Math/Stat 341: Probability
First Lecture

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http://www.williams.edu/Mathematics/sjmiller/public_html/341

Bronfman 106
Williams College, February 6, 2015

Introduction and Objectives

Introduction / Objectives

Probability theory: model the real world, predict likelihood of events.

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Objectives

- Obviously learn probability.
- Emphasize techniques / asking the right questions.
- Model problems and analyze model.
- Elegant solutions vs brute force (parameters in closed form versus numerical solutions).
- Looking at equations and getting a sense: log -5 Method: $\frac{p \pm pq}{p+q \pm 2pq}$.

Types of Problems

- Biology: will a species survive?
- Physics / Chemistry / Number Theory: Random Matrix Theory.
- Gambling: Double-plus-one.
- Economics: Stock market / economy.
- Finance: Monte Carlo integration.
- Marketing: Movie schedules.
- Cryptography: Markov Chain Monte Carlo.
- 8 ever 9 never (bridge).

My (applied) experiences

- Marketing: parameters for linear programming (SilverScreener).
- Data integrity: detecting fraud with Benford's Law (IRS, Iranian elections).
- Sabermetrics: Pythagorean Won-Loss Theorem.

Course Mechanics

Grading / Administrative

- Move at fast pace, responsible for reading before class: 5% of grade. HW: 15%. Writing: 10%. Midterm: 30% (if there are two exams only best counts). ‘Final’ exam: 40%. You may also do a project for 10% of your grade (which reduces all other categories proportionally).
- Pre-reqs: Calc III, basic combinatorics / set theory, linear algebra.

Office hours / feedback

- MWF 8:40-9:30am, Tues 1-2, Thur 2:30-3:30pm and when I’m in my office (schedule online)
- Feedback ephsmath@gmail.com, password williams1793.

Other

- Webpage: numerous handouts, additional comments each day (mix of review and optional advanced material).
- Clickers: see how well we can estimate probabilities, always anonymous.
- Probability Lifesaver: opportunity to help write a book, lots of worked examples.
- Creating HW problems: mix of ones you can solve and ones you want to learn about.
- Gather and analyze some data set of interest.
- PREPARE FOR CLASS! Must do readings before each class.

Being Prepared

Never know when an opportunity presents itself....

S. J. Miller at the Sarnak 61^{st} Dinner

(copyright C. J. Mozzochi, Princeton N.J)

Being Prepared

- Your Job:
  - ◇ Be prepared for class: do reading, think about material.
  - ◇ Come to me, the TAs and each other with questions.
- My/TAs Job:
  - ◇ Provide resources, guiding questions.
  - ◇ Be available.

Other: Advice from Jeff Miller

- Party less than the person next to you.

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Happy to do practice interviews, adjust deadlines....

Gambling

Football Wager

2007: Friend of a favorite student bet $500 at 1000:1 odds on Patriots going undefeated and winning the Superbowl.

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Football Wager

2008: In third quarter, Pats leading, Vegas offers to buy back the bet at 300:1, told no....

WHAT WAS THE BETTOR'S MISTAKE?

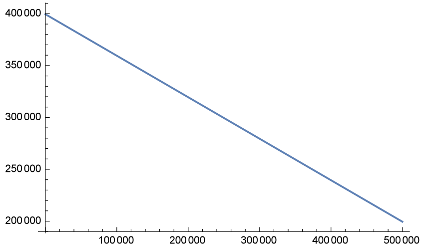
Hedging

Pats win with probability p, Giants $q = 1 - p$.

Bet \$1 bet on Giants, if they win get $\$x$.
Already bet \$500 on Patriots, now bet $\$B$ on the Giants.

