GAMMA AND BETA FUNCTIONS + DISTRIBUTIONS

GAMMA FUNCTION

$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y}\, dy$ for $Re(\alpha) > 0$

- **NOTE:** $\Gamma(1) = 1$ and $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ for $Re(\alpha) > 0$

↳ $\Gamma(1) = \int_0^\infty e^{-y}\, dy = 1$

$\Gamma(\alpha+1) = \int_0^\infty y^\alpha e^{-y}\, dy$ integrate by parts:

$$\begin{cases} u = y^\alpha & dv = e^{-y}\, dy \\ du = \alpha y^{\alpha-1} & v = -e^{-y} \end{cases}$$

$$= -y^\alpha e^{-y}\Big|_0^\infty + \int_0^\infty \alpha y^{\alpha-1} e^{-y}\, dy$$

$$= \alpha \int_0^\infty y^{\alpha-1} e^{-y}\, dy$$

$$= \alpha\,\Gamma(\alpha)$$

- **NOTE:** If $\alpha = n$ (a positive integer) Then $\Gamma(n+1) = n!$
This follows from repeated use of $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ and $\Gamma(1) = 1$

THM: $\Gamma(1/2) = \sqrt{\pi}$

we shall use this technique again in studying the Beta distribution.
Let $a \in (0,1)$ so $\Gamma(a)$, $\Gamma(1-a)$ exist and are finite. As all integrands below are positive, all change of variables can be justified.

$$\Gamma(a)\Gamma(1-a) = \int_0^\infty\int_0^\infty x^{a-1} e^{-x} y^{1-a-1} e^{-y}\, dx\, dy$$

$$= \int_0^\infty\int_0^\infty \left(\frac{x}{y}\right)^{a-1} e^{-\left(\frac{x}{y}\right)y}\, y^{-1} e^{-y}\, dx\, dy$$

Change variables. Fix y, set $z = \frac{x}{y}$, $dz = \frac{dx}{y} = y^{-1}dx$

$$= \int_0^\infty\int_0^\infty z^{a-1} e^{-zy} e^{-y}\, dz\, dy$$

$$= \int_0^\infty\int_0^\infty z^{a-1} e^{-(z+1)y}\, dz\, dy \quad \begin{cases} \text{Change Vars: } u = (z+1)y, & du = (z+1)\,dy \\ \text{Fix } z \end{cases}$$

$$= \int_0^\infty\int_0^\infty z^{a-1} e^{-u} \frac{du}{z+1}\, dz = \int_0^\infty \frac{z^{a-1}}{z+1} \underbrace{\int_0^\infty e^{-u}\, du}_{=\Gamma(1)=1}\, dz$$

$$= \int_0^\infty \frac{z^{a-1}}{1+z}\, dz \quad \text{Change vars: } z = u^2,\ dz = 2u\,du$$

$$= \int_0^\infty u^{2a-2}/(1+u^2) \cdot 2u\, du \quad \text{Change vars: } u = \tan\theta,\ du = \sec^2\theta\, d\theta$$

$1 + \tan^2\theta = \sec^2\theta$, $u = 0 \to \theta = 0$, $u = \infty \to \theta = \pi/2$

$$= 2\int_0^{\pi/2} \frac{(\tan\theta)^{2a-1}}{\sec^2\theta} \sec^2\theta\, d\theta : \text{ If } a = 1/2 \text{ get } 2\int_0^{\pi/2} d\theta = \pi$$

BETA DISTRIBUTION

$$\text{Density } f(x; \alpha, \beta) = \begin{cases} \dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cdot\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$$

We must show this is the correct integration constant:

This is equivalent to showing

$$\Gamma(\alpha)\Gamma(\beta) = \Gamma(\alpha+\beta)\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx$$

It turns out to be easier to expand the LHS than the RHS

$$\Gamma(\alpha)\Gamma(\beta) = \int_0^\infty\int_0^\infty y^{\alpha-1}e^{-y} z^{\beta-1}e^{-z}\,dy\,dz \qquad \text{• Fix } z, \text{ Send } y \to zu,\ dy \to z\,du$$

$$= \int_0^\infty\int_0^\infty (zu)^{\alpha-1}e^{-zu} z^{\beta-1}e^{-z} z\,du\,dz$$

$$= \int_0^\infty\int_0^\infty z^{\alpha+\beta-1}e^{-(1+u)z} u^{\alpha-1}\,du\,dz \qquad \text{• Fix } u. \text{ Let } t=(u+1)z,\ dt=(u+1)\,dz$$

$$= \int_0^\infty\int_0^\infty \left(\frac{t}{u+1}\right)^{\alpha+\beta-1} e^{-t} u^{\alpha-1} \cdot \frac{1}{(u+1)}\,dt\,du$$

$$= \int_0^\infty u^{\alpha-1}\cdot\left(\frac{1}{u+1}\right)^{\beta+1} \underbrace{\int_0^\infty t^{\alpha+\beta-1}e^{-t}\,dt}_{\text{This is } \Gamma(\alpha+\beta)}\,du$$

$$= \Gamma(\alpha+\beta)\int_0^\infty \left(\frac{u}{u+1}\right)^{\alpha-1}\cdot\left(\frac{1}{u+1}\right)^{\beta+1}du$$

$$\text{let } x = \frac{u}{u+1},\quad 1-x = \frac{1}{u+1},\quad dx = \frac{1}{(u+1)^2}\,du \qquad u=0, x=0;\ u=\infty, x=1$$

$$= \Gamma(\alpha+\beta)\int_0^1 x^{\alpha-1}(1-x)^{\beta+1}(1-x)^{-2}\,dx$$

$$= \Gamma(\alpha+\beta)\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx \quad \text{as desired.}$$

Note: all change of variables can be justified (integrals of positive fns, all converges)

We can use this to calculate $\Gamma(1/2)$. Recall $\Gamma(1)=1$ and from the def, $\Gamma(1/2)>0$

$$1 = \int_0^1 \frac{\Gamma(\frac12+\frac12)}{\Gamma(1/2)\Gamma(1/2)} x^{\frac12-1}(1-x)^{\frac12-1}\,dx \longrightarrow \Gamma(\tfrac12)^2 = \int_0^1 x^{-\frac12}(1-x)^{-\frac12}\,dx$$

Change variables: $x=u^2$ $\quad dx = 2u\,du$ so $x^{-\frac12}dx = 2\,du$

$$\Longrightarrow \Gamma(1/2)^2 = \int_0^1 (1-u^2)^{-1/2}\,2\,du$$

Change variables: $u=\sin\theta$, $du=\cos\theta\,d\theta$, $u=0\to\theta=0$, $u=1\to\theta=\pi/2$

$$\Longrightarrow \Gamma(1/2)^2 = 2\int_0^{\pi/2}(1-\sin^2\theta)^{-1/2}\cos\theta\,d\theta = 2\int_0^{\pi/2}\frac{\cos\theta\,d\theta}{(\cos^2\theta)^{1/2}} = \pi$$

As $\Gamma(1/2)$ is positive and $\Gamma(1/2)^2=\pi$, we have $\Gamma(1/2)=\sqrt{\pi}$

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