

FIRST THREE EXTRA PROBLEMS

MATH 341

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1. START OF FIRST SET OF PROPOSED PROBLEMS

2. ARNOSTI, NICK

We first describe the game Dominion. Each player begins with a deck of 10 cards: seven coppers, worth 1 coin each, and three estates, which are worthless. Each turn, players draw a hand of five cards and purchase a new card to add to their deck; at the end of their turn, all cards used.... As there are 7 coins in the original deck and the entire deck is drawn in two turns, the first turn determines the second, and the first two turns will either consist of a turn with 3 coins and a turn with 4 coins or a turn with 2 coins and a turn with 5 coins. This is because, in our first turn, we can have at most 3 estate cards, and thus have either 2, 3, 4 or 5 coins on the first pick, and therefore the second pick has the remaining, i.e., 5, 4, 3 or 2 coins. Thus the number of coins in the two turns are either $\{2, 5\}$ or $\{3, 4\}$.

Exercise 2.1. *What is the probability of a 4/3 (or 3/4) split? In other words, what is the probability that a player will have at least three coppers in each of their first two hands?*

Solution: Observe that $\mathbb{P}(4/3 \text{ split}) = 1 - \mathbb{P}(5/2 \text{ split})$, and that $\mathbb{P}(5 \text{ coppers on first turn}) = \mathbb{P}(3 \text{ estates on second turn})$. There are $\binom{10}{3}$ ways to order the ten cards (counting each estate as identical and each copper as identical). Of these, $\binom{5}{3}$ have three estates in the second hand. Thus, $\mathbb{P}(5 \text{ coppers on first turn}) = \frac{\binom{5}{3}}{\binom{10}{3}} = \frac{1}{12}$. By symmetry, $\mathbb{P}(5 \text{ coppers on second turn})$ is the same. Thus, $\mathbb{P}(4/3 \text{ split}) = 1 - 2 \cdot \frac{1}{12} = \frac{5}{6}$.

Remark 2.2. *We give another proof. We add labels to each type of card so that they can be distinguished (perhaps we add a color). There are $\binom{10}{5}$ ways to choose 5 cards from the 10. If we want a hand of 5 with 3 estates, there are $\binom{3}{3} \binom{7}{2}$ ways (we must choose all three estates, and then 2 of the coppers). Thus the probability of having a 5/2 split is $\binom{3}{3} \binom{7}{2} / \binom{10}{5} = 1/12$, the same answer as above! This is a common feature, where frequently we may count the same problem many different ways.*

When purchasing cards, the possibilities are silver, which costs three copper to add to your deck and is worth 2 coin, and gold, which costs six copper to add to your deck and is worth 3 coin. After purchasing a card, it, along with ALL cards from your hand (including those used to purchase it) are placed in your discard pile. When your deck runs out, you shuffle the discard pile and draw from it.

Exercise 2.3. *Suppose a player purchases only silver (when possible), and let X be a random variable representing the first turn that a player can afford gold. What is the distribution function for X ?*

Solution:

3. ATKINSON, BEN

Exercise 3.1. *The Yankees have been on a winning streak, and they have won 3 of their last 4 games. Assume, however, that their true winning percentage is $p > 1/2$, and all games are independent of each other. What is the probability the Yankees have more wins than losses in their last 7 games?*

Solution: There are two ways to solve this. The possible outcomes of their last four games are

$$WWWL, WWLW, WLWW, LWWW.$$

So long as they have won one of the remaining 3 games, they will have more wins than losses; equivalently, by the Law of Total Probability we just need to compute the probability that they have won none of these three games. The probability they lost all three games is $(1 - p)^3$, and thus the probability that they won at least one of the games (and thus have won more than half of the last 7 games) is $1 - (1 - p)^3$.

Alternatively, we could say that the only outcomes for their past five games are

$$LWWWL, WLWWL, WWLWL, WWWLL.$$

The reason this might be true is that if they had won five games ago, we would have said they've won 4 out of 5. In fact, such logic would eliminate $WWWLL$, as in this case we would just say they have won three in a row. Regardless, in all these situations they are 3–2 in the last 5 games, and hence the only way they could have more losses than wins is to have lost the two previous games, which happens with probability $(1 - p)^2$. Thus the probability that they have more wins than losses is $1 - (1 - p)^2$.

Exercise 3.2. *The Yankees must have a 25 man roster; however due to their enormous payroll and the fact that many ballplayers want to play for a team with such a distinguished history, let us assume that they have 30 players of equal talent who should make the team. Therefore, they devise the following scheme to determine who makes the roster. A bag contains 5 black marbles and 25 white marbles, and each player chooses, without replacement, a ball from the bag. The 5 players who choose black marbles are off the roster for the next week. The order of choosing rotates each week, and each player loses 1 spot (i.e. pick 1 goes to 2, 29 goes to 1, etc.). Given that a player has the number 2 choice, what is the probability he sits 2 weeks in a row?*

Solution: There are a lot of red herrings in the statement of this problem. A red herring is shorthand for facts in the problem that are not relevant to the solution. For this exercise, all that matters is that each player (by symmetry) has a $5/30 = 1/6$ chance of sitting out each week. Therefore the odds of sitting out two weeks in a row is just $(1/6)^2 = 1/36$. It doesn't matter when they pick; however, the solution would be different if we were told what ball the people with higher picks chose.

4. BERRY, JACK

Exercise 4.1. *From a standard deck of 52 cards, you deal yourself 5 cards:*

$$A\heartsuit 4\clubsuit 8\spadesuit A\heartsuit 2\clubsuit$$

You are now allowed to exchange any number of cards in your hand for the same number of cards from the deck. You decide to exchange the $4\clubsuit$, $8\spadesuit$, and $2\clubsuit$ for three new cards. What is the probability that you will get three of a kind or better?

Solution: The number of ways to draw 3 cards from the remaining 47 cards is

$$\binom{47}{3}$$

The number of ways to draw the remaining 2 aces is

$$\binom{2}{2} \binom{45}{1}$$

Finally, there are

$$\binom{2}{1} \binom{45}{2}$$

ways to choose 1 ace. Thus, the probability of getting three or four aces is

$$\frac{\binom{2}{2} \binom{45}{1} + \binom{2}{1} \binom{45}{2}}{\binom{47}{3}};$$

note, however, that this is *not* the answer to the question asked. We want to find all hands that are at least as good as three of a kind. We could also have a full house from drawing three cards with the same denomination (for example, three Jacks). The number of ways of drawing three of a kind is complicated, depending on whether or not it is a 2, 4 or 8. For those three denominations, there is only one way to do it (as one of each of these cards is out of play). For the remaining 9 denominations (3, 5, 6, 7, 9, 10, Jack, Queen and King), there are $\binom{4}{3} = 4$ ways. Thus the number of ways of getting three of a kind is

$$\binom{3}{1} \binom{3}{3} + \binom{9}{1} \binom{4}{3},$$

so the probability of getting at least three of a kind is

$$\frac{\binom{2}{2} \binom{45}{1} + \binom{2}{1} \binom{45}{2} + \binom{3}{1} \binom{3}{3} + \binom{9}{1} \binom{4}{3}}{\binom{47}{3}} = \frac{688}{5405} \approx .127.$$

Exercise 4.2. In Texas Holdem' Poker, each player is dealt two private cards, and must make a 5 card poker hand combining these two cards and a selection of cards from a 5 card community. In the following hand, four out of the five community cards have been revealed:

$$A\clubsuit 2\diamond J\diamond A\diamond$$

You know your private cards are:

$$2\spadesuit K\diamond$$

Your only remaining opponent bets aggressively, telling you that there is a 0.8 probability that he has three of a kind or better. In addition, based on the betting of other players earlier in the hand, you estimate that there is a 0.4 probability that another player had an Ace (but he has since folded) What is the probability that you will win this hand?

Solution: Not surprisingly, the solution to a problem like this is quite difficult. Even if we know our opponent has three of a kind or better, this could mean he has three of a kind, a straight, a flush, a full house (three of a kind and a pair), four of a kind or a straight flush. Further, his hand could also improve with the additional dealt card. If his three of a kind is coming from holding an ace, that is going to be tough to beat. It seems our best bet to win is to go for a flush (if he has a pair, then even if an ace is turned then we might both have a full house, and almost surely if so his pair would beat our pair of 2's). Of course, the probability of a flush in diamonds depends on whether or not our opponent (as well as those who folded) have diamonds. If we assume there are

k diamonds remaining in the deck, then our probability of getting a flush in diamonds is k/N , where N is the number of remaining cards.

5. BINDER, ARI

Exercise 5.1. Assume that there are α tables in the Paresky cafeteria. In a choice between tables T and U , where T has t spots empty and U has u spots empty, and $u \geq t$, a student is 2^{u-t} times more likely to choose table T over table U (this extends to more than two tables: for instance, if there are three tables, with 2, 1, and 1 spots left, the student will choose the first table with probability $1/2$ and the other two with probability $1/4$ each). Further, assume each table seats four students, and that right now, the cafeteria is empty. A group of four students enters the cafeteria, and the students choose places to sit, one by one. What is the greatest α for which the probability that all four end up at the same table remains above 1 percent?

Solution: Solving $.01 = \frac{2}{\alpha+1} \cdot \frac{4}{\alpha+3} \cdot \frac{8}{\alpha+7}$ for α and taking the floor of the result yields $\alpha = 14$. We arrive at this solution as follows: after the first student chooses a table at which to sit, we must calculate the probability the second student joins the first. There are α tables, and the student is twice as likely to choose the table at which the first student is sitting as one of the other tables. Thus, the probability that the second student joins the first is $\frac{2}{\alpha+1}$. This is because the probability the student chooses one of the $\alpha - 1$ other tables has shrunk from $\frac{1}{\alpha}$ to $\frac{1}{\alpha+1}$, reflecting the fact he is more likely to join his friend than sit alone. Similarly, the probability that the third student joins the first two is $\frac{2^2}{\alpha-2^2+1} = \frac{4}{\alpha+3}$, because the third student is four times more likely to join his two friends than to sit alone. The probability that the fourth student joins the first three is $\frac{2^3}{\alpha-2^3+1} = \frac{8}{\alpha+7}$, because he is eight times more likely to join his three friends than to sit alone.

Exercise 5.2. Now assume each table seats β people, k students come upon an empty cafeteria and they choose places to sit, one by one again. What is the minimum k , in terms of α and β , such that the probability that the last two students sit at the same table is at least $1/k$?

Solution: Maybe choosing a fixed value, in terms of α and β , instead of $1/k$, will make things more tractable.

6. BROWN, CHAD

Exercise 6.1. There are m players playing a game of Texas hold'em. Before the first round of betting, each player is dealt two cards face down. What is the probability that each player is dealt a unique pair (called a "pocket pair")?

Solution: We assume $m \leq 13$, as otherwise the problem is trivial. Let $A_1, A_2, \dots, A_m$ be the event that player 1, 2, $\dots$, m is dealt a unique pocket pair. We would like to

calculate $\mathbb{P}(A_1 \cap A_2 \cap \dots \cap A_m)$.

$$\begin{aligned}
 \mathbb{P}(A_1 \cap A_2 \cap \dots \cap A_m) &= \mathbb{P}(A_1 \cap (A_2 \cap A_3 \cap \dots \cap A_m)) \\
 &= \mathbb{P}(A_1) * \mathbb{P}(A_2 \cap A_3 \cap \dots \cap A_m | A_1) \\
 &= \mathbb{P}(A_1) * \mathbb{P}(A_2 \cap (A_3 \cap \dots \cap A_m) | A_1) \\
 &= \mathbb{P}(A_1) * \mathbb{P}(A_2 | A_1) * \mathbb{P}(A_3 \cap \dots \cap A_m | A_1 \cap A_2) \\
 &\vdots \\
 &= \mathbb{P}(A_1) * \mathbb{P}(A_2 | A_1) * \mathbb{P}(A_3 | A_1 \cap A_2) * \dots * \mathbb{P}(A_m | A_1 \cap A_2 \cap \dots \cap A_{m-1}) \\
 &= (1) \left(\frac{3}{51}\right) * \left(\frac{48}{50}\right) \left(\frac{3}{49}\right) * \left(\frac{44}{48}\right) \left(\frac{3}{47}\right) * \dots * \left(\frac{52-4m}{52-2m}\right) \left(\frac{1}{52-2m-1}\right) \\
 &= \prod_{n=1}^m \frac{52-4n}{52-2n} \frac{1}{52-2n-1}.
 \end{aligned}$$

Remark 6.2. *I don't believe the analysis here is correct. What does it mean for one person to be dealt a unique pair?*

Solution: How many ways are there to deal cards? It makes a big difference whether or not order matters. There are $\binom{52}{2m}$ ways to choose $2m$ cards where order does not count, and there are $\binom{52}{2m} (2m)!$ ways to choose $2m$ cards so that order counts. If we choose the cards with order mattering, then the first two cards go to the first person, the next two to the second person, and so on. There are $\binom{13}{1} \binom{4}{2} 2!$ ways to assign a pair to the first person, with order mattering. The second person has $\binom{12}{1} \binom{4}{2} 2!$ ways (as they cannot have the same pair). Continuing we find the number of ways is just $m! \binom{13}{m} (2 \binom{4}{2})^m$, so the probability is $m! \binom{13}{m} (2 \binom{4}{2})^m / (\binom{52}{2m} (2m)!)$.

Exercise 6.3. *The previous problem wasn't so bad. However, what if we try to calculate the probability of each player being dealt ANY pocket pair. Now, for example, both player 1 and player 2 could have a pair of aces. This problem is much harder because we now have to keep track of which pairs are left. If the player 1 has a pair of aces, it is now less likely that player 2 will be dealt a pair of aces than, say, a pair of kings.*

Solution:

7. BUSTARD, TODD

Exercise 7.1. *Determine the probability that a 5 card poker hand will be three of a kind or a full house (three of a kind and a pair).*

Solution: As there are $\binom{52}{5}$ ways to pick 5 cards, it is natural to say that the probability is just

$$\begin{aligned}\mathbb{P}(\text{three of a kind or full house}) &= \mathbb{P}(\text{exactly three cards with the same number}) \\ &= \frac{\binom{13}{1} \binom{4}{3} \binom{48}{2}}{\binom{52}{5}} \\ &= \frac{94}{4165} \approx .022569,\end{aligned}$$

where $\binom{13}{1}$ arises from having to choose one of the 13 values, then there are $\binom{4}{3}$ ways to choose three cards from this value, and then $\binom{48}{2}$ ways to choose two remaining cards. The key word is *exactly* – we want to make sure we do not have four of a kind. It doesn't matter if we have three of a kind or a full house (three of a kind and a pair); that is why the last factor is $\binom{48}{2}$, as it does not matter whether or not the two remaining cards are equal to each other.

Whenever doing problems such as this, the best way to avoid making mistakes is to split the counting based on how often we have each key value. For us, we need to either have three of a kind and two different cards (from each other and the three of a kind), or three of a kind and a pair. The probability is

$$\frac{\binom{13}{1} \binom{4}{3} \binom{12}{2} \binom{4}{1} \binom{4}{1} + \binom{13}{1} \binom{4}{3} \binom{12}{1} \binom{4}{2}}{\binom{52}{5}} = \frac{94}{4165} \approx .022569.$$

We are calculating the same quantity two different ways; the first way is faster and better. The argument generalizes, and if we had a deck of $4N$ cards with N denominations, we would still have agreement. It is a nice exercise to show this. If we want to calculate the probability that our hand has at least three cards with the same number, that would be

$$\frac{\binom{13}{1} \binom{4}{3} \binom{48}{2} + \binom{13}{1} \binom{4}{4} \binom{48}{1}}{\binom{52}{5}} = \frac{19}{833} \approx .022809;$$

note this is close to, but not the same as, the probability above.

Exercise 7.2. Determine the probability that a hand in Texas Holdem' Poker will be a full house. Note that in Texas Holdem', there are seven cards (2 in your hand, and 5 community cards).

Solution: There are $\binom{52}{7}$ ways to choose 7 cards from 52. There are two ways of interpreting this question: either the best hand we can form is a full house, or we can form at least a full house. We shall adopt the latter interpretation. There are three ways this can be done: (1) we can have three of a kind twice and then another card; (2) we can have three of a kind once and two different pairs; (3) we can have one three of a kind, one pair, and then two different cards. The number of ways of doing these are: (1): we first choose two of the thirteen numbers, then for each of these choose 3 of the four, and then choose one of the remaining 44, for $\binom{13}{2} \binom{4}{3} \binom{4}{3} \binom{44}{1}$ ways; for (2) it is $\binom{13}{1} \binom{4}{3} \binom{12}{2} \binom{4}{2} \binom{4}{2}$, while for (3) it is $\binom{13}{1} \binom{4}{3} \binom{12}{1} \binom{4}{2} \binom{11}{2} \binom{4}{1} \binom{4}{1}$. The solution is found by adding the three and dividing by $\binom{52}{7}$.

8. CHO, JAEHONG

Exercise 8.1. *A fair coin is tossed 5 times. If a head comes up for a given toss, I win the amount that I bet and if a tail comes up then I lose the amount bet. Assume I bet a dollar on the first toss,, two dollars on the second toss and so on and so forth. What is the probability that I will make a profit of more than three dollars?*

Solution: There are a total 32 possible outcomes. If we cared about how much we expect to make on average then we would have to analyze each in detail; however, the analysis is simplified because we only care about how often we make more than \$3.

- (1) There is $\binom{5}{0}$ possible cases of 5 tails.
- (2) There are $\binom{5}{1}$ possible cases of 1 heads and 4 tails
- (3) There are $\binom{5}{2}$ possible cases of 2 heads and 3 tails
- (4) There are $\binom{5}{3}$ possible cases of 3 heads and 2 tails
- (5) There are $\binom{5}{4}$ possible cases of 4 heads and 1 tail
- (6) There is $\binom{5}{5}$ possible cases of 5 heads

We clearly make no money in case (0). In all the subcases of (2) and (3), the maximum profit that can be generated comes from (T,T,T,H,H), in which case the profit is less than 3. Among all the possible subcases in (4), (5) and (6) the following lists all the cases in which the ending profit is greater than 3

- (HTTHH), with a profit of 5
- (THHHT), with a profit of 5
- (THTHH), with a profit of 7
- (TTHHH), with a profit of 9
- (HHHHT), with a profit of 5
- (HHHTH), with a profit of 7
- (HHTHH), with a profit of 9
- (HTHHH), with a profit of 11
- (THHHH), with a profit of 13
- (HHHHH), with a profit of 15.

Therefore, the probability we win more than three dollars is $10/32$.

Exercise 8.2. *Consider a game of Texas Hold em, which is played as follows. There are a total of 4 players at a table. A neutral and proficient dealer shuffles the deck M times using the same shuffling technique before handing the cards out. In our game, there is no betting and the person with the best hand wins the hand. For simplicity, the numbers in the following discussion refers to the order in which each card was placed after the shuffling has occurred. Each player receives two cards. The topmost card and the 5th card is dealt to Player 1, the 2nd and the 6th card is dealt to Player 2, the 3rd and 7th to Player 3 and the 4th and 8th to Player 4. After the hands are dealt to the respective players, the 9th card in the original deck gets burnt and the next three (10th, 11th, and 12th) flops are shown on the table. The dealer then burns the another card, the 13th, and shows the turn (14th). The dealer then burns yet another card (15th) and shows the river (16th). The 5 cards of flops, turn and river (i.e., the cards not burned but turned over) are community cards, meaning that all four players share those 5 cards. At the*

end of the game, a player's hand is the best possible combination of 5 cards out of the total of 7 cards (the 5 community cards and their two cards).

