# Summary of "A Probabilistic Proof of Wallis's Formula for $\pi$ by Steven Miller

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Miller's paper begins with examples of various famous formulas for $\pi$ that serve to illustrate several methods for finding $\pi$, ranging from using the derivative of arctan to complex analysis to the infinite product expansion of $\sin x$. After mentioning several of these examples, the heart of the paper begins with an elementary formula for $\pi$ that relies on standard facts of probability distributions. This is Wallis's forumula for $\pi$, which states the following:

$$\frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{2n \cdot 2n}{(2n-1)(2n+1)}.$$

After giving the standard definition for a continuous distribution function, Miller notes that for every non-negative continuous function whose integral is finite, there exists a constant such that the product of the constant and the function is a probability distribution.

Next he examines the Gamma function and the Student $t$-distribution. The Gamma function is important because $\Gamma(1/2) = \sqrt{\pi}$, a fact that can be proven using standard integration techniques involving a $u$-substitution followed by a switch to polar coordinates.

It is apparent that the density of the $t$-distribution is continuous and nonnegative, so to prove that the density is a continuous probability density he just needs to show that it integrates to 1, which can be done using two properties of the Beta function. A characteristic of the $t$-distribution is that as the sample size increases to infinity, the distribution converges to the standard normal. To show this, he can either use the Central Limit Theorem, or prove it directly for this particular case. He chooses the latter option, and claims that the product of the integral of the $t$-distribution and some constant is equal to one. This relies on the fact presented above. After integrating, he solves for this constant

$$c = \frac{1}{\sqrt{2\pi}}.$$

Now all that is left is to expand the gamma function and substitute it into the above equation, to reach Wallis's formula for $\pi$.