Expected Winning:

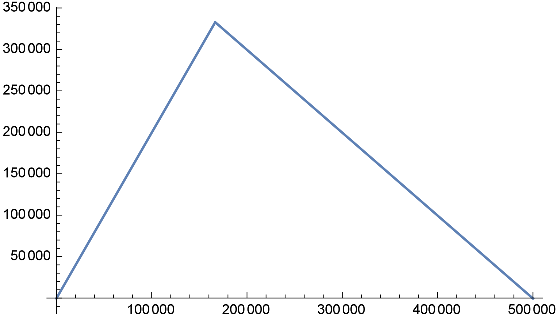
$$f(p, x, B) \;=\; p \cdot 500000 + (1 - p)Bx - 500 - B.$$



Guaranteed Winnings

By hedging can ensure some winnings:

$$g(p, x, B) = \min(500000, Bx) - 500 - B.$$

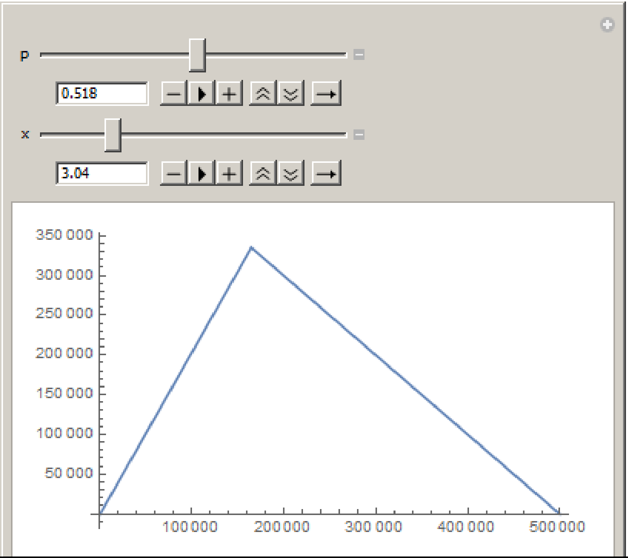


Here $p = .8, x = 3.$

Mathematica Code

```
f[p_, x_, B_] := 500000 p + (1 - p) B x - 500 - B
g[p_, x_, B_] := Min[500000, B x] - 500 - B
Plot[f[.8, 3, B], {B, 0, 500000}]
Plot[g[.8, 3, B], {B, 0, 500000}]
Manipulate[Plot[g[p, x, B], {B, 0, 500000}], {p, 0, 1}, {x, 1, 10}]
```

Mathematica Code



Sabermetrics Club at Williams....

http://fivethirtyeight.com/features/
a-head-coach-botched-the-end-of-the-super-bowl-and-it-wasnt-pete-carroll/

Clicker Problems

Birthday Problem I

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How large must N be for there to be at least a 50% probability that two of the N people share a birthday?

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Birthday Problem

How large must N be for there to be at least a 50% probability that two of the N people share a birthday?

- (A) 11 people
- (B) 22 people
- (C) 33 people
- (D) 44 people
- (E) 90 people
- (F) 180 people
- (G) 365 people
- (H) 500 people.

Birthday Problem I

Birthday Problem

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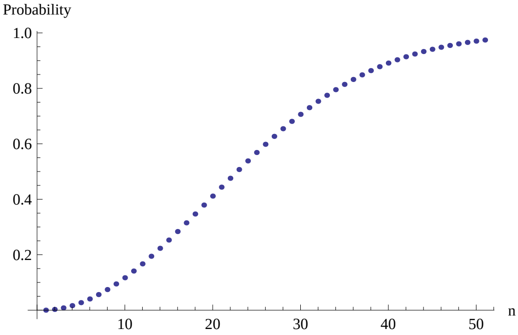

Birthday Problem II

How large must N be for there to be at least a 50% probability that two of N Plutonians share a birthday?

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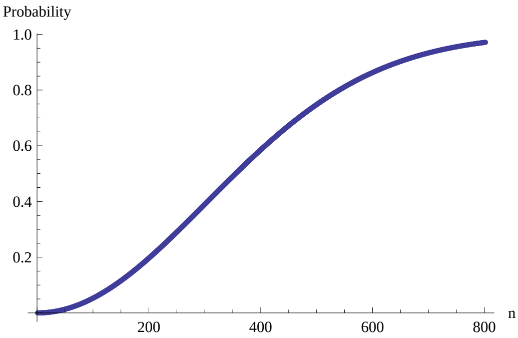
- (A) 110 people
- (B) 220 people
- (C) 330 people
- (D) 440 people
- (E) 1,000 people
- (F) 5,000 people
- (G) 10,000 people
- (H) 20,000 people
- (I) more than 30,000 people.