Initially we start with a new deck. We assume it is ordered as follows: all four aces are on the top, then all four twos, and so on and so forth, with all four kings at the very bottom. The order of suits is as follows: Clubs, Diamonds, Hearts and Spades.

After each game, the dealer collects the played cards in order such that the order of the deck, before they were dealt, is preserved.

The dealer only uses the the Riffle shuffle after each hand to randomize the deck for the next round. There are two steps in a Riffle shuffle. (1) The dealer splits the deck in half, the top half goes to his left hand and the bottom half goes to his right. (2) Starting from the bottom, each card on the right hand will be placed on top of the left hand. For example, after the first shuffle, the order of cards will look like this (starting from the top, with numbers indicating the original position before the shuffle):

$$1, 27, 2, 28, 3, 29, \dots, 51, 26, 52.$$

What is the smallest possible number M such that when the dealer shuffles the deck M times before each game, the probability that each player wins the hand is equal in the long run. Of course, we assume that we can play the game infinitely often. (Note that unlike the conventional Texas Hold em, suits matter in our case, so a pair $9\spadesuit 9\heartsuit$ beats the pair $9\diamondsuit 9\clubsuit$).

Solution:

9. CITRO, BRIAN

Exercise 9.1. Consider three buckets labeled A , B , and C . Bucket A contains 2 red balls, bucket B contains 2 blue balls, and bucket C contains 1 red ball and 1 blue ball. A bucket is randomly selected, and a ball is randomly selected from that bucket. The chosen ball happens to be red. Given this information, what is the probability that the ball was chosen from bucket A ?

Solution: By the rules of conditional probability, we know that $P(A|R) = \frac{P(A \cap R)}{P(R)}$. $P(A \cap R) = 1/3$, since $P(A) = 1/3$ and if A is selected, the ball must be red. $P(R) = 1/2$, since there are 3 red balls and 3 blue balls, and each is equally likely to be chosen. Therefore, $P(A|R) = \frac{1/3}{1/2} = 2/3$.

Exercise 9.2. One popular Roulette strategy is the Labouchere system. In this system, the player first chooses the amount of money x that they are trying to win. They then write a list of numbers which sum to x . Each time they place a bet, the player bets an amount equal to the sum of the first and last numbers on the list on either red or black. If the player wins, they cross off the two numbers on the list and move to the next numbers. If they lose a bet, they add a new number to the end of the list equal to the amount lost in the bet. When the player has crossed off every number on the list, he has won $\$x$.

Assume a player is attempting to win $\$100$, so he writes down a list of ten $\$10$ s. What is the probability that he will complete the list and win? Assume an American roulette wheel, with 18 black numbers, 18 red numbers, and 2 green numbers; assume a table

limit (maximum bet) of \$500; and assume that if the player ever reaches a point where he has to make a bet above \$500 he gives up.

Solution:

10. FISH, CROSBY

Exercise 10.1. Given a randomly shuffled standard deck of playing cards, what is the probability that the first four cards are of the same suit?

Solution: There are $\binom{4}{1}$ ways to choose the suit. There are $\binom{13}{4}$ ways of choosing four cards from a given suit, and $\binom{52}{4}$ ways of choosing four cards. Thus the probability is just

$$\frac{\binom{4}{1} \binom{13}{4}}{\binom{52}{4}} = \frac{44}{4165} \approx .0105642.$$

We could also calculate this by counting how many ways we can construct such a hand, where the order in which the cards is drawn matters. In this case, the probability is just

$$4 \frac{13}{52} \cdot \frac{12}{51} \cdot \frac{11}{50} \cdot \frac{10}{49} = 4 \cdot \frac{13!48!}{9!52!} = \binom{4}{1} \frac{13!}{9!4!} \frac{48!4!}{52!} = \frac{\binom{4}{1} \binom{13}{4}}{\binom{52}{4}},$$

which is just what we had before.

It is interesting to calculate the probability that the four cards are all in different suits. That is

$$\mathbb{P}(\text{All different suits}) = \frac{\binom{52}{1} \binom{39}{1} \binom{26}{1} \binom{13}{1}}{52 \cdot 51 \cdot 50 \cdot 49} = \frac{2197}{20825},$$

which is about .105498. This is interesting, meaning that the probability of getting four cards in the same suit is almost the same as getting four cards in four different suits.

Exercise 10.2. Given a randomly shuffled standard deck of playing cards, what is the probability that somewhere in the deck there are at least 4 consecutive cards of the same suit?

Solution: One way to attack this problem is to use binary indicator random variables and Chebyshev's theorem. We can compute the expected number of times we have four cards in a suit easily. Let $X_1, \dots, X_{48}$ be the random variables where $X_i = 1$ if cards i through $i + 3$ are the same suit, and 0 otherwise. We know $\mathbb{E}[X_i] = 44/4165$, and so by linearity of expectation we have the expected number of times we have four consecutive cards in a suit is

$$\mathbb{E}[X_1] + \dots + \mathbb{E}[X_{48}] = 48 \cdot \frac{44}{4165} \approx .507083,$$

which is almost exactly between 0 and 1. To compute the variance requires us to understand the correlations between the X_i 's.

11. FORD, AARON

Exercise 11.1. *Your 12 red checker pieces are randomly placed, one by one, onto your 8×8 checker board's eligible squares (in other words, they are placed only on the 32 black squares). What is the probability that they are placed in a legitimate opening configuration (in other words, that they are all in the first three rows or all in the last three rows)?*

Solution: We give two solutions. For the first, we assume the 12 pieces are distinguishable, and calculate the probability that the checkers end up in the 12 squares in the first three rows; multiplying by 2 gives the answer. The first piece has a $\frac{12}{32}$ chance of being placed onto a correct square. The next piece has a $\frac{11}{31}$ chance, the next a $\frac{10}{30}$ chance, and so on, down to a $\frac{1}{21}$ chance for the final piece, if all of the previous ones are set up correctly. So the probability is equal to

$$\frac{2 \cdot 12!}{\frac{32!}{20!}} \approx 8.858 \cdot 10^{-9}$$

For the non-distinguishable case, we either fill up the three bottom or the three top rows. There are only two ways of doing so, and the number of ways of choosing 12 squares from 32 is $\binom{32}{12} = 32!/(12!20!)$. Thus the probability is $2/\binom{32}{12}$, which after some algebra we note is the same as $2 \cdot 12!/(32!/20!)$.

Exercise 11.2. *Let's say that you are playing blackjack at a casino where the dealer is using an infinite number of decks of cards, and each card has an equal probability of being drawn. Let's say that you must keep hitting, or asking for another card, until you hit 21 or go over. What is the chance that you land on 21?*

Solution: The natural way to interpret this is that, at each draw, we are equally likely to get any card. We need to look at how many ways there are to get exactly 21 or bust. As the largest denomination of a card is 10, if we bust we must have had somewhere between a 12 and a 20, and we end somewhere between a 22 and a 30, given that we were never at 21. It is already a difficult problem just calculating how many ways we can add up to exactly 21. This is equivalent to solving the following equation:

$$1x_1 + 2x_2 + 3x_3 + \cdots + 9x_9 + 10x_{10} + 10x_{11} + 10x_{12} + 10x_{13} = 21,$$

where $x_i \geq 0$ is the number of cards we have with the value i (unless $i \in \{11, 12, 13\}$ in which case it is 10). This is an example of a Diophantine equation; in general these are very hard to solve (and this is just one case of what we need!).

12. GROVER, MICHAEL

Exercise 12.1. *Let $F(x, y) = xy$ where $0 \leq x, y \leq 1$. What is the marginal distribution function of X ?*

Solution: If we let y take its maximum value, 1, $F(x, y) = x$. This is the marginal distribution function of X .

Exercise 12.2. Suppose a consumer lives for an infinite number of periods. The consumer is endowed with 1 unit of the only good at period zero. The consumer has preferences represented by the following utility function:

$$u(x_0, x_1, x_2, \dots) = \sum_{t=0}^{\infty} \beta^t \log(x_t)$$

Assume the consumer has access to a market that provides a return. Specifically, if amount k of the good is not eaten in period t , it turns into $(1+r_i)k$ of the good in period $t+1$ with probability p_i where $i \in \mathbb{Z}^+$ is finite. Characterize the consumption stream as a function of p_i and r_i . Note that although the consumer's utility function dictates risk aversion, he will still invest some money in the market as long as $\sum_i x_i p_i > 0$.

Solution:

13. JACKSON, STEVEN

Consider a bounded random walk in $[0, 1]$ starting at $x \in [0, 1]$ and returning after n steps of fixed size. Furthermore, suppose that the number and type of nonzero steps that are taken affect some later calculation.

Exercise 13.1. Suppose the fixed step sizes are $\{0, \pm \frac{1}{2}\}$. How many ways are there to return to the start while taking k positive steps?

Solution: Except for the special case of $x \in \{0, 1/2, 1\}$, each choice of x determines a unique first step, and then the next nonzero step must be in the opposite direction. For simplicity, suppose $x \in (0, 1/2)$, so that the first nonzero step will be $1/2$, and the next nonzero step will be $-1/2$. If we take k steps to the right, we will have k such paired steps.

The only thing left to figure out is how to distribute the zeros among these. The easiest way to look at this is to say that we are choosing $n - 2k$ of the n spots to be zeros, hence there are $\binom{n}{n-2k} = \binom{n}{2k}$ ways to have a walk with k steps to the right while returning to the starting point $x \in (0, 1/2)$, where $0 \leq k \leq n/2$.

Exercise 13.2. Suppose the fixed step sizes are $\{0, \pm \frac{1}{2^\ell}, \pm \frac{2}{2^\ell}, \dots, \pm \frac{2^\ell-1}{2^\ell}\}$, for $\ell \in \mathbb{N}$. Now how many ways are there to return to the start while taking k_1 steps of $+\frac{1}{2^\ell}$, k_2 steps of $+\frac{2}{2^\ell}$, $\dots$, $k_{2^\ell-1}$ steps of $+\frac{2^\ell-1}{2^\ell}$, and k_{-1} steps of size $-\frac{1}{2^\ell}$, etc.?

Solution: (Actually, Partial Solution.) We can apply the same trick as in the first question to a rather small subset of the possible walks, where we just alternate positive and negative steps of a given magnitude. There are also certain cases of mixing that are relatively easy to analyze, but calculating the number for a completely general walk is fiendishly difficult.

14. KOLOGLU, MURAT

Exercise 14.1 (Thursday Dice Fever). Two Williams students, M and N , decide to have a Thursday night probabilistic contest—first of many to come. They have in their possession one m sided and one n sided fair dice, where $m \geq n$. M rolls the m sided die and N rolls the n sided die. The higher roll wins and in the case of a tie they reroll until either one wins. What is the probability that N wins in terms of n and m ?

Solution: Consider N 's roll first. He has a $1/n$ probability of rolling each number from 1 to n . Assume he rolls some number r . Now M has a $\frac{r-1}{m}$ chance of rolling less than r and thus losing and a $1/m$ probability of tying. Now, if M rolls an r , they reroll and, since the rolls are independent the probability that N wins is the same probability as in the initial case. Let us count these probabilities over r . Now any r itself has probability $1/n$, so

$$\mathbb{P}(N \text{ wins}) = \frac{1}{n} \sum_{r=1}^n \frac{r-1}{m} + \frac{1}{m} \mathbb{P}(N \text{ wins})$$

Bringing the probability that N wins to the left hand side, we find

$$\begin{aligned} \frac{m-1}{m} \mathbb{P}(N \text{ wins}) &= \frac{1}{n} \sum_{r=1}^n \frac{r-1}{m} \\ \mathbb{P}(N \text{ wins}) &= \frac{m}{n(m-1)} \sum_{r=1}^n \frac{r-1}{m} \\ &= \frac{1}{n(m-1)} \sum_{r=1}^n (r-1) \\ &= \frac{1}{n(m-1)} \frac{(n-1)n}{2} \\ &= \frac{n-1}{2(m-1)}, \end{aligned}$$

where we used $\sum_{i=0}^K i = K(K+1)/2$.

As a corollary, we have that the probability M wins is, after some algebra,

$$\mathbb{P}(M \text{ wins}) = \frac{2m-n-1}{2(m-1)}.$$

Exercise 14.2. Let X be a continuous random variable with uniform density function $1/n$ for $0 < x \leq n$ where $n \in \mathbb{Z}^+$ and 0 otherwise. What is the probability that the n^{th} digit of π^X is prime? Another version would be, what is the probability that the n^{th} digit of π^n is prime where $n \in \mathbb{Z}^+$?

Solution:

15. KUNG, ANDREW

Exercise 15.1. In the Mega Millions lottery game, players pick six numbers from two separate pools of numbers - five different numbers from 1 to 56 and one number from 1 to 46. You win the jackpot by matching all six winning numbers in a drawing. There are also smaller prizes for matching some of the six winning numbers. What is the probability of winning the \$10,000 prize by matching four of the numbers from 1 to 56 and the one number from 1 to 46?

Solution: To win the \$10,000, you must match four of the five numbers from 1 to 56 and match the one number from 1 to 46, while not matching five numbers from 1 to 56.

The total number of possible number combinations is 175,711,536 (thus the probability of winning the Jackpot is 1 in 175,711,536):

$$\binom{56}{5} \binom{46}{1} = 175,711,536.$$

The number of combinations that satisfy the \$10,000 condition can be thought of this way: to match four of the five numbers from 1 to 56, you must get one number wrong. For this wrong number, there are 56 - 5 different numbers you can have instead of the right number, since there are 56 total numbers. Thus there are $\binom{5}{4} \binom{51}{1} = 255$ ways to get four of the five correct, and thus the probability of winning the \$10,000 is

$$\frac{225}{175,711,536},$$

or about 1 in 689,065.

Exercise 15.2. *From a probabilistic standpoint, which chance-based casino game offers the greatest payout (or smallest loss) for players given optimal betting strategy and standard house limitations: slots, roulette, craps, or baccarat? That is, if you were to take \$N to a casino to start, which game would offer the most play before you run out of money, assuming you are betting the same unit every time you play a game?*

Solution:

16. LIU, ANDREW

Exercise 16.1. *You are playing European Roulette, with 36 numbers and 1 zero, with the payout for each number being 1:35. Assume no maximum bet limits and that you have \$122400. Under what circumstances would you win \$100 with at least 99% probability?*

Solution: If you placed a \$100 bet on 34 of 37 spots, you lose with a probability of 3/37. Then, if you place a bet of \$3500 on 34 of 37 spots, you lose with a probability of 3/37. Thus, the probability of losing both since they are independent events is $(3/37)^2$, so the probability of winning at least one is $1 - (3/37)^2 = 0.9934$. If you won in either case, you would win \$100. This was inspired by the following blog post: <http://gregmankiw.blogspot.com/2009/06/was-keynes-really-savvy-investor.html> The problem with this method, of course, is what happens when you lose? You either lose big or you lose enormous! All the small little wins are not enough to make up for the one gigantic loss - if it occurs, it can be devastating.

Exercise 16.2. *You are playing Texas Hold 'Em and your two cards are a King and a seven. Assume no blinds (i.e. pot starts at \$0) and that your opponent will always call your bet. How much should you bet if you want to have an expected winnings of \$100 after this round?*

Solution:

17. LORENZO, ANTONIO

Exercise 17.1. *A color blind goat is grazing on wild flowers. There are five different colored flowers for the goat to choose from: blue, red, green, yellow, and orange. What is the probability the goat will eat a red flower given there are twenty flowers of each color and the goat is equally likely to eat any of them?*

Solution: The probability is $\frac{20}{100} = \frac{1}{5}$.

Exercise 17.2. *You are one of four players at a table playing Texas Hold 'em poker. What is the probability your best hand after the flop is four of a kind while the three other players do not have a hand better than a pair?*

Solution:

18. MOORE, DAVID

Exercise 18.1. *In a game of Texas Hold Em poker, you are dealt the king and ace of spades. What is the probability that you will be able to make a royal flush in spades? (In Texas Hold Em, each player is dealt two cards, and then five cards are dealt face up on the table; each player chooses the best five-card poker hand out of the seven cards they can see. In this case, getting a royal flush would specifically require the ten, jack, and queen of spades to be among the face-up five cards.)*

Solution: There are three cards we are interested in seeing dealt, namely the ten, jack, and queen of spades. The number of ways to get these three cards and then two others is just $\binom{3}{3} \binom{47}{2}$; as there are $\binom{50}{5}$ ways to choose five cards from the remaining 50, we see the probability is just $\binom{3}{3} \binom{47}{2} / \binom{50}{5}$, or 1/1960.

Exercise 18.2. *Suppose we have n monkeys typing at n typewriters. The typewriters are fed by infinite rolls of paper and there is no limit to how long the monkeys can continue typing. At each discrete time step, beginning at $t = 1$, each of the monkeys types a single key on their typewriter, selected uniformly at random and independently of the other monkeys, from a set of 50 possibilities characters (consisting of letters, numbers, and punctuation). One freely available edition of Hamlet contains 157,929 characters. A monkey is said to have typed Hamlet if any consecutive sequence of 157,929 characters on their paper corresponds to the 157,929 characters of Hamlet. As a function of n , what is expected number of time steps until at least one monkey has typed Hamlet?*

Remark 18.3. *My suspicion is that knowledge of the actual text of Hamlet is required in order to solve this problem exactly. It is easy to calculate the probability of any single trial of 157929 characters being precisely those of Hamlet, but a single monkey can be said to go through many such trials – beginning with characters 1 to 157929, then 2 to 157930, 3 to 157931, etc. – which are highly dependent on each other and presumably also on the desired text. Obviously we can get an upper bound by simply ignoring overlapping characters and taking 1 to 157929, 157930 to 315858, etc. as our trials along the lines of the “Murphy’s Law” proof, but I would be interested to know if there is a tractable general method for getting an exact solution.*

Solution:

Exercise 18.4. Consider a graph with n nodes and, initially, no edges. Choose a pair of nodes uniformly at random from the set of all pairs of nodes which do not already have an edge connecting them directly, and add an edge between the two nodes. Repeat the previous step until there are n edges in the graph. What is the probability that the graph is connected?

Solution:

19. PEGADO, SEAN

Exercise 19.1 (Playing Weatherman). The Williams College Mountain Day Committee is trying to decide which of the first three Fridays in October to declare Mountain Day, and it is the Thursday before the first Friday of October. The committee wants to make the decision that gives the best chance of sunny skies.

As of this first Thursday, there is a 30% chance of rain for the first Friday. The forecast predicts a 30% – 50% chance of rain for the second Friday; there is a degree of uncertainty in the forecast because the second Friday is currently a week away. Similarly, the third Friday of the month has a current forecast of 5% – 45%, with even more uncertainty because it is two weeks away.

If the committee does not declare the first Friday Mountain Day, it has up until the following Thursday to make a decision. Each Thursday brings a decrease in the weather's uncertainty, so by the second Thursday the forecast for the second Friday will not have uncertainty (that is, it will be a specific probability between 30% and 50%), and the forecast for the final Thursday will be some 20% range between 5% and 45% (for example, it could be between 15% and 35%, or it could be between 5% and 25%).

If the committee does not declare one of the first two Fridays Mountain Day, the third Friday will be automatically chosen. Assume when given an uncertain forecast that every percentage is equally likely to be the correct forecast.

Which Friday in October should the committee declare Mountain Day in order to minimize the chance of rain and maximize the chance of sunny skies?

Solution:

Question 19.2. What is the difference between the two formulations? Is the new version clearer than the original?

Exercise 19.3 (Sweet decisions.). Once Mountain Day has been decided, the committee must order doughnuts and warm apple cider for the celebrations. There are three different types of doughnuts: plain, sugar coated, and glazed. There is a budget for 800 doughnuts, and the committee must purchase at least 100 of each type. How many Mountain Days would it take to exhaust every possible combination of doughnuts?

