^{-1/2}S(X, n)$ ,  $n \in \mathbf{N}$  is tight.

Now for the proof of property (D) in Theorem 1.1, suppose  $Q$  is an infinite subset of  $\mathbf{N}$ . Because of tightness, there exists an infinite set  $T \subset Q$  and a probability measure  $\mu$  on  $\mathbf{R}$  (both  $T$  and  $\mu$  henceforth fixed) such that

$$n^{-1/2}S(X, n) \implies \mu \text{ as } n \rightarrow \infty, n \in T. \quad (3.24)$$

To complete the proof of property (D) in Theorem 1.1, our task now is to show that  $\mu$  is neither degenerate nor normal.

Because of (3.23), (3.24), and [1, p. 338, the Corollary], one has by (3.21) and (3.22) that

$$\int_{x \in \mathbf{R}} x \mu(dx) = 0 \quad \text{and} \quad \int_{x \in \mathbf{R}} x^2 \mu(dx) = 1. \quad (3.25)$$

Hence the probability measure  $\mu$  has positive variance and is therefore nondegenerate.

If  $\mu$  were normal, then by (3.25) it would have to be the  $N(0, 1)$  distribution. Because of (3.24) and [1, p. 334, Corollary 1, and p. 338, Theorem 25.11], to show that  $\mu$  fails to be normal, it now suffices to show (see the sentence preceding Lemma 3.2) that

$$\forall n \geq 2L, \quad E \left( n^{-1/2} S(X, n) \right)^L \leq EZ^L - 8^{-L/2} \cdot L! \cdot L^{-L}. \quad (3.26)$$

Let  $M \geq 2L$  be arbitrary but fixed. Let  $m \in \mathbf{N}$  be the integer such that

$$2L^m \leq M < 2L^{m+1}. \quad (3.27)$$

By (3.27) and Lemma 3.7, for all  $n \geq m$ ,

$$E \left[ S(X^{(n)}, M) \right]^L \leq E \left[ M^{1/2} Z \right]^L - 2^{-L} \cdot L! \cdot L^{(m-1)L/2}. \quad (3.28)$$

Now for each  $n \in \mathbf{N}$ ,  $[S(X^{(n)}, M)]^L \leq [3^{1/2} M]^L$  a.s. (since  $|X_k^{(n)}| \leq 3^{1/2}$  a.s. for each  $k \in \mathbf{Z}$  by Lemma 3.6), and hence by (3.19) and dominated convergence,  $E[S(X^{(n)}, M)]^L \rightarrow E[S(X, M)]^L$  as  $n \rightarrow \infty$ . Hence by (3.28),

$$E[S(X, M)]^L \leq E[M^{1/2} Z]^L - 2^{-L} \cdot L! \cdot L^{(m-1)L/2}.$$

(That also follows simply from Fatou's Lemma.) Hence by the second inequality in (3.27),

$$\begin{aligned} E \left[ M^{-1/2} S(X, M) \right]^L &\leq EZ^L - 2^{-L} \cdot L! \cdot L^{(m-1)L/2} \cdot M^{-L/2} \\ &\leq EZ^L - 2^{-L} \cdot L! \cdot L^{(m-1)L/2} \cdot (2L^{m+1})^{-L/2} \\ &= EZ^L - 2^{-3L/2} \cdot L! \cdot L^{-L}. \end{aligned}$$

Since  $M \geq 2L$  was arbitrary, eq. (3.26) holds. That completes the proof that  $\mu$  is non-normal, and that property (D) in Theorem 1.1 holds. The proof of Theorem 1.1 is complete.

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