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A Probabilistic Proof of Wallis' Formula for π - Steven J. Miller

The article begins by presenting some of the many formulas one can use as expressions for π. One of these formulas is Wallis' Formula, which can be written as follows:

$$\frac{\pi}{2} = \frac{2 \cdot 2}{1 \cdot 3} \cdot \frac{4 \cdot 4}{3 \cdot 5} \cdot \frac{6 \cdot 6}{\cdot 5 \cdot 7} \cdots = \prod_{n=1}^{\infty} \frac{2n \cdot 2n}{(2n-1)(2n+1)} \tag{0.1}$$

The formula can be proven in multiple ways. Rather than using the infinite product expansion for $sin(x)$ or induction for integral powers of $sin(x)$, the formula is proven using only basic probability/statistics knowledge.

A fundamental idea which is used in the proof is the theory of normalizing constants. If a continuous function f is nonnegative everywhere and $\int_{-\infty}^{\infty} f(x)dx < \infty$, we can multiply f by a constant c to create a valid probability density function cf which integrates to 1. Additionally, it is shown (with and without using the Central Limit Theorem) that the Student-t distribution converges to the standard normal distribution as the degrees of freedom approach infinity. Combining these ideas, it is shown that the normalizing constant of the Student-t distribution approaches the normalizing constant of the standard normal ($\frac{1}{\sqrt{2\pi}}$) as the degrees of freedom approach infinity.

The last important result necessary to complete the proof is that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ which is proven by relating the integral of the gamma function to the integral of the standard normal distribution. The claim follows from the famous result $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} = 1$.

Armed with the above results, the proof of Wallis' formula is simple. Consider the Student-t distribution with $2m$ degrees of freedom. Using the formula for the Student-t distribution, the normalizing constant c_{2m} is found in terms of the gamma function and m. The terms are rewritten into a product from 1 to m resembling the infinite product in Wallis' formula. Recalling that the normalizing constant of the Student-t converges to the normalizing constant of the standard normal ($\frac{1}{\sqrt{2\pi}}$), Wallis' formula is proven by taking the limit as m goes to infinity.

The article concludes by providing an application of this result. Namely, proving a new formula for $log(\frac{\pi}{2})$.