Solution: It will take as many Mountain Days as there are combinations. Since we must have at least 100 of each of the three types of doughnuts, we purchase those 300 doughnuts and ask a simpler question: how many ways can we buy 500 doughnuts of three different varieties?

Think of all 500 of our doughnuts as plain and being sorted into three different boxes. One box will remain plain, one will add sugar to all the doughnuts inside, and one will add a glaze to every doughnut. We now wish to calculate the number of ways we can

put our 500 doughnuts into these three boxes. Since the doughnuts are identical before being placed in the boxes, and each box is distinguished, we know from combinatorics that there are $\binom{500+3-1}{3-1} = \binom{502}{2} = 125,751$ ways this can happen. Thus it would take 125,751 Mountain Days to exhaust every possibility; not a task soon accomplished if Mountain Day remains an annual event.

20. PHAM, VINCENT

Exercise 20.1. *Given a monic quadratic polynomial $f(x) = x^2 + ax + b$ where a, b are independent uniform random variables on $[-1, 1]$, what is the probability that f has two real roots?*

Solution: Our polynomial has real roots when $a^2 - 4b \geq 0$, which means $a^2/4 \geq b$. We break the analysis into different cases, depending on the value of b . This is always true if $b \leq 0$, while if $b > 1/4$ then this never holds. We are thus left with b such that $0 \leq b \leq 1/4$. For such b , the probability we choose an a sufficiently large is just, by the law of total probability, the probability that $|a| \leq 2\sqrt{b}$. Thus the probability the two roots are distinct is just

$$\int_{-1}^0 \frac{1}{2} db \cdot 1 + \int_0^{1/4} \frac{1}{2} 2\sqrt{b} db = \frac{1}{2} + \frac{1}{3} \left(\frac{1}{4}\right)^{3/2}$$

Exercise 20.2 (The 21-card Poker). *A game of 21-card Poker consists of 4 tens, 4 Jacks, 4 Queens, 4 Kings and 1 Joker (a wild card that can represent any of the other 20 cards). The hand rankings are the same with normal Poker except now there is a five-of-a-kind hand that triumphs over all. There are two players playing against one another in this game. At the begin of each game, the dealer shuffles the desk then lets each player choose a number of cards to be cut. Those cards are moved to the bottom of the desk and then the dealer deals out five cards to each player. There are two round of betting. Between those two rounds, each player can choose to discard any number of cards in their hands for new cards. Also, let us assume that the players are professional poker-player and play the game with their best interest and to the best possible strategy.*

- (1) *Assume that you got the Joker in your initial hand. Which one has the higher chance of winning now? Can we calculate that chance?*
- (2) *Assume that your opponent is psychic and he knows the exact location of the Joker in the desk before the cuts so that he always cuts in order for the Joker to come to him. Now assume also that no one gets the Joker in their initial hands. Which one has the higher chance of winning now?*
- (3) *How the answer in a) and b) change if those discarded cards are shuffled back into the desk?*

(A win is when you have the better hand regardless of whether you or your opponent decide to flush or not during the betting round.)

Solution:

21. SATOPÄÄ, VILLE

Exercise 21.1. *First roll a fair die. Then flip a fair coin as many times as shown by the face of the die. Let X be the number of heads. What is $\mathbb{E}[X]$?*

Solution:

$$\mathbb{E}[X] = \frac{1}{6} \sum_{i=1}^6 \frac{i}{2} = \frac{1}{6} \cdot \frac{21}{2} = \frac{21}{12} = 1.75.$$

Exercise 21.2. Two computers play Tic-tac-toe on a 3×3 grid. They take turns. Both computers decide to use the Random Tic-tac-toe Strategy: place the next mark randomly in the remaining free spaces. What is the probability that the starting computer wins?

Solution:

22. SHEA, MEGHAN

Exercise 22.1. Consider this gamble. You are asked to flip a coin 5 times. If the coin lands on heads, you win 100 dollars but if it lands on tails, you lose 100 dollars. What is the probability that you walk away a winner (i.e., you made money on the gamble)?

Solution: We give two solutions. Since each flip is independent and therefore one flip does not depend on the result of the other, you can think of it as 5 independent gambles. The probability of winning each gamble is $1/2$ and therefore the probability of losing each gamble is also $1/2$. In order to finish with a net gain, you need to win a majority of the flips so since there are 5 total flips, you need to win at least 3. Therefore if let X denote the number of heads, we want to find $P(X \geq 3)$, which is just $P(X = 3) + P(X = 4) + P(X = 5)$. So

$$P(X \geq 3) = \binom{5}{3} \frac{1}{2}^3 \frac{1}{2}^2 + \binom{5}{4} \frac{1}{2}^4 \frac{1}{2}^1 + \binom{5}{5} \frac{1}{2}^5 \frac{1}{2}^0$$

$$P(X \geq 3) = \frac{5!}{(3!)(2!)(32)} + \frac{5!}{(4!)(32)} + \frac{5!}{(5!)(32)}$$

Therefore the probability of walking away a winner is

$$\frac{10}{32} + \frac{5}{32} + \frac{1}{32} = \frac{16}{32} = \frac{1}{2}.$$

For the second solution, as we are equally likely to have a tail as to have a head, the probability we win must be $1/2$ by symmetry. We gave the first proof as we *must* argue along those lines if the probability of winning is not $1/2$, as then these techniques break down.

Exercise 22.2. Now consider this same gamble as above, but with n tosses. The game now ends when there are five tosses in a row with the same outcome, that is either 5 heads in a row or 5 tails in a row ends the game. Now what is the probability that you walk away a winner?

Solution: For a fair coin, we are just as likely to have 5 heads in a row as 5 tails, and so again the probability is just $1/2$.

23. SHIN, GEA

Exercise 23.1. *How many different ways are there to arrange 20 people around a round table (all seats are equivalent)? Assume that there is one couple among 20 people. What is the probability that the couple sits next to each other?*

Solution: This is a very interesting question. The first guess might be $20!$. But this counts the same arrangements of people around the table for multiple times. The clever way to solve this question is to artificially give orders to the people by a simple trick. For instance, make one person to sit on any chair around a table. Whichever chair he sits on, they are all equal cases. But after making one person to sit, we now have an order. If a second person sits two seats to the right of the first person, we now have the order between the first and the second person. Thus, after putting the first person to a seat, the problem now became just ordering 19 people in a straight line, which is simply $19!$.

Now, let us think about the probability that the couple sits next to each other. Since the couple would always sit next to each other, we can treat them as a one person. Then, using the logic from the previous paragraph, it seems that there are $18!$ ways for them to sit next to each other. But we actually have to think about the couple changing seats within themselves. Thus, we should multiply by $2!$, and the probability comes out as $\frac{2 \cdot 18!}{19!} = \frac{2}{19}$.

Exercise 23.2. *You are playing 5 card poker with 3 other people. It is the last round of betting and the pot is \$1000, of which you have contributed \$250. Another player bets \$100, the other two players call, and it is finally your turn to call (pot is \$1300), but your hand is not ideal. You have a pair Kings (clubs and diamonds), a Queen of spades, and a 2 of clubs, as well as a 5 of clubs. Suppose that you call when the probability of your winning is greater than $\frac{1}{2}$ but you fold otherwise, should you call or fold?*

24. SHIRKOVA, TERESA

Exercise 24.1. *Suppose you have an $n^2 \times n^2$ Sudoku board, where $n \geq 2$ is a positive integer (1×1 Sudoku is very well understood!). We place an integer $m \in \{1, \dots, n\}$ randomly on two distinct squares subject to the rules of Sudoku, namely that the two m 's are not in the same row, column or $n \times n$ box. Let $\ell \in \{1, \dots, n\}$ be distinct from m : find the probability that ℓ is in the same row or column as one of the m 's, as well as the probability that it is in the intersection of a row and a column of the two m 's.*

Solution: There are n^4 squares on the Sudoku board, n^2 squares in each row and n^2 in each column. Thus to be in the same row or column as one of the m 's means we must be in one of $2(n^2 - 1)$ squares. As there are exactly two squares that are in a row of one of the m 's and a column of the other, there are $4(n^2 - 1) - 2$ squares that are in a row or column of one of the m 's. Thus the probability that we place the ℓ in the same row or column as one of the m 's is just $\frac{4(n^2 - 1) - 2}{n^4 - 2} = \frac{4n^2 - 6}{n^4 - 2}$ (since there are $n^4 - 2$ squares still open on the board), and the probability that it is in the intersection of a row and column is $\frac{2}{n^4 - 2}$.

Exercise 24.2. *A jar contain 5 pencils and 7 pens. We pull out two objects, with all pairs equally likely to be chosen. Assume any pencil works with probability $1/7$, and*

pens only work if they are paired with a pencil that doesn't work. What is the probability that at least one object picked out of the jar works?

Solution: If we pull two pens, which happens with probability $\binom{7}{2} / \binom{12}{2}$, then neither works as there is no pencil. If we pull two pencils, which happens with probability $\binom{5}{2} / \binom{12}{2}$, then each pencil works with probability $1/7$, so the probability at least one works is $1 - (6/7)^2$. If we pull one pencil and one pen, which happens with probability $\binom{5}{1} \binom{7}{1} / \binom{12}{2}$, then the exactly one of the two work. Thus the probability that at least one object works is

$$\frac{\binom{5}{2}}{\binom{12}{2}} \cdot 1 - \left(\frac{6}{7}\right)^2 + \frac{\binom{5}{1} \binom{7}{1}}{\binom{12}{2}} \cdot 1 = \frac{615}{1078} \approx .570501.$$

25. XIE, ZOE

Exercise 25.1. Suppose there are 200 possible essay questions on a standardized test. Every test contains two questions randomly selected from the pool of 200, and a test taker has to choose one of the two questions to answer. If someone prepares 20 out of these 200 questions, then what is the probability that he will have one of these 20 questions on the actual test?

Solution: With each of the 20 questions, there are a number of ways to include at least one of these 20 questions. Without loss of generality we can number the 200 questions 1 to 200 and the 20 prepared questions 1 to 20. Not to double count we have

Q1	(1, 2)	(1, 3)	(1, 4)	...	(1, 200)	199 ways
Q2	(2, 3)	(2, 4)	(2, 5)	...	(2, 200)	198 ways
				...		
Q20	(20, 21)	(20, 22)	(20, 23)	...	(20, 200)	180 ways

$$\begin{aligned} & \mathbb{P}(\text{at least one prepared question on the test}) \\ &= \frac{\text{pairs with at least one of the 20 prepared questions}}{\text{total number of possible pairs}} \\ &= \frac{199 + 198 + \cdots + 180}{\binom{200}{2}} \\ &\approx 19.05\% \end{aligned}$$

Alternative (and quicker) answer to question 1

$$\begin{aligned} & \mathbb{P}(\text{at least one prepared question on the test}) \\ &= 1 - \mathbb{P}(\text{no prepared question on the test}) \\ &= 1 - \frac{\text{ways to choose 2 Qs from outside the 20 prepared Qs}}{\text{total ways to choose the 2 questions}} \\ &= 1 - \frac{\binom{180}{2}}{\binom{200}{2}} \\ &\approx 19.05\% \end{aligned}$$

Exercise 25.2. A box has three drawers; one contains two gold coins, one contains two silver coins, and one contains one gold coin and one silver coin. Assume that one drawer is selected randomly and that a randomly selected coin from that drawer turns out to be gold. What is the probability that the chosen drawer is one that contains two gold coins?

Solution: Let A be the event that the draw contains two gold coins, and B be the event that a randomly selected coin is gold. Note $\mathbb{P}(A) = 1/3$, $\mathbb{P}(B) = 1/2$ and $\mathbb{P}(A|B) = \mathbb{P}(A \cap B)/\mathbb{P}(B)$; however, $A \cap B = A$, so $\mathbb{P}(A|B) = \frac{1/3}{1/2} = \frac{2}{3}$, and of course $\mathbb{P}(A|B)$ is the probability that we have selected the draw with two gold coins given that we see a gold coin.

26. XIONG, WENTAO

Exercise 26.1. If we randomly choose 3 of the 8 vertices of a cube, what is the probability that these 3 points form an isosceles right triangle?

Solution: The number of possible ways to choose 3 vertices on a cube is $\binom{8}{3} = \frac{8!}{3!5!} = 56$.

To form a right isosceles triangle, we must choose three vertices on the same face. There are $\binom{6}{1}$ ways to choose a face, and then $\binom{4}{3}$ ways to choose three vertices on that face, for $\binom{6}{1} \binom{4}{3} = 24$ possibilities. Thus the probability is just $24/56$, or about 42%.

Exercise 26.2. We again choose 3 of the 8 vertices of our cube to form a triangle. What is the probability that two randomly chosen triangles formed this way are in the same plane? Assume all triangles are equally likely to be chosen, and we are not allowed to choose the same triangle twice.

Solution: The total number of triangles is $\binom{8}{3} = 56$. There are $\binom{56}{2} = 1540$ ways to choose two different triangles. If our two triangles are in the same plane, they must be in one of the planes that contain 4 of 8 vertices of the cube. In each such plane, there are $\binom{4}{3} = 4$ possible ways to form a triangle, and thus $\binom{4}{2} = 6$ possible ways to select 2 out of the 4 triangles on each plane. There are $6 + 6 = 12$ such planes in a parallelepiped, and thus $12 * 6 = 72$ possible ways that two randomly-chosen triangles formed this way are in the same plane. Therefore, the probability is $\frac{72}{1540}$, or about 4.7%.

27. ZHANG, LIYANG

Exercise 27.1. If you roll a fair die $6n$ times, what is the probability that each number occurs the same number of times?

Solution: The solution involves the multinomial coefficient. The probability we have each number exactly n times is $\binom{(6n)!}{n! \dots n!} \left(\frac{1}{6}\right)^n \cdots \left(\frac{1}{6}\right)^n$, where we have $n!$ six times in the bottom of the multinomial coefficient, and six factors of $(1/6)^n$.

Exercise 27.2. If you roll a fair dice $6n$ times, what is the probability that the average number is exactly a 4?

Solution: Let $X_1, \dots, X_{6n}$ denote the outcomes of the $6n$ rolls. For the average to be exactly 4, we need

$$X_1 + \dots + X_{6n} = 4 \cdot 6n = 24n.$$

While this problem is very similar to the birthday problem, it is complicated by the fact that each X_i is at most 6.

28

Exercise 28.1. *We start at the origin and every minute move right one unit with probability p and left one unit with probability $1 - p$. What is the probability of being 5 units to the right after 10 steps? After 11 steps?*

Solution: To be 5 units to the right, we must have taken 5 more steps to the right than to the left. If L is the number of steps to the left and R to the right, then $R + L = 10$ and $R - L = 5$; solving gives $R = 15/2$, which is impossible. If now $R + L = 11$, then solving gives $R = 8$ and $L = 3$. In this case, the probability is just $\binom{11}{8} p^8 (1 - p)^3$.

Exercise 28.2. *There are n lights with binary switches. Each minute one has to pick one light to turn on or off, and every light is equally likely to be picked. After m times, what is the probability that there are exactly k lights on? Assume initially all lights are off.*

Solution:

29

Exercise 29.1. *There are five Fridays in October 2009. One of these Fridays will be Mountain Day, when all Williams' classes are canceled and everybody is supposed to climb up the mountain and drink hot cider together. Assume that the probability that the weather is good for climbing is 0.6 and the probability that each Friday will be Mountain Day is 0.2. The probability that they will run out of hot cider on the mountain is 0.1. Mr. Lazy loves climbing up the mountain and loves drinking hot cider; however, he always gets up late. The probability that he may get up too late on Mountain Day is 0.35. What is the probability that Mr. Lazy can make it to the top of the mountain and get to drink hot cider given that the second Friday is Mountain Day?*

Solution: The probability is just $.65 \cdot .6 \cdot .9$, which comes from the probability he gets up early enough times the probability it is good climbing weather times the probability there is enough cider. (Note several assumptions are made in computing this probability!)

Exercise 29.2. *There is a wedding tradition in Thailand that the groom has to pay some dowry to the bride's parents as a thank you present for raising. Mr. Handsome currently has a Thai girlfriend and wants to get married to her anytime before he turns 30. His girlfriend's parents are quite conservative and will not let him marry her unless he holds the wedding in the traditional Thai style in Thailand. Therefore, he plans to start saving \$1000 a month to pay for the dowry, the wedding party, and the plane ticket to Thailand. Similar to other couples, Mr. Handsome has a 30% probability of quarreling with his girlfriend each month. If the quarrel happens, there will be a 50% chance that it is his fault, in which case he has to spend \$250 to buy her a big bouquet of flowers*

and bring her to a fancy restaurant for an apology. In other words, if there is a quarrel and it is Mr. Handsome's fault, he only saves \$750 in that month. If he still does not have enough money after he is 30, he will feel upset and hopeless, and thus decide to break up with his girlfriend. Suppose that now he is 23 years old and he needs to save \$75,000 dollars. What is the probability that he will marry his girlfriend? Assume all dollar amounts are fixed for the seven years.

Solution: There are 8 years to save the \$75,000. Each month he saves at least \$750, which comes to \$72000; thus he only needs to save an additional \$3000. Every month where he is not responsible for a quarrel means he saves an additional \$250, so he only needs 12 months (out of the 96 months) where he is not responsible for a quarrel. For each month, the probability that there is a quarrel which is his fault is .15. Thus the probability that he saves enough money is equal to the probability that there are at most 84 months where the quarrel is his fault. This is a binomial problem, where $n = 96$ and $p = .15$; the probability of at most 84 months with a quarrel that is his fault is

$$\sum_{k=0}^{84} \binom{96}{k} .15^k (1 - .15)^{96-k},$$

which is extremely close to 1 (the odds are less than 1 in 10^{50} that he won't save enough money).

30. START OF SECOND SET OF PROPOSED PROBLEMS

31. ARNOSTI, NICK

Exercise 31.1. *In the game of Settlers of Catan, each player rolls two standard dice to begin their turn. All players then receive resources based on the outcome of the roll and the locations of their cities. If a seven is rolled, any player with over 7 resources must discard half of their hand, and then the player who rolled the seven may steal a resource from one of the other three players. Suppose that a player receives resources in the following quantities: one on a roll of 4, 5, or 10, two on a roll of 3, three on a roll of 6, and zero otherwise. If this player ends his turn with two resources, what is the probability that after four rolls he will have had to discard due to the roll of a 7?*

Solution: The player cannot have over seven cards until after the second roll. Thus, the third roll is the first that can cause him to discard. He discards on the third roll if and only if he gained six or more cards on the first two rolls and the third roll is a seven. This can occur in only one way: rolls of 6, 6, 7, which has probability $(\frac{5}{36})^2 \cdot \frac{1}{6} = \frac{25}{7776}$. He discards on the fourth roll if and only if he gained six or more resources on the first three rolls and the fourth roll is a seven. He can gain six or more resources on three rolls in the following ways (in the vector (x_1, x_2, x_3) , x_i represents the number of resources gained on roll i): $(1, 2, 3)$; $(0, 3, 3)$; $(1, 3, 3)$; $(2, 2, 3)$; $(2, 3, 3)$; $(3, 3, 3)$ and permutations of these. On a given roll, $\mathbb{P}(1 \text{ resource}) = \mathbb{P}(4) + \mathbb{P}(5) + \mathbb{P}(10) = \frac{10}{36} = \frac{5}{18}$, $\mathbb{P}(2 \text{ resources}) = \mathbb{P}(3) = \frac{1}{12}$, $\mathbb{P}(3 \text{ resources}) = \mathbb{P}(6) = \frac{5}{36}$, and $\mathbb{P}(0 \text{ resources}) = 1 - \frac{5}{18} - \frac{1}{12} - \frac{5}{36} = \frac{1}{2}$. Thus,

$$\begin{aligned} \mathbb{P}(\text{discard on roll 4}) &= \mathbb{P}(\text{six or more resources on first three rolls}) \\ &\quad \cdot \mathbb{P}(7 \text{ on fourth roll}) \\ &= \left[3! \cdot \frac{5}{18} \cdot \frac{1}{12} \cdot \frac{5}{36} + 3 \cdot \frac{1}{2} \cdot \left(\frac{5}{36}\right)^2 + 3 \cdot \frac{5}{18} \cdot \left(\frac{5}{36}\right)^2 \right. \\ &\quad \left. + 3 \cdot \left(\frac{1}{12}\right)^2 \cdot \frac{5}{36} + 3 \cdot \frac{1}{12} \cdot \left(\frac{5}{36}\right)^2 + \left(\frac{5}{36}\right)^3 \right] \cdot \frac{1}{6} \\ &= \frac{3485}{279936}. \end{aligned}$$

As a player cannot discard on both the third roll and the fourth roll, the answer is $\frac{3485}{279936} + \frac{25}{7776} = \frac{4385}{279936}$, or about 1.57%.