Birthday Problem II

How large must *N* be for there to be at least a 50% probability that two of *N* Plutonians share a birthday? ‘Recall’ one Plutonian year is about 248 Earth years (or 90,520 days).



Voting: Democratic Primaries

During the Democratic primaries in 2008, Clinton and Obama received exactly the same number of votes in Syracuse, NY. How probable was this?

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- (A) 1 / 10
- (B) 1 / 100
- (C) 1 / 1,000
- (D) 1 / 10,000
- (E) 1 / 100,000
- (F) 1 / 1,000,000 (one in a million)
- (G) 1 / 1,000,000,000 (one in a billion).

Voting: Democratic Primaries (continued)

Syracuse University mathematics Professor Hyune-Ju Kim said the result was less than one in a million, according to the Syracuse Post-Standard, which quoted the professor as saying, "It's almost impossible." Her comments were reprinted widely, as the Associated Press picked up the story. (Carl Bialik, WSJ, 2/12/08)

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Far greater than 1/137! What's going on?

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Far greater than 1/137! What's going on?

Prof. Kim's calculation ... was based on the assumption that Syracuse voters were likely to vote in equal proportions to the state as a whole, which went for Ms. Clinton, its junior senator, 57%-40%. .... Prof. Kim said she had little time to make the calculation, so she made the questionable assumption ... for simplicity.

From Shooting Hoops
to the Geometric Series Formula

Simpler Game: Hoops

Game of hoops: first basket wins, alternate shooting.

Simpler Game: Hoops: Mathematical Formulation

Bird and **Magic** (I'm old!) alternate shooting; first basket wins.

- **Bird** always gets basket with probability p.
- **Magic** always gets basket with probability q.

Let x be the probability **Bird** wins – what is x?

Solving the Hoop Game

Classic solution involves the geometric series.

Break into cases:

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- **Bird** wins on 1^{st} shot: p.
- **Bird** wins on 2^{nd} shot: $(1-p)(1-q) \cdot p$.

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Break into cases:

- **Bird** wins on 1st shot: p.
- **Bird** wins on 2nd shot: $(1-p)(1-q) \cdot p$.
- **Bird** wins on 3rd shot: $(1-p)(1-q) \cdot (1-p)(1-q) \cdot p$.

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- **Bird** wins on 3^{rd} shot: $(1-p)(1-q)\cdot(1-p)(1-q)\cdot p$.
- **Bird** wins on n^{th} shot:
  $(1-p)(1-q)\cdot(1-p)(1-q)\cdots(1-p)(1-q)\cdot p$.

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- **Bird** wins on nth shot: $(1-p)(1-q)\cdot(1-p)(1-q)\cdots(1-p)(1-q)\cdot p$.

Let $r=(1-p)(1-q)$. Then

$$\begin{aligned} x &= \text{Prob}(\textbf{Bird}\text{ wins}) \\ &= p+rp+r^2p+r^3p+\cdots \\ &= p\left(1+r+r^2+r^3+\cdots\right), \end{aligned}$$

the geometric series.

Solving the Hoop Game: The Power of Perspective

Showed

$$x = \text{Prob}(\textbf{Bird} \text{ wins}) = p(1 + r + r^2 + r^3 + \cdots);$$

will solve without the geometric series formula.

Solving the Hoop Game: The Power of Perspective

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Have

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will solve without the geometric series formula.

Have

$$x = \text{Prob}(\textbf{Bird} \text{ wins}) = p + (1 - p)(1 - q)$$

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Have

$$x = \text{Prob}(\textbf{Bird} \text{ wins}) = p + (1-p)(1-q)x$$

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$$x = \text{Prob}(\textbf{Bird}\text{ wins}) = p + (1-p)(1-q)x = p + rx.$$

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Thus

$$(1-r)x = p \quad \text{or} \quad x = \frac{p}{1-r}.$$

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Thus

$$(1-r)x = p \quad \text{or} \quad x = \frac{p}{1-r}.$$

As $x = p(1 + r + r^2 + r^3 + \cdots)$, find

$$1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r}.$$

Lessons from Hoop Problem

- Power of Perspective: Memoryless process.
- Can circumvent algebra with deeper understanding! (Hard)
- Depth of a problem not always what expect.
- Importance of knowing more than the minimum: connections.
- Math is fun!