Exercise 31.2. *Suppose that if a Math-Science Resource Center (MSRC) tutor is helping a group of n students on a problem, the amount of time it takes to work through it is $\ln(n+1)$ minutes (thus the more students present the longer it takes as there are more people with potential questions; however, the time cost of each additional student is getting smaller and smaller). Suppose further (and idealistically) that any students who are in the MSRC and have not worked on a given problem come to the tutor as a group, and that students who have not worked on either problem do them in order. Assume that when students come to the tutor as a group, he works exclusively with them until they understand the problem, and then moves on to the next group of students (if any). The probability of N students entering in a given minute is $\frac{e^{1-N}}{e-1}$, and each has probability*

p_1 of having a question on problem 1 and independent probability p_2 of having a problem on question 2. Let w_i denote the time that student i must wait before understanding both problems. Is the average of w_i over all students finite?

32. ATKINSON, BEN

Exercise 32.1. Consider a standard 52 card deck, with 4 suits each encompassing 13 cards numbered 2, 3, ..., 10, J, Q, K, A. What is the probability that a hand of 5 cards contains all cards of the same suit (a flush)?

Solution: The first card can be anything, and then the remaining must be in that suit. One way to compute the answer is

$$\frac{52}{52} \frac{12}{51} \frac{11}{50} \frac{10}{49} \frac{9}{48} = \frac{33}{16660} \approx 0.00198079.$$

In our counting for this solution, order matters. Another way is to say there are $\binom{4}{1}$ ways to choose the suit, and then $\binom{13}{5}$ ways to choose 5 cards in that suit. As there are $\binom{52}{5}$ ways to choose 5 cards, we find the probability is

$$\frac{\binom{4}{1} \binom{13}{5}}{\binom{52}{5}} = \frac{33}{16660}.$$

Exercise 32.2. General Mills is having a new promotion for their cereal. Each box of cereal contains exactly one of a purple cow, an Ephraim Williams action figure, or a Lord Jeff action figure; each box is equally likely to have any of the three prizes. A Williams professor would like to obtain two purple cows and two Ephraim Williams action figures to give to his children (Cameron and Kayla). He does not care about the Lord Jeff action figures, and promptly throws them in the garbage when he obtains one. What is the expected number of boxes this professor must open to obtain the desired number and type of toys?

Solution:

33. BERRY, JACK

Exercise 33.1. An organism with a single chromosome has a genome size of G nucleotides with m single nucleotide mutations in the genome. Assume G is at least 5 orders of magnitude larger than m . A proofreading enzyme starts at one end of the DNA and travels along until it finds a mutation, which it will then correct. On average, how many nucleotides must the enzyme check before it finds a mutation?

Solution: Imagine the genome is divided into runs of nucleotides, $R(n)$, separated by a mutation so that there are $m + 1$ runs of correct nucleotides. Each run has an equal probability of having a certain length,

$$E[R(1)] = E[R(2)] = \cdots = E[R(m + 1)].$$

Also, we know the total length of all the runs, $R(1) + R(2) + \dots + R(m+1)$, equals the number of correct nucleotides, or $G - m$. Thus

$$E[(1)] = \frac{G - m}{m + 1},$$

and hence the proofreading enzyme must read $(G - m)/(m + 1)$ nucleotides before encountering a mutation.

Exercise 33.2. Now suppose the occurrence of a mutation increases the likelihood of another mutation being close by so that if a mutation occurs at nucleotide k , $N(k)$, then there is a .8 probability of there being another mutation within $G/(5m)$. In other words, there is a probability of 0.8 that a mutation is between $N(k) - G/(5m)$ and $N(k) + G/(5m)$. What is the average number of nucleotides the proofreading enzyme must check before encountering a mutation?

Solution:

34. BINDER, ARI

Exercise 34.1. There are 128 players in the US Open draw. Assume that player α beats any of the other 127 players with probability p . Let A correspond to the number of matches α wins at the Open. Find A 's mass function and expected value.

Solution: First, note that the $128 = 2^7$ players are divided into 2^6 pairs. The two players in each pair play against each other, and the winner moves on to the next round, in which there are 2^5 pairs. Play continues until $1 = 2^0$ player remains; he is the winner. Thus, α must win 7 matches to win the tournament. We can view this as the reverse of a geometric distribution; as long as we have successes, we keep playing, stopping after the first failure or when we have 7 successes. Hence,

$$f(A) = \begin{cases} p^n(1-p) & \text{if } 0 \leq n \leq 6 \\ p^n & \text{if } n = 7 \\ 0 & \text{otherwise,} \end{cases}$$

and therefore

$$\begin{aligned} \mathbb{E}[A] &= \sum_{n=0}^6 n f(n) \\ &= p(1-p) + 2p^2(1-p) + 3p^3(1-p) + 4p^4(1-p) + 5p^5(1-p) \\ &\quad + 6p^6(1-p) + 7p^7 \\ &= p(1 + p + p^2 + p^3 + p^4 + p^5 + p^6). \end{aligned}$$

Whenever we have a long, involved answer such as the one above, it is worth checking extreme cases to see how reasonable it is. The most natural and easiest cases to check are $p = 1$ and $p = 0$. Not surprisingly, as $p \rightarrow 1$ the expected number of games won converges to 7, while if $p \rightarrow 0$ it converges to 0. It's easier to see that the expected number of wins is an increasing function of p (perhaps the simplest way to see this is to compute its derivative with respect to p , and note that it is always non-negative).

The next value to use to test our answer is $p = 1/2$, though this is a bit harder to judge. When $p = 1/2$ we get $127/128$, a little less than 1. As our player loses his first

match half the time, and is eliminated after the second match another 25% of the time, such an answer is eminently reasonable; however, there is a better way of looking at the problem. Let's count the total number of wins in the tournament. There are 64 matches in the first round, leading to 64 wins. The next rounds generate 32, 16, 8, 4, 2 and 1 additional wins, for a grand total of 127 wins. If $p = 1/2$ for every player, then they all have the same expected number of wins. Thus we must divide the 127 wins among 128 people, which leads to an expected value of $127/128$ wins per person, exactly what we find. (If we had an infinite tournament, then the expected number of wins when $p = 1/2$ would be 2, as this would be a geometric series.)

While these arguments of course do not prove that our calculation above is correct, they do provide strong support, and illustrate the value of these checks. (The original solution had a typo, and the proposed answer failed these tests.)

Exercise 34.2. Now consider an n -round “full feed in” draw, where $n \geq 3$. The format of this is as follows: Play until you lose. If you win your first n matches, you win. If you win your first $n - 1$ matches and lose, you finish second. If you win your first $n - 2$ matches and lose, you play an additional match for third place. If you lose before the semifinal round, you go from the main draw into the backdraw. The backdraw works as follows: The 2^{n-1} first round losers play against each other, and the 2^{n-2} losers of this round are out of the tournament. The 2^{n-2} winners of this round play against the 2^{n-2} losers from the second round of the main draw. The 2^{n-2} winners of this round play against each other, and the 2^{n-3} winners that result play against the 2^{n-3} losers from the third round of the main draw. The pattern repeats until there are four players remaining in the backdraw (i.e. the four winners that result from the four maindraw quarterfinal losers playing against the four players who won their previous backdraw round), and then these four players play a single elimination tournament down to one. That player finishes fifth in the overall tournament, and is the winner of the backdraw. To summarize, you are done when you either: win the tournament, lose in the finals, make the semis and then win or lose the third place match, lose in the maindraw before the semis but then win the backdraw, or lose in the main draw before the semis and lose again in the backdraw. Verify that a player who loses in the first round and then proceeds to win the backdraw wins $2n - 4$ matches. Find the mass function and expected value of A .

Solution:

35. BROWN, CHAD

Exercise 35.1. Calculate the probability of an MLB team sweeping every game of the playoffs, given the following assumptions:

- (1) Every team is evenly matched.
- (2) The probability of the home team winning any particular game is p (and thus the probability of the away team winning is $1 - p$).
- (3) Assume that all eight playoff teams are randomly assigned a number from 1 to 8, corresponding to their seed, rather than using regular season stats.
- (4) The first series (LDS) is best of 5, with the first, second, and fifth game played at the home of the team with the higher seed. The second series (LCS) and third

series (WS) are best of seven, with the first, second, six, and seventh games at the home of the team with the higher seed.

(5) *The outcome of each game is independent to the outcome of any previous games.*

Solution: Only one team can sweep each series they play in. The LCS and WS are particularly easy to analyze. For a sweep to occur, one team must win each of the first four games, two of which are at their home and two of which are away. Thus the probability that there is a sweep in the LCS is $p^2(1-p)^2$; similarly the probability that there is a sweep in the WS is also $p^2(1-p)^2$.

The probability that the lower seed sweeps a LDS is $(1-p)^2p$, while the probability that the higher seed sweeps the LDS is $p^2(1-p)$. We thus find that the probability that a lower seed sweeps all three series is

$$(1-p)^2p \cdot p^2(1-p)^2 \cdot p^2(1-p)^2 = p^5(1-p)^6,$$

while for a higher seed it is

$$p^2(1-p) \cdot p^2(1-p)^2 \cdot p^2(1-p)^2 = p^6(1-p)^5.$$

There are 8 teams (4 low seeds, 4 high seeds), and thus the total probability that one of the 8 teams sweeps each series it plays is just

$$4p^5(1-p)^6 + 4p^6(1-p)^5 = 4p^5(1-p)^5.$$

For $p = 1/2$ the probability is $4 \cdot 2^{-10} = 2^{-8}$ or 1 in 256.

Exercise 35.2. *What is the probability of a team sweeping every game of the playoffs if the LCS and WS were changed to a 3-3-1 format, where the first, second, third, and seventh game of these series were played at the home of the higher seeded team?*

Solution: This problem can be readily solved by brute force, as there are not too many possibilities to consider. Unlike the previous case, however, the analysis of the LCS and WS is significantly harder, as it matters whether or not the low or high seed is sweeping. There are seven series to consider (four LDS, two LCS and one WS). Let's consider just the American League. Let us assume that the top seed plays the worst and the second seed plays the third. The probability of the first or second sweeping their series is $p^2(1-p)$, while the probability that the third or fourth sweep their series is $(1-p)^2p$. If the fourth seed sweeps, then it will clearly be the lower seed in the next round; similarly if the top seed wins it will clearly be at home. Thus the probability that the top seed sweeps its two American League series is just $p^2(1-p) \cdot p^3(1-p)$, while the probability that the fourth seed sweeps both is $(1-p)^2p \cdot (1-p)^3p$. For the second or third seed to sweep the second round, however, it matters whether or not they are playing the top or the bottom seed. Thus we need to know how likely it is for the top seed to win its series. As they need to win three out of five, the possibilities are either WWW (winning in three games), $LWWW$, $WLWW$, $WWLW$ (winning in four games), or $LLWWW$, $WLLWW$, $WWLLW$, $LWWLW$, $WLWLW$, $WWLLW$ (winning in five games); the probabilities of these events are readily calculated. For example, in $WLLWW$ the top seed wins games 1 and 5 at home, wins game 4 on the road, and loses game 2 at home and loses game 3 on the road. The probability of this happening is $p(1-p)p(1-p)p$. We can find the probability that the fourth seed wins its series by the law of total probability, as it is just one minus the probability the top seed

wins. We then have a conditional probability for the second or third team sweeping, depending on whom their opponent is.

36. BUSTARD, TODD

Exercise 36.1. *Consider postseason baseball. This year the record for most consecutive games with an RBI was tied twice. Assuming that a player gets four at bats per game, plays in n postseason games and has a batting average p_0 and a probability p_1 of hitting a home run. Also, assume that a team has probability q of having a runner in scoring position every time our batter steps up to the plate. What is the probability that this player gets an RBI in each of his first k games?*

Solution: Obviously the model above has numerous simplifying assumptions. The knowledgeable baseball fan will note that there is one very important piece of information missing above, namely that if the player walks with the bases loaded, then this too counts as an RBI! For simplicity we assume that this and other ways of getting an RBI without getting a hit (such as a sacrifice fly) do not happen. Equivalently, we confine ourselves to RBIs from hits.

Even under this assumption, the problem is not well-defined. It makes a difference whether or not the runner is on second or third, as a single should score a runner from third but may not score a runner from second. For simplicity, we assume any runner on second or third (i.e., a runner in scoring position) scores on any hit, while a runner on first only scores on a home-run.

The probability that our batter gets a hitting RBI in an at-bat is thus $p_1 + q(p_0 - p_1)$. We compute this as follows: if he hits a home run, he clearly receives an RBI; if he does not hit a home run but does get a hit (which happens with probability $p_0 - p_1$) then he earns an RBI if and only if someone is in scoring position. Thus the probability that he does not get an RBI in an at-bat is $1 - (p_1 + q(p_0 - p_1))$, so the probability that he does not get an RBI in the game is $(1 - (p_1 + q(p_0 - p_1)))^4$ (since he has four at-bats).

Therefore

$$\text{Probability at least 1 RBI in a game} = 1 - (1 - (p_1 + q(p_0 - p_1)))^4;$$

thus the probability that he gets an RBI in each of the first k games is simply the above to the k^{th} power.

Exercise 36.2. *Generalize the above model to include factors such as walking with the bases loaded (admittedly rare), sacrificing with a runner on third and less than two outs (not so rare), et cetera.*

37. CHO, JAEHONG

38. CITRO, BRIAN

Exercise 38.1. *You have a total of six socks in your drawer, which are a mixture of black and white. You pull out two socks randomly. The probability that you get a pair of two white socks is $2/3$. What is the probability that you get a pair of two black socks?*

Solution: The probability of getting two black socks is 0. Since you know there are 6 socks and the probability of getting two white socks is $\frac{2}{3}$, there must be 5 white socks

and one black sock, because the probability of drawing two white socks would then be $\frac{5}{6} * \frac{4}{5} = \frac{2}{3}$.

Another way to look at this is the following. Assume we have w white socks and b black socks, with $w + b = 6$ of course. The number of ways of drawing two socks (order does not matter) is $\binom{6}{2}$, while the number of ways of drawing two white socks is $\binom{w}{2}$. Thus the probability we get two white socks is just

$$\frac{\binom{w}{2}}{\binom{6}{2}} = \frac{\frac{w!}{2!(w-2)!}}{\frac{6!}{4!2!}} = \frac{w(w-1)}{6 \cdot 5};$$

if we want this to equal $2/3$, then we need $w = 5$. This implies there is only one black sock, and thus the probability of a pair of black socks is zero.

Exercise 38.2. *There is a line of 100 people waiting to board an airplane. Each of the 100 people in line are assigned a seat on the plane corresponding to their position in line; i.e., the first person in line is assigned seat 1, the last person in line is assigned seat 100. Upon boarding the plane, the first person in line decides to not sit in his assigned seat but to instead randomly pick a seat to sit in from the remaining group of 99 seats. Each successive passenger sits in his/her assigned seat unless it is taken by someone, in which case they randomly choose from the remaining open seats. What is the probability that the 100th passenger sits in his assigned seat?*

Solution:

39. FISH, CROSBY

Exercise 39.1. *Suppose that a student is signing up for a winter study class. The probability of getting into any choice between 1 and 4 is given by $f(x_i) = \frac{x_i}{10}$. Which choice should the student expect to get?*

Solution: Let X take values 1 through 4, then we take

$$\mathbb{E}(X) = \sum_{x=1}^4 x \left(\frac{x}{10}\right) = 1 \cdot \frac{1}{10} + 2 \cdot \frac{2}{10} + 3 \cdot \frac{3}{10} + 4 \cdot \frac{4}{10} = 3$$

Exercise 39.2. *In a game of rock, paper, scissors, a rock beats scissors, scissors beat paper, and paper beats rock. Assume each player randomly chooses one of rock, paper, scissors, with each choice equally likely. Assume every time a player wins they get 1 point, and every time they lose they lose 1 point; if the two players choose the same object, it is a tie and no one gains or loses a point. The first person to reach 10 points wins. What is the probability that the first player wins within the first 12 games?*

Solution: The probability that you win a point is $1/3$, the probability that you lose a point is $1/3$, and the probability that there is a tie is $1/3$. There is only one way to win in the first 10 games: you must win each time, which happens with probability $(1/3)^{10}$.

To win in exactly 11 games, you must win 10 times and have one tie; however, the tie must occur before the eleventh game, as otherwise you would win in 10. There are $\binom{10}{1}$ ways to choose one of the first 10 games to be a tie, and thus the probability you win in exactly 11 games is just

$$\binom{10}{1} \left(\frac{1}{3}\right)^9 \frac{1}{3} \cdot \frac{1}{3} = \frac{10}{3^{11}}.$$

To win in exactly 12 games, either you win 10 times and have two ties (both of which must happen by the eleventh game) or you win 11 times and have one loss (and the loss must be in the first ten games). Thus this probability is simply

$$\binom{11}{2} \left(\frac{1}{3}\right)^9 \left(\frac{1}{3}\right)^2 \cdot \frac{1}{3} + \binom{10}{1} \left(\frac{1}{3}\right)^9 \frac{1}{3} \cdot \left(\frac{1}{3}\right)^2 = \frac{65}{3^{12}}.$$

Combining everything, we see the probability the first person wins in the first 12 games is

$$\frac{1}{3^{10}} + \frac{10}{3^{11}} + \frac{65}{3^{12}} = \frac{104}{531441}.$$

Let's consider another approach, where we try to arrive at this answer by considering all sequences of 12 tosses where player one nets *at least* 10 points. (In other words, there is no mercy rule, and once player one has 10 points he can continue playing and taking player 2's money!). For player 1 to net 10, he can either have 10 wins and 2 ties, 11 wins and 1 tie, 12 wins, or 11 wins and 1 loss. Thus the probability one of these happens is

$$\begin{aligned} & \binom{12}{2} \left(\frac{1}{3}\right)^{10} \left(\frac{1}{3}\right)^2 + \binom{12}{1} \left(\frac{1}{3}\right)^{11} \left(\frac{1}{3}\right) + \binom{12}{0} \left(\frac{1}{3}\right)^{12} + \binom{12}{1} \left(\frac{1}{3}\right)^{11} \left(\frac{1}{3}\right) \\ &= \frac{91}{531441}. \end{aligned}$$

What went wrong? The problem is we forgot a few cases. We might have had 10 points by the tenth or eleventh toss, but ended with less. Thus we could end with as few as 8 points if we have 10 wins followed by two losses (which happens one way) or 9 points if we have 10 wins a loss and a tie. As he only has 10 wins, the last round must be either a tie or a loss, and in fact the loss cannot occur earlier than the eleventh round (though the tie can occur anywhere). If the loss is in the last spot, we have $\binom{11}{1}$ ways to choose where the tie is, while if the loss is in the second to last spot then the last spot must be a tie, in which case there is one way to do this. All told, we find there are 1+11+1 more possibilities. As $3^{12} = 531441$, the probability is

$$\frac{91}{531441} + \frac{13}{531441} = \frac{104}{531441},$$

the same answer as before!

40. FORD, AARON

Exercise 40.1. *There are m people in a circle. Alice begins by flipping n independent, biased coins, each with probability p of landing on a heads. She passes the coins that are heads to the right, and the process is repeated, except that the second person passes the coins that come up tails. We keep alternating, with odd numbered people passing on heads and even numbered people passing on tails. What is the expected number of coins that return to Alice?*

Solution: For a coin to return to Alice, the sequence of tosses must be $HTHTHT \dots$, ending in a tail if there are an even number of people and a head if there are an odd

number of people. We set

$$p = \begin{cases} p(1-p)p(1-p) \cdots p(1-p) = p^{\frac{m}{2}}(1-p)^{\frac{m}{2}} & \text{if } n \text{ is even} \\ p(1-p)p(1-p) \cdots p(1-p)p = p^{\frac{m+1}{2}}(1-p)^{\frac{m-1}{2}} & \text{if } n \text{ is odd.} \end{cases}$$

We thus have a binomial process with n events and probability p of success, implying the expected value is np , or $np^{\frac{m}{2}}(1-p)^{\frac{m}{2}}$ if m is even and $np^{\frac{m+1}{2}}(1-p)^{\frac{m-1}{2}}$ if m is odd.

Remark 40.2. *Is the following approach correct? Alice begins with n coins. We expect her to pass np coins to the second person. Flipping np coins, we expect this person to pass $np(1-p)$ coins, the 3rd person to pass $np(1-p)p$ coins, and so on. Thus if m is even, we expect Alice to get $np^{\frac{m}{2}}(1-p)^{\frac{m}{2}}$ coins, while if m is odd we expect Alice to get $np^{\frac{m+1}{2}}(1-p)^{\frac{m-1}{2}}$ coins in return, the same answer as above.*

Exercise 40.3. *Continuing the problem above, assume now $p = 1/2$. As a function of n , how many people have to be in the group if we expect Alice to have no coins returned to her?*

Solution: Note that in the special case that $p = 1/2$ the expected number of coins returned to Alice is just $n/2^m$. Thus if $n < 2^m$, we expect Alice to get fewer than 1 coin. If, however, $n = 2^m - 1$ with m large, then for all intents and purposes we expect Alice to get one coin back. Thus the ‘right’ way to interpret this problem is that we want $n/2^m < .5$ so that the expected number of coins returned to Alice is less than 1. In other words, $n < 2^{m-1}$.

41. GROVER, MICHAEL

Exercise 41.1. *Consider the following game: we roll a fair dice until we either roll a one or choose to stop. Our strategy will be a ‘pure’ strategy as follows: we choose some number c and stop if we get a c or higher, and continue if we roll less than c (and of course greater than 1). Fix an integer k , and the payoff is defined as follows: we receive r^k dollars when we stop on r . Let $S_c(k)$ represent the strategy of stopping at c or more. If $k = 1$ then the expected value of the strategies are \$3.5 for $S_6(1)$, \$4 for $S_5(1)$ and $S_4(1)$, and less for $S_3(1)$, $S_2(1)$ and $S_1(1)$. What must k equal for strategy $S_6(k)$ to be superior to $S_5(k)$?*

Solution: Clearly if k is sufficiently large then strategy $S_6(k)$ is better than $S_5(k)$, so there is such a k . To find the smallest k , note that for strategy $S_6(k)$ that we are equally likely to stop rolling on a 6 or a 1, while for $S_5(k)$ we stop on a 6, 5 or 1 each with probability $1/3$. To see this, consider for example strategy $S_5(1)$. We keep rolling until we get either a 1, 5 or 6; by symmetry each of these outcomes is equally likely, and thus each occurs with probability $1/3$.

Thus the expected payoff in strategy $S_6(k)$ is

$$6^k \cdot \frac{1}{2} + 1^k \cdot \frac{1}{2},$$

while for strategy $S_5(k)$ it is

$$6^k \cdot \frac{1}{3} + 5^k \cdot \frac{1}{3} + 1^k \cdot \frac{1}{3}.$$

A straightforward calculation shows $S_6(3) < S_5(3)$ but $S_6(4) \geq S_5(4)$.

Exercise 41.2. *Prove or disprove: If X is a random variable with a Poisson distribution with parameter N with N a random variable with a Poisson distribution with parameter λ , then X is a Poisson random variable with parameter λ .*

Solution:

42. JACKSON, STEVEN

Exercise 42.1. *Suppose that there are n air molecules inside a box of volume 1 m^3 . What is the probability that no molecule occupies a specific cubic centimeter (which is 10^{-6} cubic meters) of the box?*

Solution: In order to actually “solve” this, we need some assumptions. The first is that each air molecule’s probability of being in that space is independent of the others, and further that our molecules may be regarded as points (and thus there is no danger of half of the molecule being inside and half outside our region). In a somewhat related manner, we also assume that each air molecule behaves essentially the same, so that there is no difference between O_2 , N_2 , and CO_2 molecules. Finally, we assume that there is nothing special about the chosen cubic centimeter.

With these assumptions, the problem is simple. Each molecule will have a probability of $(1 - 10^{-6})$ of not occupying the specific cubic centimeter, by the complement rule. As they are independent, the joint probability will simply be $(1 - 10^{-6})^n$.

Of course, given our assumptions, this will simply be an approximation, and hopefully a good one. However, we are free to wonder about how much our approximations cost us. While gasses are diffuse, they do collide with each other, and thus surely aren’t truly independent. Moreover, some molecules will be polar, hence there can also be longer range electromagnetic interactions. The question is, how much is the assumption of independence violated by these?

We can also ask whether any cubic centimeter must automatically be equivalent. It doesn’t seem unreasonable that a cubic centimeter in the corner of the box and one in the center of the box will be somewhat different, but again the question is how much will this affect our answer? These are terrific questions to ask, and lead to interesting models.

Exercise 42.2. *In the past three years, there has been a torrential downpour in Williamstown on the Saturday of the fourth weekend in October. What is the probability of a torrential downpour on that Saturday this year?*

Solution: As weather models are typically untrustworthy more than two weeks away, none of our models allow us to avail ourselves of these results more than to approximate the probability of raining in the middle of October.

43. WISA KITICHAIWAT

Exercise 43.1. *Suppose that each student has a 20% chance of failing a midterm in a probability class, independent of everyone else. If there are 35 students in this class, what is the probability that the number of students who fail the midterm will be between 7 and 15?*

Solution: Letting X be the number of students who fail the midterm, we find

$$\mathbb{P}(7 \leq X \leq 15) = \sum_{m=7}^{15} \binom{35}{m} (0.2)^m (0.8)^{35-m} \approx 0.566634.$$

Exercise 43.2. *Professor Miller just won the first prize of the lottery and decided to generously donate some of his wealth to a random charity. As a probability professor from Williams, he thinks it is too boring to just give some money to an organization. Therefore, he goes to a charity and randomly gives two die to its director; the first one being purple and the second being gold. He asks the director to simultaneously roll both die as many times as that person wants; however, she has to stop rolling the die when the sum equals to 8. The director can choose to stop rolling both die whenever she wants. The amount of money that Professor Miller donates is $10^{x_1} + 100x_2$, where x_1 is the outcome of the first dice and x_2 the outcome of the second. If x_1 is the payoff of the pink die and x_2 is the pay off of the blue die. What is the best exit strategy for the director to play in order to get as much money as possible?*

Solution: If we were to accept any roll, the expected value is \$185,535. If we stop whenever the first die is a 6, then we stop on one of the following pairs: (6,1), (6,2), (6,3), (6,4), (6,5), (6,6), (2,6), (3,5), (4,4), (5,3). There are 10 pairs, and thus the probability of stopping on each is just 1/10 (as all pairs are equally likely). In this case the expected value is 611500. As the most money possible with a first roll of a 5 is $10^5 + 600 < 611500$, the expected payout decreases if we stop before a 6 on the first roll.

44. KOLOGLU, MURAT

Exercise 44.1. The Timekeeper. *The Timekeeper likes to use a traditional clock, with the hour and minute hands continuously rotating circularly, with one full rotation of the hour hand signifying 12 hours and one full rotation of the minute hand signifying 60 minutes (1 hour). Of course, being the Timekeeper, his clock is perfect, and there is no margin for error. At some point in time, he glances at his clock and records the small angle between the two hands to be $x\pi$. Some random moment in the next 24 hours, he glances for a second time. What is the probability that the smaller angle between the hour and the minute hands the second time he looks at his clock is less than $x\pi$?*

Solution: There are several key observations to make. We know that the positions of the two hands are dependent. By position, all we actually mean is the angle from some set axis passing through the center of the circle. One might define this axis to be the line along 12 and 6 on the clock's face, so on and so forth. Now, the hour hand completes one rotation in 12 hours and the minute hand completes one rotation in 1 hour. We can say that their angular velocities are then:

$$v_{\text{Hour hand}} = \frac{2\pi}{12 \text{ hours}}$$

$$v_{\text{Minute hand}} = \frac{2\pi}{1 \text{ hour}}$$

One can infer that the angle between the two hands of the clock changes at angular velocity

$$v_{\vartheta} = v_{\text{Minute hand}} - v_{\text{Hour hand}}$$

$$v_{\vartheta} = 11 \left(\frac{2\pi}{12 \text{ hours}} \right)$$

Observe that due to radial symmetry the particular choice of an axis does not matter. If we transfer ourselves to the frame of the hour hand and choose our axis to be along the hour hand, then it becomes apparent that the angle ϑ completes one rotation in $\frac{12}{11}$ hours. This means that in 12 hours the angle ϑ completes 11 rotations. In fact, this is an extra bit of information that we don't need to solve this problem. The key concept is that the rate of change of the angle is constant, which means that the value of the angle is uniformly distributed in time.

Going back to the question asked, it is easy to see that the probability of getting a certain x is irrelevant. What is being asked is essentially $\mathbb{P}(\vartheta' \leq x\pi)$, where ϑ' is the small angle between the two hands, in painted language. We have inferred that ϑ is uniformly distributed and we know it has to span the full 2π . Then the small angle ϑ' is uniformly distributed on $[0, \pi]$. Therefore $\mathbb{P}(\vartheta' \leq x\pi) = x$ for $0 \leq x \leq 1$.

Exercise 44.2. Modern Times. *The Timekeeper has realized that he needs heightened precision to keep up with modern times and technology. So he has installed a seconds hand and a milliseconds hand on his perfect clock. Unlike the seconds hands of lesser, more traditional clocks, neither of the seconds and milliseconds hands 'tick'; they move continuously. Being the Timekeeper he has infinite time. So he decided that it would be fun to do some time-consuming math. He derived the probability of the greatest angle of the quadrilateral formed by the points where the 4 hands of his clock would intersect the circular frame of his clock being greater than $x\pi$ when he looks at his clock at a random moment in time. What expression did he get? Using this probability, what is the expected greatest angle of the quadrilateral?*

Solution:

45. KUNG, ANDREW

Exercise 45.1. *In the Mega Millions lottery game, players pick six numbers from two separate pools of numbers - five different numbers from 1 to 56 (white balls) and one number from 1 to 46 (white ball). The payout structure, rounded to the nearest whole dollar, is as follows:*

- Five white and one yellow: Jackpot (Odds: 1 in 175,711,536)
- Five white and no yellow: \$250,000 (Odds: 1 in 3,904,701)
- Four white and one yellow: \$10,000 (Odds: 1 in 15,313)
- Four white and no yellow: \$150 (Odds: 1 in 13,781)
- Three white and one yellow: \$150 (Odds: 1 in 13,781)
- Three white and no yellow: \$7 (Odds: 1 in 306)
- Two white and one yellow: \$10 (Odds: 1 in 844)
- One white and one yellow: \$3 (Odds: 1 in 141)
- No white and one yellow: \$2 (Odds: 1 in 75)

$$\frac{\binom{5}{3} \binom{51}{2} \binom{1}{1}}{\binom{56}{5} \binom{46}{1}} = \frac{2125}{29285256},$$

Approximately how large must the Jackpot be before a \$1 lottery ticket is a good investment? That is, when is the expected value greater than \$1?

$$\text{Jackpot} \cdot \frac{1}{175,711,536} + 250,000 \cdot \frac{1}{3,904,701} + 10,000 \cdot \frac{1}{689,065} + 150 \cdot \frac{1}{15,313} + 150 \cdot \frac{1}{13,781} + 7 \cdot \frac{1}{306} + 10 \cdot \frac{1}{844} + 3 \cdot \frac{1}{141} + 2 \cdot \frac{1}{75},$$

Exercise 45.2. *Two people, Andrew and Jack, are in a fantasy football league together. Through six weeks, Andrew's team is 5-1 with 725 total points (first place out of ten), and Jack's team is 0-6 with 538 total points (ninth place out of ten). Yahoo! projections has Andrew's team at 118 points this week and Jack's team at 106 points. If Andrew and Jack are facing each other this week, what is the probability that Andrew wins? What factors must be taken into account to do such a calculation, and how reliable/accurate will these calculations be?*

46. LIU, ANDREW

Solution: There are 25 prime numbers between 1 and 100 and you have a $\frac{1}{100}$ chance of rowing any particular prime, so our first guess might naturally be that the answer is just $\frac{1}{100^{25}}$. The error with this is that it does not matter the order in which we roll the 25 primes, only that our first 25 rolls are distinct primes at most 100. We must take into account the fact that there are $25!$ ways to order the first 25 prime numbers, and thus the probability is

$$25! \cdot \frac{1}{100^{25}} = \frac{236682282155319}{152587890625000000000000000000000000} \approx 1.55112 \cdot 10^{-25}.$$

Exercise 46.2. Suppose you run an auction site that sells luxury handbags. You start all auctions at 2 cents, each bid increases the price of the auction by another 2 cents, and the auction is 5 days long. However, each bid costs the bidder X cents to make, and each bid extends the length of the auction by 20 seconds. Additionally, the winner

of the auction needs to pay the final price of the item. Given a \$1000 bag, number of bidders Y who act optimally (only bid at the last second), are determined (never ever sleep and are always watching the auction), and thrifty (will not spend more than \$50 total) find the relationship between X and Y to make your expected profit 0 (i.e., profit equals how much you get from sale value and bid money minus cost of bag).

Solution: The idea taken from <http://www.maxloren.com>. The site uses this strategy and is obviously very profitable (they charge 60 cents a bid).

47. LORENZO, ANTONIO

Exercise 47.1. Suppose you roll two identical, normal, fair, six-sided dice at the same time. Find the expected value of the sum of the two dice after one roll.

Solution: One way to do this is by brute force:

$$2 \cdot \frac{1}{36} + 3 \cdot \frac{2}{36} + \cdots + 6 \cdot \frac{5}{36} + 7 \cdot \frac{6}{36} + 8 \cdot \frac{5}{36} + \cdots + 11 \cdot \frac{2}{36} + 12 \cdot \frac{1}{36} = 7.$$

Another way to do this is to use linearity of expectation. Let X_i be the outcome of die i , and let $X = X_1 + X_2$. Then $\mathbb{E}[X] = \mathbb{E}[X_1] + \mathbb{E}[X_2]$, and

$$\mathbb{E}[X_i] = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} = 3.5;$$

thus we again obtain 7.

Alternatively, we can solve this problem by noting that there are 21 possible pairs of outcomes where order does not matter, ranging from (1,1), (1,2) et cetera all the way to (5,6), (6,6). We then note that there are two ways of getting each outcome except for the six diagonal possibilities ((1,1), (2,2), ..., (6,6)); note $(21 - 6) \cdot 2 + 6 = 15 \cdot 2 + 6 = 36$, which accounts for all the events.

Exercise 47.2. Suppose you roll n fair dice at the same time a total of m times. Find the expected value of the sum of all the rolls.

Solution: We could again determine the answer by brute force, but it is much easier to use linearity of expectation. This is equivalent to rolling nm die. If we let $X = X_1 + \cdots + X_{nm}$, then the expected value of the sum is the sum of the expected values, and thus the expected value of our sum is $nm \cdot 3.5$.

48. MOORE, DAVID

Exercise 48.1. Log-binomial distribution. Let $X \sim \text{Bin}(n, p)$, and define $Y = 2^X$. Give the mass function for Y .

Solution: We know the mass function for X is

$$f_X(x) = \mathbb{P}(X = x) = \binom{n}{x} p^x (1 - p)^{1-x}.$$

Substituting $x = \log_2 y$ produces

$$\begin{aligned} f_Y(y) &= \mathbb{P}(Y = y) \\ &= \mathbb{P}(2^X = y) \\ &= \mathbb{P}(X = \log_2 y) \\ &= \binom{n}{\log_2 y} p^{\log_2 y} (1-p)^{1-\log_2 y} \end{aligned}$$

Note that unlike the continuous case, in which we need to work with the cumulative distribution function, here we can substitute directly into the mass function.

Exercise 48.2. *You and a friend have identical coins which land on heads with probability p and tails otherwise. At some regular time interval, you and your friend flip your coins simultaneously. If you throw a head, you mark down a point for yourself, and if your friend throws a head, they get a point. This continues until both of you have at least ten points. What is the expected number of time steps that the game runs?*

Solution:

49. PEGADO, SEAN

Exercise 49.1. Game time. *Consider the following game a friend proposes to you. You are shown an integer k , $1 \leq k \leq 100$. A random integer X is drawn from the uniform distribution on the integers $\{1, 2, \dots, 100\}$ (that is, the probability X equals n is $\frac{1}{100}$ if $n \in \{1, 2, \dots, 100\}$ and 0 otherwise) is generated. If $X \leq k + 10$, you win \$1; otherwise, you lose \$2. For what values of k should you play?*

Solution: One of the standard ways of determining whether or not it is worth playing a game is to compute the expected value; if it is positive it is worth playing in general, while if it is negative it isn't. (This is not entirely true, as some people are risk averse or risk seeking; imagine losing one million dollars with probability one in a million and otherwise winning two dollars otherwise; although the expected value is positive, for many people the danger of losing a million dollars is not worth the possibility of gaining two dollars.) For each k , we compute the expected value of the game. The probability we win grows with k ; clearly the probability we win ranges from a high of 1 if $k \geq 90$ to a low of .11 if $k = 1$. In general, the probability we win is $\frac{\min(k+10, 100)}{100}$ for $1 \leq k \leq 100$. Thus the expected value is

$$1 \cdot \frac{\min(k+10, 100)}{100} - 2 \cdot \left(1 - \frac{\min(k+10, 100)}{100}\right).$$

The expected value is negative when $k = 66$ and positive when $k = 67$, so it is worth playing whenever we see a $k \geq 67$.

Exercise 49.2. *Now assume we must play. If k is chosen uniformly from $\{1, \dots, 100\}$, what is the expected value from playing?*

Solution: If $k \geq 90$ (which happens with probability .1) our expected value is 1. For the other k the probability of winning is $\frac{k+10}{100}$, and thus the expected value is

$$1 \cdot \frac{1}{10} + \frac{1}{100} \sum_{k=1}^{89} \left(1 \cdot \frac{k+10}{100} - 2 \left(1 - \frac{k+10}{100} \right) \right) = -\frac{423}{2000};$$

thus the best strategy in this case is not to play!

50. PESKOE, BEN

Exercise 50.1. *It is fall in Williamstown, and the big tree in the center of the science quad is losing its leaves. The leaves happen to be identical ℓ by ℓ squares, where ℓ is probably around .1 meters or so. The tree is very big, and distributes the leaves evenly over the quad. Since they are flat, they can overlap on the ground. The quad is exactly 100m by 50m. If I am standing on a given point on the quad after n leaves have fallen, what is the probability that I am standing on a clear patch of grass?*

Solution: We can start by finding the probability that I am standing on a given leaf, say leaf i . The probability that I am standing on a line parallel with Route 2 through leaf i is $\frac{\ell}{100}$, and the probability that I am standing on a line perpendicular with Route 2 through leaf n is $\frac{\ell}{50}$. These two events are independent, so the probability that I am standing on leaf i is given by $\frac{\ell}{100} \cdot \frac{\ell}{50} = \frac{\ell^2}{5000}$.

Now, since the locations of all the leaves are independent, we can treat this situation as n identical independent random variables. To figure out the probability of standing on any one of them, we must calculate the probability that we are not standing on any of them and take the complement. That is, $1 - (1 - (\frac{\ell^2}{5000}))^n$.

Exercise 50.2. *Assume the same tree continues to shed leaves forever. What is the expected number of leaves it would take to cover the entire science quad?*

Solution:

Exercise 50.3. *Now assume that we have studied the tree more closely and determined some patterns regarding its leaves falling. Namely, the time between leaves falling has a Poisson distribution with parameter $\lambda = 1$ second. We also found that leaves fall for exactly 20 days (1,728,000 seconds). What is the probability that the science quad will be completely covered at the end of the fall?*

Solution:

51. PHAM, VINCENT

Exercise 51.1. *N men and N women are seated at a round table at random. What is the probability that they are seated alternatively by gender?*

We know that there are $(2N - 1)!$ way to seat $2N$ people on a round table when all we care about is the relative ordering of the people. This is because, by symmetry, we might as well assign the first person to sit at spot 1. Now we only need to find the number of ways N men and N women can be seated alternatively by gender. We first seat the N men at the table at every other seat, which gives $(N - 1)!$ different ways as we might as well place the first man at seat 1. We then seat the N women at the round

table. Here we notice that since the men are different, it does matter where we sit the very first women; thus there are $N!$ ways to seat the women at the N seats between the men at the table. Thus the probability is just $\frac{(N-1)!N!}{(2N-1)!}$. When $N = 2$ this probability is just $1/3$; this falls to $1/10$ when $N = 3$ and $1/35$ when $N = 5$.

Exercise 51.2. N people are seated at a round table for two seminars. After the first seminar they take a break and then are seated again at the round table randomly. What is the probability that there are at least two persons whose number of people set between them are the same before and after the break.

Solution:

52. SATOPÄÄ, VILLE

Exercise 52.1. In Finnish lottery the participants choose 7 unique numbers between 1 and 39. Then the next Saturday the lottery committee randomly chooses 7 numbers. Each number is equally likely to be selected. If these 7 numbers match your 7 numbers, you win a million euros. A lottery ticket costs 1 euro. If you participate every week for a total of 50 years, what is your expected return? You can assume that every year a total of 52 lotteries are organized.

Solution: Since the probability of winning in any week is

$$P(\text{win}) = \frac{1}{\binom{39}{7}} = \frac{1}{15380937},$$

the expected return in a single lottery is

$$1000000 \cdot \frac{1}{15380937} - 1 \cdot \frac{15380936}{15380937} = -\frac{14380936}{15380937} \approx -0.934984 \text{ euros.}$$

Thus the total expected return in 50 years is just

$$50 \cdot 52 \cdot \frac{-14380936}{15380937} \approx -2430.96.$$

Of course, many lotteries also have smaller prizes as well, so the expected return could be better than this.

Exercise 52.2. You flip a biased coin that lands heads with probability $p \in [0, 1]$. If it lands heads, you win. Otherwise your friend wins. How would you simulate such a biased coin and determine the winner using a fair coin?

Solution: Assume we have a fair coin. We can simulate any number $p \in (0, 1)$ as follows: write p in base 2 (if there are two expansions for p , namely a finite and an infinite one, choose the finite one). We flip our coin N times and create a number x by letting the n^{th} binary digit be a 1 if toss n is a heads and a 0 if toss n is a tail. We choose N so that if we continue tossing, no matter what we get we will not change which side of p our number lies on. For example, imagine $p = \sqrt{2}/2 \approx 0.707107$ and our sequence of tosses starts $HHTHTTHTH$. Then

$$x = \frac{1}{2} + \frac{1}{4} + \frac{1}{16} + \frac{1}{128} + \frac{1}{512} = \frac{421}{512} \approx 0.822266;$$

as this number is greater than p , we would consider this a toss that landed on a tail.

Exercise 52.3. *Is it physically possible to construct a coin that lands heads with probability p , where $p \notin \{0, 0.5, 1\}$, i.e. the problem is non-trivial?*

Solution: One possibility is to have one side of the coin heavier than another, and investigate the affect this would have on the probability of the coin landing on heads or tails. Of course, we need to be clear on the groundrules. For example, do we always toss the coin from the same initial configuration (say head up) but with variable force and spin? How are we changing these? There are people who practice tossing coins, and can toss a coin to land a given way with probability greater than 50%.

53. SHEA, MEGHAN

Exercise 53.1. *There are 3 boxes, each meant for a specific person. The boxes are distributed among 5 people randomly, with the 3 people the boxes were meant for in the group of 5 and each box equally likely to end up with each person. What is the probability that at least 1 of the boxes is given to the right person?*

Solution: Since the boxes are distributed randomly, the probability a box is given to the correct person is just $\frac{1}{5}$. There are two ways to compute the probability that at least one box is correctly assigned. The first is brute force, where we compute the probability exactly one, two or three people are given the correct box. The second is to use the law of total probability; if we can determine the probability no one gets the right box, then the probability that at least one person does is just one minus this.

As the probability of each box going to the correct person is $1/5$, the probability that none of the boxes go to the correct person is $(4/5)^3$, and hence the probability that at least one person gets the correct box is

$$1 - (4/5)^3 = \frac{61}{125} = .488.$$

Exercise 53.2. *Imagine we have k boxes. Assume there are n distinct toys, and that in each box there is exactly one action figure, and each box is equally likely to have any action figure. If we pick 3 boxes at random, what is the probability that we get at least 2 of the same action figure?*

Solution: Again it is easier to find one minus the probability that we have 3 distinct figures. There are $3! \binom{n}{3}$ ways to choose three distinct action figures (with order mattering), and n^3 ways to choose 3 action figures (with order mattering). Thus the probability we have at least two of the same action figure is just

$$1 - \frac{3! \binom{n}{3}}{n^3} = 1 - \frac{n(n-1)(n-2)}{n^3}.$$

Implicit in the analysis above is that $n \geq 3$. If $n < 3$ then we must have at least two copies of the same action figure. The binomial coefficient notation actually knows this. We define $\binom{n}{r}$ by

$$\binom{n}{r} = \frac{n(n-1)(n-2) \cdots (n-(r-1))}{r(r-1)(r-2) \cdots 1};$$

if $r > n$ then one of the factors in the numerator is zero, and thus the binomial coefficient vanishes. For example,

$$\binom{3}{5} = \frac{3 \cdot 2 \cdot 1 \cdot 0 \cdot (-1)}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 0.$$

While we can choose anything we wish for notation, some notations are clearly superior to others. In this case, there should be no way of choosing 5 objects from 3, regardless of whether or not order matters! Fortunately the binomial coefficient realizes this absurdity, and is defined in such a way that it vanishes in these cases. Thus our formula above is true for all n , and not just $n \geq 3$.

We should of course remark that if $n \geq r$ then our binomial coefficient is the same as $n!/r!(n-r)!$.

54. SHIN, GEA

Exercise 54.1. Consider a smart probability student detained in an underwater facility with a trapdoor. Each day the student has one chance to try and open the trapdoor, with independent probability 0.01 of successfully opening the door. Fortunately, he has a magic machine, given to him by his professor, that increases his probability of breaking down the door by 10% each day (so if his probability was 50%, the next day it rises to 55%). Find the expected number of days that he will be detained.

Solution: We can solve this problem through brute force. The probability that he breaks away on the first day is 0.01. The probability that he breaks away on the second day is $(1 - 0.01)(0.01)(1.1)$. As $(0.01)(1.1)^{49} > 1$, he can definitely break away on 50th day; as $(.01)(1.1)^{48} < 1$, there is a positive chance he is still imprisoned after 49 days. Let X be the number of days he is detained. Set $p = 0.01$ and $k = 1.1$. Then, the expected number of days that he will be detained is

$$\begin{aligned} E(X) = & 0 \cdot p + 1 \cdot (1 - p)pk + 2 \cdot (1 - p)(1 - pk)pk^2 + \cdots \\ & + 48 \cdot (1 - p) \cdots (1 - pk^{47})pk^{48} + 49 \cdot (1 - p) \cdots (1 - pk^{48}) \cdot 1. \end{aligned}$$

Evaluating this sum yields ≈ 20.36488 . Thus, we expect our student to be detained for about 21 days.

For problems such as this, it is worthwhile to get some feel for the answer, some idea of approximately how many days we expect to wait. Note that if $1.1^n = 2$ then $n = \log(2)/\log(1.1) \approx 7.3$. Thus every 7.3 days (approximately) the probability of escape doubles. Therefore in about 22 days the probability of escaping is about 8%. If the probability were always 8% we would expect to need about 12.5 days to escape; as the probability is growing we expect the answer to be less. Thus, we would be surprised if the probability were greater than about 34. By doing a more careful analysis, we can get better upper and lower estimates.

Exercise 54.2. Consider again a smart probability student detained in an underwater facility with a trapdoor. Each day the student has one chance to try and open the trapdoor, with the initial probability 0.01 of successfully opening the door. Fortunately, he has now a magic wand, given by his professor, that increases his probability of breaking down the door by $(1 - p) \cdot 0.1$ each day (where p is the probability of successfully

opening the door in the previous day). Find the expected number of days that he will be detained.

Solution: Note that the probability of opening the door on the first day is 0.01, on the second day it rises to $0.01 + (1 - 0.01) \cdot 0.1 = 0.109$ and on the third day to $0.109 + (1 - 0.109) \cdot 0.1 = 0.1981$. Note the probability is growing very rapidly at first, but as time passes and it approaches 1, it grows very slowly. Unlike the previous exercise, the probability never reaches or exceeds one, so it is possible (though unlikely) for our student to be detained arbitrarily long. Arguing in a similar manner as above, we find the expected time is about 4.798. If instead of gaining 10% of the missing probability we gain only 1%, the expected time rises to about 12.517.

Exercise 54.3. Professor Miller loves birthdays so much that he wants his students to enjoy their birthdays at home without class. He has a policy of canceling the entire class if two or more people share the same birthday and the day happens to be a class day. How many students does he need to maximize the expected number of students present at his lectures for a whole semester, i.e. students times the number of lectures? Assume that there are 28 lectures in total. For example, suppose that there are 40 students in class. If two people have the same birthday on one of the lecture dates and there are no other shared birthdays, then he will only have 27 lectures, so $27 \cdot 40 = 1080$. If there were two sets of two (or more) people with the same birthday on two of the lecture dates, then he will only have 26 lectures, with $26 \cdot 40 = 1040$. We are trying to maximize the product of lectures times students.

Solution:

55. SHIRKOVA, TERESA

Exercise 55.1. Suppose you have a standard deck of 52 cards with 13 cards per suit. We deal 5 cards to a person, one at a time. We say a set of cards in the deal is in a sequence if each one numerically follows or precedes an existing card in the hand; as soon as we get a card whose number does not precede or follow a card in our hand, we say the sequence has ended. Aces are considered both high and low (so we may wrap around). Thus the following are sequences:

324A, 56473, KQJA2, QJAK2, 34567, 54637,

while the following are not

243, 43625, QJA2K, 8531K.

What is the probability we have a sequence of length at least 3 starting with the first card dealt?

Solution: One way to solve this problem is to break it up into three cases: we have a sequence of length exactly 5, exactly 4, and exactly 3. As we are allowing wrap-arounds (aces high and low), the first card is entirely arbitrary. All of our sequences must start off with the first three cards in a sequence. After the first card is chosen, the second card must be either one above or one below; the probability of that happening is $8/51$. There is again exactly two choices for the number of the next card (which must be either directly below the lowest term or directly above the highest), so the probability is $8/50$.

If we have a sequence of exactly three, the next card cannot be above the highest or below the lowest. Thus of the remaining 49 cards in the deck, we must avoid 8. Hence the probability that the fourth card does not continue the sequence is $41/49$, and the remaining card can be anything. Thus the probability of having a sequence of length exactly 3 is

$$1 \cdot \frac{8}{51} \cdot \frac{8}{50} \cdot \frac{41}{49} \cdot 1 = \frac{1312}{62475}.$$

Similarly the probability of a sequence of length exactly 4 is

$$1 \cdot \frac{8}{51} \cdot \frac{8}{50} \cdot \frac{8}{49} \cdot \frac{40}{48} = \frac{128}{37485},$$

while having a sequence of length exactly 5 is

$$1 \cdot \frac{8}{51} \cdot \frac{8}{50} \cdot \frac{8}{49} \cdot \frac{8}{48} = \frac{128}{187425}.$$

Thus the total probability of having a sequence of length at least 3 is

$$\frac{1312}{62475} + \frac{128}{37485} + \frac{128}{187425} = \frac{32}{1275} \approx 0.025098.$$

There is an easier way to compute this probability, though. If we only care that our sequence have length at least three and starts with the first card, then once the first three cards are in sequence it does not matter what the remaining two cards are. The answer is thus

$$1 \cdot \frac{8}{51} \cdot \frac{8}{50} \cdot 1 \cdot 1 = \frac{32}{1275}.$$

We chose to give the longer proof first for several reasons. First, if we care about a more general question (such as having a sequence of length exactly 4), we need to argue along those lines. Second, seeing the long way helps us appreciate the power of the second.

There are a myriad of problems we could ask along these lines. The next would be to have a sequence of length at least three, starting anywhere. If we let S denote cards that are in a sequence of length at least three together, N for cards that are not in that sequence, and D for it not mattering what value the card has, our possibilities are the following: $SSSDD, NSSSD, NNSSS$.

Exercise 55.2. Assume we have a cereal box with n possible different prizes (labeled $c_1, c_2, \dots, c_n$), and that each box contains exactly one prize (which is equally likely to be any of the prizes). Consider the following buying pattern. You buy a box, which has some prize, say c_i . Now you buy the second box. If the second box has the prize c_i , when you go shopping again you buy a number of boxes equal to the number of boxes you currently have squared (which in this case is 2^2 or 4); if the second box does not have the same prize as the first, the next time you go shopping you purchase just one box. In general, if none of the latest purchases have a new prize you only buy one additional box the next time; if, however, at least one of the latest box purchased has a prize you already have you get to buy the square of the number of boxes you currently have. Find the expected number of days it would take to get a full set of objects.

56. XIE, ZOE

Exercise 56.1. *The probability that a child's parents divorce increases with time, and (sadly) once parents divorce assume they never remarry each other. Assume that the cumulative distribution function of the parents being divorced when the child is $y \leq 18$ years old is $\sum_{t=1}^y \frac{1}{10^t}$ for $y = 1, 2, \dots$ and 0 for $y = 0$ (at birth). Assume that the child's income when he turns 23 depends on her parents' marital status before she turns 18. More specifically, let's posit that her hourly wage at age 23 follows a normal distribution with mean 20 dollars and standard deviation $\sqrt{2}$ dollars if her parents remain married to each other when she is 18, and a normal distribution with mean 15 dollars and standard deviation $\sqrt{7}$ dollars otherwise (we take a negative wage as meaning that we owe money to the landlord). What is the average (mean) wage and its standard deviation when the child is 23?*

Solution: We first compute the probability the parents are divorced when the child is 18:

$$\mathbb{P}(\text{parents divorced when child is 18}) = \sum_{t=1}^{18} \frac{1}{10^t} = \frac{\frac{1}{10} - (\frac{1}{10})^{18}}{1 - \frac{1}{10}} = \frac{1}{9}.$$

Thus the probability that the parents are still married is

$$\mathbb{P}(\text{parents married when child is 18}) = 1 - \frac{1}{9} = \frac{8}{9},$$

and hence the expected wage is simply

$$\mathbb{E}(\text{wage of child at 23}) = \frac{1}{9}(15) + \frac{8}{9}(20) = \$19.44.$$

What about the standard deviation?

Exercise 56.2. *Alice, Bob, Charlie and Dixon play a badminton doubles game. Alice serves first and continues to serve as long as she is winning; then her partner Bob serves until he loses a point; then Charlie serves until he loses a point and then Dixon. A point is scored only when a player is serving and wins. The first pair of players to win 21 points wins the game. Assume that each player wins a point with probability .6 when she or her partner serves and with probability 0.4 when one of the opponents serves. What is the probability that Alice and Bob will win the game?*

Solution: The first simplification we may make is that we may regard each team as getting two serve attempts. We can set up a recurrence or difference equation. Let $x_{i,j}$ be the probability that Alice and Bob win given that they are serving and have i points to j points for Charlie and Dixon, and similarly let $y_{k,\ell}$ be the probability that Charlie and Dixon win given that they are serving and have ℓ points to k points for Alice and Bob. Note $x_{i,j} = y_{i,j}$ by symmetry.

It's actually convenient to break the probabilities up even further. Let $a_{i,j}$ be the probability that Alice and Bob win given that they have i points to j points for their opposition *and* Alice is serving; let $b_{i,j}$ be the same situation except now Bob is serving. Call the similar quantities for Charlie and Dixon $c_{i,j}$ and $d_{i,j}$; again by symmetry we have $c_{i,j} = a_{i,j}$ and $d_{i,j} = b_{i,j}$.

Then

$$a_{i,j} = .6a_{i+1,j} + .4b_{i,j}$$

and

$$b_{i,j} = .6b_{i+1,j} + .4(1 - a_{i,j}),$$

with initial conditions

$$a_{21,j} = b_{21,j} = 1$$

for $0 \leq j \leq 20$. Why are these equations true? For the first, Alice and Bob win her serve with probability .6, and they then have $i + 1$ points to the i points for Charlie and Dixon; similarly if they lose the first point (which happens with probability .4) then they are in the situation of Bob serving with a score of i to j , and by definition the probability that they win in this case is just $b_{i,j}$. The second equation is a bit harder; the last expression is really $.4(1 - c_{i,j})$ (as the probability Alice and Bob win is just 1 minus the probability that Charlie and Dixon win); we can simplify this by recalling that $c_{i,j} = a_{i,j}$. The boundary equations are apparent, as the game ends once a team reaches 21.

57. XIONG, WENTAO

Exercise 57.1 (The Monty Hall Problem). *In the game show ‘Let’s make a deal’, a contestant is given the choice of three doors. Behind one door is a car; behind the others, goats. The car and the goats were placed randomly behind the doors before the show. After the contestant has chosen a door, the host (Monty Hall) opens one of the remaining two doors. Monty Hall knows what is behind each door; if he has a choice of opening a door revealing a car or a goat he will always open the door revealing the goat, but if goats are behind both doors he randomly opens one of them. After Monty Hall opens a door with a goat, our contestant is asked whether or not he wants to switch doors. Should he?*

Solution: Yes! Without loss of generality, imagine our contestant chooses door 1. The following 3 cases are equally likely (each occurring with probability $\frac{1}{3}$):

- Door 1: goat; Door 2: goat; Door 3: car.
- Door 1: goat; Door 2: car; Door 3: goat.
- Door 1: car; Door 2: goat; Door 3: goat.

If we are in case 1, Monty Hall must open door 2, and our contestant wins the car if he switches, and loses if he stays with door 1.

If we are in case 2, Monty Hall must open door 3, and our contestant again wins the car if he switches, and loses if he stays with door 1.

Finally, if we are in case 3 then Monty Hall can open either door. If our contestant switches then he loses, while if he stays with door 1 he wins the car.

Thus, if he switches he wins 2 out of 3 times, while if he stays with his door he wins only one out of three.

‘Cecil Adams’ has a great way of thinking about this problem: in effect, you are given the opportunity of either sticking with your original door, or having *both* of the other two doors. Thus, if you switch you should win two-thirds of the time. This problem has generated lots of discussion and controversy over the years; see

http://en.wikipedia.org/wiki/Monty_Hall_problem

for more details.

Exercise 57.2. Wentao starts with \$ k and gambles with his professor, who owns an infinite sum of money, in order to purchase some authentic Chinese food that costs \$ N ($N > k$). Wentao tosses a biased coin that turns up a head with probability $p > 1/2$ and a tail with probability $1 - p$. If the coin comes up heads, Wentao wins \$ 1; if tails, Wentao loses \$ 1. This game ends when Wentao's capital becomes 0 or \$ N . What is the probability Wentao wins?

Solution: Let p_i denote the probability of Wentao's ultimate ruin starting from \$ i . Clearly $p_0 = 1$ and $p_N = 0$ (if Wentao starts with no money he is clearly bankrupt, while if he starts with \$ N he has the funds he needs). What about the other values of p_i ? These are harder to deduce. We can set up an equation relating the various p_i 's to each other. The main idea is the following: if we have \$ i with $0 < i < N$, then after we toss the coin we either have \$ $i + 1$ (which happens with probability p) or \$ $i - 1$ (which happens with probability $1 - p$). We thus find that we have the relation

$$p_i = p \cdot p_{i+1} + (1 - p) \cdot p_{i-1}$$

if $1 \leq i \leq N - 1$, with boundary conditions $p_0 = 1$, $p_N = 0$. The reason we can write this down is that we have a memoryless process. We don't care how we reach the point of having \$ i ; all that matters is that at some point we have \$ i .

The standard way to solve difference equations is the Method of Divine Inspiration; namely, we guess an answer and see if it works! The standard guess is to try $p_i = r^i$. Substituting this into

$$p_i = p \cdot p_{i+1} + (1 - p) \cdot p_{i-1}$$

gives

$$r^i = pr^{i+1} + (1 - p)r^{i-1}.$$

After some algebra we find that r satisfies the equation

$$pr^2 - r + (1 - p) = 0,$$

which has roots $r_1 = 1$ and $r_2 = \frac{1-p}{p}$. One of the most important properties of linear difference equations is that if r_1^i and r_2^i are solutions, so is $c_1 r_1^i + c_2 r_2^i$ for any c_1, c_2 . To see this, we simply substitute this into the original difference equation, and see that this also solves it. We simply must choose c_1 and c_2 to satisfy the boundary conditions, which here is that $p_0 = 1$ and $p_N = 0$.

As $p \neq \frac{1}{2}$, the two roots r_1 and r_2 are distinct and we must solve

$$\begin{aligned} c_1 1^0 + c_2 \left(\frac{1-p}{p} \right)^0 &= 1 \\ c_1 1^N + c_2 \left(\frac{1-p}{p} \right)^N &= 0. \end{aligned}$$

This is two equations in two unknowns (c_1, c_2), which should be solvable. There are two natural ways to do this. The first is to solve for c_2 in terms of c_1 in the first equation and then substitute that into the second. Thus, from the first equation we find $c_2 = 1 - c_1$.

Plugging this into the second equation gives

$$c_1 \cdot 1 + (1 - c_1) \cdot \left(\frac{1-p}{p}\right)^N = 0.$$

This implies

$$c_1 = \frac{\left(\frac{1-p}{p}\right)^N}{\left(\frac{1-p}{p}\right)^N - 1}.$$

After some algebra, we see the solution is

$$p_i = \frac{\left(\frac{1-p}{p}\right)^i - \left(\frac{1-p}{p}\right)^N}{1 - \left(\frac{1-p}{p}\right)^N}.$$

The other way to find c_1 and c_2 is through linear algebra, writing the equations as

$$\begin{pmatrix} 1 & 1 \\ 1 & \left(\frac{1-p}{p}\right)^N \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

As the matrix is invertible, we can find c_1 and c_2 .

Version 7 of Mathematica is very good at solving recurrence relations. To solve this one, simply type

`RSolve[{a[i] == p a[i + 1] + (1 - p) a[i - 1], a[0] == 1, a[M] == 0}, a[i], i]`

In the code above, we have changed p_i to $a[i]$ and instead of having the boundary at N we have it at M (Mathematica has reserved N as a symbol for numerics, and thus that is unavailable).

Exercise 57.3. *In the previous problem we calculated the probability that Wentao eventually won (or lost) the game. In that analysis, it did not matter how long we played, only what the final result was. What is the expected number of tosses until the game ends? Remember our coin is biased, and $p \neq 1/2$.*

Solution: Let T_i be the expected number of tosses until Wentao reaches either \$ 0 or \$ N when we start counting with him having \$ i . Note $T_0 = T_N = 0$, and for $1 \leq i \leq N-1$ we have

$$T_i = (1 + T_{i+1})p + (1 + T_{i-1})(1 - p),$$

which after some algebra yields the recurrence relation

$$T_{i+1} = \frac{1}{p}T_i - \frac{1-p}{p}T_{i-1} - \frac{1}{p}.$$

We again turn to the Method of Divine Inspiration. If we didn't have the $-1/p$ at the end, we could solve it by guessing $T_i = r^i$. Let us write $T_i = U_i + a$ for some constant a . We find

$$U_{i+1} = \frac{1}{p}U_i - \frac{1-p}{p}U_{i-1} + a \left(\frac{1}{p} - \frac{1-p}{p} - 1 \right) - \frac{1}{p} = \frac{1}{p}U_i - \frac{1-p}{p}U_{i-1} - \frac{1}{p};$$

unfortunately the constant in front of a cancels, and thus this guess does not work.

We must therefore guess again. We need something a bit more complicated than $T_i = U_i + a$ with a constant, as the coefficients above led to a catastrophic cancelation of the a term. We try $T_i = U_i + ia$, which after some more algebra gives

$$U_{i+1} = \frac{1}{p}U_i - \frac{1-p}{p}U_{i-1} + (i-1)a \left(\frac{1}{p} - \frac{1-p}{p} - 1 \right) + a \left(\frac{1}{p} - 2 \right) - \frac{1}{p},$$

where we wrote $T_i = U_i + (i-1)a + a$ and $T_{i+1} = U_{i+1} + (i-1)a + 2a$ to simplify the algebra, as the $(i-1)a$ term will have a coefficient of zero. We see this is a better choice, and for any $p \neq 2$ there is a solution, namely

$$a = \frac{\frac{1}{p}}{\frac{1}{p} - 2} = \frac{1}{1 - 2p}.$$

Now that we know a , we can solve for U_i as before.

Typing

Simplify[RSolve[{T[i] == p (T[i + 1] + 1) + (1 - p) (T[i - 1] + 1), T[0] == 0, T[M] == 0}, T[i], i]]

into Mathematica yields

$$T_i = \frac{i + M \left(\left(\frac{1-p}{p} \right)^i - 1 \right) - i \left(\frac{1-p}{p} \right)^M}{\left(\left(\frac{1-p}{p} \right)^M - 1 \right) (2p - 1)}.$$

Note in the formula that it is essential that the coin is biased; if $p = 1/2$ the denominator is zero.

58. ZHANG, LIYANG

Exercise 58.1. *There are four mathematicians sitting around a table. The magic hat generator generates black and white hats with equal probability 1/2. They win a million dollars if they can guess the color of at least one hat correctly. They lose a million dollars if anyone who speaks is wrong. Given that one can only see the colors of other people's hats but not their own hat's color, can they come up with a winning strategy? (Note that if everyone is silent, they neither win nor lose.)*

Solution: Let's look at all of the 16 possible combinations of colors of hats:

WWWW WWWB WWBW WBWW BWWW WWBB WBBW BBWW WBWB
BWBW BWBB BBBW BBWB BWBB WBBB BBBB

Here is a strategy which has a positive expected value: If one sees all other three people's hats have the same color, then he calls the opposite color. If one sees two different colors, remain silent. Thus we have a 8/16 chance to win and 2/16 to lose. The expected value is thus

$$\frac{1}{2} \cdot 10^6 - \frac{2}{16} \cdot 10^6 = 375000.$$

Is there a better strategy? If we forget about the fourth person and consider just the first three, what happens if we reuse the three-person strategy? In that case, if you see two hats of an opposite color you say nothing and otherwise if you see two hats of the

same color you say the opposite. In that case the probability of being right is $6/8$ and the probability of being wrong is $2/8$, for an expected value of

$$\frac{6}{8} \cdot 10^6 - \frac{2}{8} \cdot 10^6 = 500000,$$

which is superior. Is there an even better strategy?

Exercise 58.2. *This was another variant of the Monty Hall problem.*

59

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61. START OF THIRD SET OF PROPOSED PROBLEMS

62. ARNOSTI, NICK

Exercise 62.1. *If X is a Poisson distributed random variable with parameter λ , use generating functions to find the third moment of X .*

Solution: The generating function for X is

$$\begin{aligned} G_X(s) &= \mathbb{E}[s^X] \\ &= \sum_{k=0}^{\infty} s^k \frac{\lambda^k e^{-\lambda}}{k!} \\ &= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(s\lambda)^k}{k!} \\ &= e^{-\lambda} e^{s\lambda} \\ &= e^{\lambda(s-1)}. \end{aligned}$$

Straightforward differentiation yields

$$\begin{aligned} G'_X(s) &= \lambda e^{\lambda(s-1)} \\ G''_X(s) &= \lambda^2 e^{\lambda(s-1)} \\ G'''_X(s) &= \lambda^3 e^{\lambda(s-1)}. \end{aligned} \tag{62.1}$$

The mean is now readily calculated; it is

$$\mathbb{E}[X] = G'_X(1) = \lambda e^{\lambda(1-1)} = \lambda.$$

Further,

$$\mathbb{E}[X(X-1)] = G''_X(1) = \lambda^2 e^{\lambda(1-1)} = \lambda^2,$$

so

$$\mathbb{E}[X^2] = \lambda^2 + \mathbb{E}[X] = \lambda^2 + \lambda.$$

Finally,

$$\begin{aligned}\mathbb{E}[X(X-1)(X-2)] &= \mathbb{E}[X^3] - 3\mathbb{E}[X^2] + 2\mathbb{E}[X] \\ &= G_X'''(1) \\ &= \lambda^3,\end{aligned}\tag{62.2}$$

and so

$$\mathbb{E}[X^3] = \lambda^3 + 3(\lambda^2 + \lambda) - 2\lambda = \lambda^3 + 3\lambda^2 + \lambda.$$

63. ATKINSON, BEN

Exercise 63.1. Let X and Y be independent $\text{Unif}(0, 1)$ random variables. Find the joint density of $Z = X + Y$.

Solution: The densities are $f_X(x) = f_Y(y) = 1$ if $0 \leq x, y \leq 1$ and 0 otherwise. The density of Z is given by the convolution of the two densities, so

$$\begin{aligned}f_Z(z) &= (f_X * f_Y)(z) \\ &= \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) dx\end{aligned}$$

We must determine the bounds of the integration. The factor $f_X(x)$ is zero unless $x \in [0, 1]$, while the second factor $f_Y(z-x)$ is zero unless $z-x \in [0, 1]$, or $x \in [z-1, z]$. Thus the density is

$$f_Z(z) = \int_{\max(0, z-1)}^{\min(1, z)} dx.$$

It's easiest to see what is happens by breaking the above into two cases. If $z \leq 1$ then the integration is from 0 to z , while if $z \geq 1$ the integration is from $z-1$ to 1. Therefore in the first case, namely when $z \leq 1$, we have

$$f_Z(z) = \int_0^z dx = z,$$

while in the second case, namely $z \geq 1$, we have

$$f_Z(z) = \int_{z-1}^1 dx = 2 - z.$$

Collecting the two cases, we find the answer is

$$f_Z(z) = \begin{cases} z & \text{if } |z| \leq 1 \\ 2 - z & \text{if } |z| \geq 1. \end{cases}$$

Our answer has many properties we would expect, such as it is symmetric about $z = 1$ and the greatest probability is when $z = 1$. Why do we expect it to be symmetric? Note that if X and Y are $\text{Unif}(0, 1)$, then so too are $1 - X$ and $1 - Y$, and hence the distribution of $X + Y$ should be the same as that of $(1 - X) + (1 - Y) = 2 - (X + Y)$. It shouldn't be surprising that the largest probability is when $X + Y = 1$ if we think back to rolling die (and, in fact, this will also predict the triangular shape we see).

If we roll two fair die, we can obtain any integer from 2 to 12. There is only one way to get a 2 or a 12, but there are six ways to get a 7 (no matter what is rolled on the first

die, there is always exactly one roll on the second die that will give us a 7). In general, the probability of rolling a $k \in \{2, \dots, 12\}$ is

$$\text{Prob}(\text{two rolls sum to } k) = \frac{6 - |k - 6|}{36}.$$

We may interpret the roll of two die as adding two discrete uniform random variables, and thus this exercise is good intuition for the continuous case.

64. RAN BI

Exercise 64.1. Let $X_1 \dots X_n$ be independent Cauchy random variables. Find the density function for $X_1 + X_2$.

Solution: Consider the sum of two independent Cauchy random variables, say $Y = X_1 + X_2$. Using convolutions, we have

$$f_Y(x) = (f_{X_1} * f_{X_2})(x) = \int_{-\infty}^{\infty} f_{X_1}(t) f_{X_2}(x - t) dt.$$

As the random variables are both Cauchy, $f_{X_1} = f_{X_2}$ and we find

$$\begin{aligned} f_Y(x) &= \int_{-\infty}^{\infty} \frac{1}{\pi(1+t^2)} \cdot \frac{1}{\pi(1+(x-t)^2)} dt \\ &= \frac{1}{\pi^2} \int_{-\infty}^{\infty} \frac{1}{1+(u+x/2)^2} \cdot \frac{1}{1+(u-x/2)^2} du, \end{aligned}$$

where we changed variables by setting $t = u + x/2$ to symmetrize the integrand.

As

$$(1 + (u + x/2)^2) - (1 + (u - x/2)^2) = 2ux,$$

we have

$$f_Y(x) = \frac{1}{\pi^2} \int_{-\infty}^{\infty} \frac{1}{2ux} \left[\frac{1}{1+(u-x/2)^2} - \frac{1}{1+(u+x/2)^2} \right] du,$$

or changing variables back to t we have

$$f_Y(x) = \frac{1}{\pi^2} \int_{-\infty}^{\infty} \frac{1}{2tx} \left[\frac{1}{1+(t-x/2)^2} - \frac{1}{1+(t+x/2)^2} \right] dt.$$

Can we integrate this? Let's study the simpler case

$$I(a) = \int_{-\infty}^{\infty} \frac{1}{t} \frac{1}{1+(t+a)^2} dt.$$

Let

$$\begin{aligned} u &= 1/t, \quad du = -dt/t^2 \\ dv &= \frac{dt}{1+(t+a)^2}, \quad v = \arctan(t+a). \end{aligned}$$

Then we find

$$I(a) = \left. \frac{\arctan(t+a)}{t} \right|_{t=-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{\arctan(t+a)}{t^2} dt.$$

Unfortunately it seems as if we are going around and around in circles. Resorting to a (good) Table of Integrals, we find

$$f_Y(x) = \frac{2x \arctan(\frac{t}{2} - x) + 2x \arctan(\frac{t}{2} + x) + \log(1 + (\frac{t}{2} + x)) - \log(1 + (\frac{t}{2} - x)^2)}{\pi^2 4x(1 + x^2)} \Big|_{-\infty}^{\infty}.$$

Evaluating the logarithm terms at the boundaries gives zero as they cancel, when the arctangent terms reinforce. As $\arctan(\pm\infty) = \pm\pi/2$, we obtain

$$f_Y(x) = \frac{4\pi x}{\pi^2 4x(1 + x^2)} = \frac{1}{\pi(1 + x^2)};$$

I think this is wrong, as I thought it was $(X_1 + X_2)/2$ is the standard Cauchy.

65. BINDER, ARI

Exercise 65.1. Consider a player's performance in two consecutive US Opens. Find the mass function for the number of matches won in these two tournaments. Recall that the mass function for number of matches won in one tournament is

$$\mathbb{P}(A = n) = f(n) = \begin{cases} p^n(1 - p) & \text{if } 0 \leq n \leq 6 \\ p^7 & \text{if } n = 7 \\ 0 & \text{otherwise.} \end{cases}$$

Solution: Let $B = A_1 + A_2$, where A_i denotes the number of wins in tournament i . One way to look at the problem is that we have 3 cases to consider: we fail to win both tournaments, we win exactly one tournament, or we win both tournaments.

Unfortunately, this is not the best way to break up the analysis. The reason is that we are interested in the probability of getting exactly 8 wins; for this purpose, we don't care if we win 7 in the first tournament and 1 in the second, or 4 in each. We break the analysis into cases, doing the easier ones first.

There are no ways to win fewer than 0 games or more than 14. There is only one way to win 14 games; that requires us to win both tournaments, which happens with probability $p^7 \cdot p^7 = p^{14}$.

If we win 6 games or fewer, then we cannot win either tournament. We must win a_1 games in the first tournament and $b - a_1$ in the second, where $a_1 \in \{0, 1, \dots, b\}$. Thus, when $b \leq 6$, the probability of winning exactly b games is

$$f_B(b) = \sum_{a_1=0}^b p^{a_1}(1 - p) \cdot p^{b-a_1}(1 - p) = (1 - p)^2 p^b \sum_{a_1=0}^b 1 = (b + 1)(1 - p)^2 p^b.$$

We are now left with the interesting case of winning b games, with $7 \leq b \leq 13$. Note that we can win exactly one tournament, but we do not need to win a tournament unless $b = 13$. The probability of getting 13 wins is just

$$f_B(13) = \binom{2}{1} p^7 \cdot p^6(1 - p) = 2p^{13}(1 - p).$$

For winning between 7 and 12 games, the probability is

$$\begin{aligned}
 f_B(b) &= \binom{2}{1} p^7 \cdot p^{b-7}(1-p) + \sum_{a_1=b-6}^6 p^{a_1}(1-p) \cdot p^{b-a_1}(1-p) \\
 &= 2p^b(1-p) + p^b(1-p)^2 \sum_{a_1=b-6}^6 1 \\
 &= 2p^b(1-p) + (13-b)p^b(1-p)^2
 \end{aligned}$$

(it's $13-b$ as the number of integers from $b-6$ to 6 is $6-(b-6)+1$). We therefore find

$$f_B(b) = \begin{cases} (b+1)(1-p)^2 p^b & \text{if } 0 \leq b \leq 6 \\ 2(1-p)p^b + (13-b)(1-p)^2 p^b & \text{if } 7 \leq b \leq 12 \\ 2(1-p)p^{13} & \text{if } b = 13 \\ p^{14} & \text{if } b = 14. \end{cases}$$

Whenever we have a complicated expression like this, it's best to do whatever checks we can. The most natural is to see if it sums to 1, which it must as it is a probability mass function. While we can evaluate it exactly by using the finite geometric series formula, it is often simpler to substitute specific values and have a program such as Mathematica grind out the computation. Doing both of these checks shows that the above is a probability mass function (namely, it sums to 1), and thus we have some confidence in our result. (In fact, initially I wrote $12-b$ instead of $13-b$, and had a slightly wrong answer.)

Exercise 65.2. *Notation as in the previous problem, what is the mean number of wins we expect our player to have?*

Solution: As we know the probability mass function, we can find the mean by $\mathbb{E}[X] = \sum_b b \cdot \mathbb{P}(B = b)$. After some algebra we find

$$\mathbb{E}[X] = 2p(1+p+p^2+p^3+p^4+p^5+p^6) = \frac{2p(1-p^7)}{1-p}.$$

66. BROWN, CHAD

Exercise 66.1. *Let X be a random variable distributed uniformly on $[0, 1]$. Find the density and expected value of the random variable Y , where $Y = X^3$.*

Solution: Consider the cumulative distribution function of X :

$$F_X(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x & \text{if } 0 < x \leq 1 \\ 1 & \text{if } x > 1. \end{cases}$$

Since $X = Y^{1/3}$, we have that

$$F_Y(y) = F_X(y^{1/3}) = \begin{cases} 0 & \text{if } y \leq 0 \\ y^{1/3} & \text{if } 0 < y \leq 1 \\ 1 & \text{if } y > 1. \end{cases}$$

Differentiating gives the density function, so

$$f_Y(y) = \frac{d}{dy}F_Y(y) = \begin{cases} 0 & \text{if } y \leq 0 \\ \frac{1}{3y^{2/3}} & \text{if } 0 < y \leq 1 \\ 0 & \text{if } y > 1. \end{cases}$$

The expected value of Y is just

$$\begin{aligned} \mathbb{E}[Y] &= \int_0^1 \frac{y}{3y^{2/3}} dy \\ &= \int_0^1 \frac{y^{1/3}}{3} dy \\ &= \left. \frac{y^{4/3}}{4} \right|_0^1 \\ &= \frac{1}{4}. \end{aligned} \tag{66.1}$$

67. BUSTARD, TODD

Exercise 67.1. Let $X_1, \dots, X_n$ be independent random variables where each X_k has an exponential density function with parameter $\lambda_k = k$, which means the density function of X_k is

$$f_{X_k}(x) = \begin{cases} ke^{-kx} & \text{if } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Find the moment generating function $M_{X_1+\dots+X_n}(t)$.

Solution: First we compute the moment generating function of an individual X_k , and then use our properties to find the moment generating function for the sum. We have

$$\begin{aligned} M_{X_k}(t) &= \int_0^\infty e^{tx} ke^{-kx} dx \\ &= \int_0^\infty ke^{-k(x-\frac{t}{k}x)} dx \\ &= \int_0^\infty ke^{-k(1-\frac{t}{k})x} dx \\ &= \frac{1}{(1-\frac{t}{k})} \int_0^\infty k \left(1-\frac{t}{k}\right) e^{-k(1-\frac{t}{k})x} dx \\ &= \frac{1}{(1-\frac{t}{k})} \int_0^\infty e^{-u} du \\ &= \frac{1}{(1-\frac{t}{k})}, \end{aligned}$$

so long as $t < k$. We need this restriction to ensure that we are integrating the exponential of a negative quantity, as otherwise the integral will not exist.

As the X_i 's are independent, the moment generating function of the sum is the product of the moment generating functions. Thus

$$\begin{aligned} M_{X_1+\dots+X_n}(t) &= M_{X_1}(t) \cdots M_{X_n}(t) \\ &= \prod_{k=1}^n \frac{1}{(1 - \frac{t}{k})}. \end{aligned}$$

In the product above, each factor imposes a different condition on t ; the k^{th} factor forces $t < k$. Thus in order for the above arguments to be valid, *each* of the n conditions must be satisfied, which means we must have $t < 1$.

68. FISH, CROSBY

Exercise 68.1. Find the moment generating function of a geometric random variable X with parameter p .

Solution: By definition, we have

$$M_X(t) = \mathbb{E}[e^{tX}] = \sum_{n=0}^{\infty} e^{tn} \mathbb{P}(X = n).$$

For a geometric random variable, $\mathbb{P}(X = n) = (1 - p)^{n-1}p$ for $n \geq 1$ and 0 otherwise. Thus

$$\begin{aligned} M_X(t) &= \sum_{n=0}^{\infty} e^{tn} (1 - p)^{n-1} p \\ &= \frac{p}{1 - p} \sum_{n=0}^{\infty} e^{tn} (1 - p)^n \\ &= \frac{p}{1 - p} \sum_{n=0}^{\infty} (e^t(1 - p))^n. \end{aligned}$$

The above is a geometric series, and converges for $e^t(1 - p) < 1$, yielding

$$M_X(t) = \frac{p}{1 - p} \frac{1}{1 - e^t(1 - p)}$$

for $e^t(1 - p) < 1$, or equivalently for $t < \log(\frac{1}{1-p})$.

69. FORD, AARON

Exercise 69.1. Find the moment generating function for the binomial distribution with parameters N and p , and using this, confirm that the mean is indeed Np .

Solution: Let $X \sim \text{Bin}(N, p)$. By definition we have

$$M_X(t) = \mathbb{E}[e^{tX}] = \sum_{k=0}^N e^{tk} \mathbb{P}(X = k).$$

For the binomial, $\mathbb{P}(X = k) = \binom{N}{k} p^k (1-p)^{N-k}$, and so

$$\begin{aligned} M_X(t) &= \sum_{k=0}^N e^{tk} \binom{N}{k} p^k (1-p)^{N-k} \\ &= \sum_{k=0}^N \binom{N}{k} (pe^t)^k (1-p)^{N-k} \\ &= (pe^t + (1-p))^{N-k}, \end{aligned}$$

where the last follows from the Binomial Theorem.

To find the mean, we use the formula $\mathbb{E}[X] = M'_X(1)$. As

$$M'_X(t) = N (pe^t + (1-p))^{N-1} \cdot pe^t,$$

which gives

$$M'_X(0) = N (pe^0 + (1-p))^{N-1} \cdot pe^0 = N(p + 1-p)^{N-1}p = Np.$$

70. GROVER, MICHAEL

Exercise 70.1. For $k > 1$, find the generating function for the series $a_n = 1/k^n$ for $n \in \{0, 1, \dots\}$.

Solution:

$$\begin{aligned} G_a(s) &= \sum_{n=0}^{\infty} (s^n \cdot \frac{1}{k^n}) \\ &= \sum_{n=0}^{\infty} \left(\frac{s}{k}\right)^n \\ &= \frac{1}{1 - \frac{s}{k}} \\ &= \frac{k}{k-s}, \end{aligned}$$

so long as $|s| < k$ (we need this to ensure that the series converges).

71. WISA KITICHAIWAT

Exercise 71.1. Let X_1 and X_2 be independent Cauchy variables. Find the joint density function of

$$Y_1 = \frac{X_1}{X_2} \quad \text{and} \quad Y_2 = 3X_1 + 4X_2.$$

Solution: The change of variable formula for the density says that

$$f_{Y_1, Y_2}(y_1, y_2) = f_{X_1, X_2}(x_1(y_1, y_2), x_2(y_1, y_2)) \cdot |J(y_1, y_2)|.$$

We thus need to determine how to express the x_i 's in terms of the y_j 's, and then take the Jacobian of the change of variable. Obviously we cannot compute the Jacobian until we know the inverse change of variables, so we do that first.

Let T be our map from (X_1, X_2) to (Y_1, Y_2) . Thus,

$$(y_1, y_2) = T(x_1, x_2) = \left(\frac{x_1}{x_2}, 3x_1 + 4x_2 \right).$$

We need to invert this relation and solve for x_1, x_2 in terms of y_1, y_2 . From

$$\begin{aligned} x_1 &= y_1 x_2 \\ 4x_2 &= y_2 - 3x_1 \end{aligned}$$

we obtain

$$(4 + 3y_1)x_2 = y_2$$

by replacing $3x_1$ in the second equation with $3y_1x_2$ and then bringing that over to the other side and factoring. This allows us to solve for x_2 in terms of y_1 and y_2 , and we find

$$x_2 = \frac{y_2}{4 + 3y_1}.$$

We now use the relation $x_1 = y_1x_2$ to find

$$x_1 = \frac{y_1y_2}{4 + 3y_1}.$$

We now have $T^{-1}(y_1, y_2) = (x_1, x_2) = (x_1(y_1, y_2), x_2(y_1, y_2))$, and thus we can compute the Jacobian. We find

$$J(y_1, y_2) = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_2}{\partial y_1} \\ \frac{\partial x_1}{\partial y_2} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} \frac{-3y_1y_2}{(4+3y_1)^2} + \frac{y_2}{4+3y_1} & \frac{-3y_2}{(4+3y_1)^2} \\ \frac{y_1}{4+3y_1} & \frac{1}{4+3y_1} \end{vmatrix}.$$

Taking the determinant gives

$$J(y_1, y_2) = \frac{-3y_1y_2}{(4 + 3y_1)^3} + \frac{y_2}{(4 + 3y_1)^3} - \frac{-3y_1y_2}{(4 + 3y_1)^3} = \frac{y_2}{4 + 3y_1}.$$

We now substitute into the change of variable formula, which says

$$f_{Y_1, Y_2}(y_1, y_2) = f_{X_1, X_2}(x_1(y_1, y_2), x_2(y_1, y_2)) \cdot |J(y_1, y_2)|.$$

As X_1 and X_2 are independent, $f_{X_1, X_2}(x_1, x_2) = f_{X_1}(x_1)f_{X_2}(x_2)$. Further, each f_{X_i} is the Cauchy density, so $f_{X_i}(x) = (\pi(1 + x^2))^{-1}$. Thus

$$\begin{aligned} f_{Y_1, Y_2}(y_1, y_2) &= \frac{1}{\pi(1 + x_1(y_1, y_2))^2} \frac{1}{\pi(1 + x_2(y_1, y_2))^2} \left| \frac{y_2}{(4 + 3y_1)^2} \right| \\ &= \frac{1}{\pi^2} \frac{(4 + 3y_1)^2}{(4 + 3y_1)^2 + (y_1y_2)^2} \frac{(4 + 3y_1)^2}{(4 + 3y_1)^2 + y_2^2} \left| \frac{y_2}{(4 + 3y_1)^2} \right| \\ &= \frac{1}{\pi^2} \frac{(4 + 3y_1)^2 |y_2|}{((4 + 3y_1)^2 + (y_1y_2)^2) \cdot ((4 + 3y_1)^2 + y_2^2)}. \end{aligned}$$

72. KOLOGLU, MURAT

Exercise 72.1. Envy is Green. After their bitter defeat in the Grand Battle for the Green Chicken (a grueling but enjoyable math competition), some of the defeated students attempted to regain the Green Chicken by traveling to its new home in Purpletown and trying to steal it from the Department of Purple Math and Statistics. Of course, as the Department was being guarded by diligent professors and students alike, it took a lot of courage to attempt to break in. Students would come to the doors of the Department and go in with probability $p < 1$ or ‘chicken’ out and turn back with probability $1 - p$. Every student that entered was caught, and every student that chose not to go in was pressured by and acquiesced to their fellows to make another attempt. Thus a student is constantly stealing until he is caught. It wasn’t until N students were caught that they realized they would not succeed on these missions, where N can be modeled as a random variable with the Poisson distribution with parameter λ . An amused student of Purple College decided to make a Purple Record article about the incident, where he revealed the total number of attempts by the invading students. What was the expected value of the total number of attempts?

Solution: We use generating functions to solve this question. Let X_i with $i \in \{1, 2, \dots, N\}$ be the random variables denoting the number of attempts by the i^{th} student. Observe that the X_i ’s are independent geometrically distributed random variables with parameter p of success. What makes this problem difficult (and interesting!) is that N is not fixed, but is itself a random variable with a Poisson distribution with parameter λ . Fortunately, N is independent from the identically distributed X_i ’s.

If N were fixed, the generating function of $X_1 + \dots + X_N$ would be just

$$G_{X_1 + \dots + X_N}(s) = \prod_{i=1}^N G_{X_i}(s) = G_X(s)^N,$$

where $G_X(s)$ is the common generating function; **as N is not fixed, however, this is wrong!** The correct solution uses a beautiful fact on compounding.

Theorem: Let $X_1, X_2, \dots$ be independent, identically distributed random variables with generating function $G_X(s)$, and let N be a random variable independent of the X_i ’s with generating function $G_N(s)$. If $Y = X_1 + \dots + X_N$, then $G_Y(s) = G_N(G_X(s))$.

We first use this theorem to solve the problem, and then give the proof. As N is a Poisson random variable, its generating function is $G_N(s) = e^{\lambda(s-1)}$; as the X_i ’s are geometric random variables, their generating functions are $G_X(s) = \frac{ps}{1-s(1-p)}$. Thus the theorem tells us that the generating function for $Y = X_1 + \dots + X_N$ is

$$G_Y(s) = G_N(G_X(s)) = G_N\left(\frac{ps}{1-s(1-p)}\right) = e^{\lambda\left(\frac{ps}{1-s(1-p)} - 1\right)}.$$

The mean is now easily determined by differentiation, as $\mathbb{E}[Y] = G'_Y(1)$. Thus

$$\begin{aligned}
 \mathbb{E}[Y] &= G'_Y(1) \\
 &= p\lambda \frac{(1 - s(1 - p)) + s(1 - p)}{(1 - s(1 - p))^2} e^{\lambda(\frac{ps}{1-s(1-p)}-1)} \Big|_{s=1} \\
 &= p\lambda \frac{1}{p^2} e^{\lambda(\frac{p}{p}-1)} \\
 &= \frac{\lambda}{p}.
 \end{aligned}$$

We now prove the theorem using conditional probabilities. We have

$$\begin{aligned}
 G_Y(s) &= \mathbb{E}[s^Y] \\
 &= \sum_{n=0}^{\infty} \mathbb{E}[s^Y | N = n] \mathbb{P}(N = n) \\
 &= \sum_{n=0}^{\infty} \mathbb{E}[s^{X_1 + \dots + X_n}] \mathbb{P}(N = n) \\
 &= \sum_{n=0}^{\infty} \mathbb{E}[s^{X_1} \dots s^{X_n}] \mathbb{P}(N = n) \\
 &= \sum_{n=0}^{\infty} \mathbb{E}[s^{X_1}] \dots \mathbb{E}[s^{X_n}] \mathbb{P}(N = n),
 \end{aligned}$$

where we used the independence of the X_i 's to say the expected value of the product is the product of the expected values. (We also used the fact that if g is a monotonically increasing function, then if $X_1, \dots, X_n$ are independent so too are $g(X_1), \dots, g(X_n)$.) Each $\mathbb{E}[s^{X_i}] = G_X(s)$ by definition, and thus

$$G_Y(s) = \sum_{n=0}^{\infty} G_X(s)^n \mathbb{P}(N = n) = G_N(G_X(s)),$$

with the last step following from applying the relation

$$G_N(s) = \sum_{n=0}^{\infty} s^n \mathbb{P}(N = n)$$

with $s = G_X(s)$.

73. LIU, ANDREW

Exercise 73.1. *If you roll three die, what is the probability that you get a prime number?*

Solution: We know that there are 216 possible combinations for three die (6^3) where order matters, so all that's left to do is to count all the ways. We can roll any integer from 3 to 18 inclusive. The only prime numbers in this range are 3, 5, 7, 11, 13, and 17, so we must just compute the probability of rolling each of these.

While it is possible to set up the answer in terms of convolutions, it is simple enough to just enumerate the possibilities. To simplify things, note that if all three die rolls are the same, then it counts as one way, if two numbers are the same, then it counts as three ways, and if all the numbers are different, then it counts as 6 ways.

3: (1,1,1): 1 way

5: (1,2,2), (1,1,3): 6 ways

7: (1,1,5), (1,2,4), (1,3,3), (2, 2, 3): 15 ways

11: (1,4,6), (2,3,6), (1,5,5), (2,4,5), (3,3,5), (3,4,4): 27 ways

13: (1,6,6), (2,5,6), (3,4,6), (3,5,5), (4,4,5): 21 ways

17: (5,6,6): 3 ways

So we have $(1 + 6 + 15 + 27 + 21 + 3)/216 = 71/216$.

Note that it is very easy to forget to count a valid combination. To make sure this doesn't happen, it is best to have a specific ordering in mind to ensure nothing is forgotten. The combinations listed above are in a very specific order: in the triple (a, b, c) we always have $a \leq b \leq c$. We start with the largest value of c , then take the largest value of b possible and then see if an a works; we then move to the next smallest b and repeat. For example, consider 11. We want to write $a + b + c = 11$ with $a \leq b \leq c$. The largest c can be is 6, so let's see if there are any such solutions. We first try $b = 6$, but immediately find that no a will work, as $6 + 6 = 12 > 11$. We then try $b = 5$, but again there are no solutions as a must be at least 1 and $b + c$ already equals 11 here. We then try $b = 4$, and now we do have a solution, as $1 + 4 + 6 = 11$. This will generate six solutions by re-ordering:

$$(1, 4, 6), (4, 1, 6), (1, 6, 4), (4, 6, 1), (6, 1, 4), (6, 4, 1).$$

We now try $b = 3$, and find $a = 2$ works since $2 + 3 + 6 = 11$. This will generate six additional solutions,

$$(2, 3, 6), (3, 2, 6), (2, 6, 3), (3, 6, 2), (6, 2, 3), (6, 3, 2).$$

We now continue and set $b = 2$, and see that there is no valid a that will work. We might at first think there is such an a , as taking $a = 3$ yields $3 + 2 + 6 = 11$; however, this solution has already been counted (it was one of the six solutions generated from $(2, 3, 6)$). Remember, in order to avoid double counting we are listing our solutions (a, b, c) with $a \leq b \leq c$.

74. PEGADO, SEAN

Exercise 74.1. Suppose you have a piece of paper with a horizontal line drawn across it. A friend draws a small square around the line at the very center of the paper. He makes a challenge: if you immediately hand the paper back (and do not play the game), you lose nothing but you also win nothing. If you do choose to play, you may have as

many or as few marks as you like be made on the sheet. When you return the paper, your friend will take the mean value of every mark. If that mean value is in the box, he will pay you \$1,000, but if the mark is outside the box, you must pay him \$1,000. A mark can be made by mutual friends: any friend can ever only place one mark, but you may assume you have infinitely many mutual friends (you are very popular).

Each friend will place a mark by closing his eyes and randomly choosing a point on the line. Each friend's random placement can be understood by a uniform probability density function. One friend's density is independent of the densities of all other friends.

Should you play the game? If so, what is the optimal strategy for maximizing your winnings? Assume the left endpoint of the line is marked 0, the right marked 1, and the left edge of the box marked $-r$, the right side of it marked r .

Solution: Yes, you should play. By the Central Limit Theorem if we have enough friends randomly pick marks, the sum of their marks will begin to be normally distributed. This is very convenient for us, since that means the average will eventually land inside the box, which is of length r on each side of the mean. Thus our winning strategy will be to have a sufficiently large number of friends place a mark, until the sum of the marks have a normal distribution. Then we will win and maximize our profits.

75. PESKOE, BEN

Exercise 75.1. Find the distribution function for the sum of two independent random variables X and Y , where X has the standard normal distribution and Y has the standard exponential distribution.

Solution: The two density functions are

$$\begin{aligned} f_X(x) &= \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \\ f_Y(x) &= \begin{cases} e^{-x} & \text{if } x \geq 0 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Using convolutions, we get

$$f_{X+Y}(x) = \int_{-\infty}^{\infty} f_X(t) f_Y(x-t) dx.$$

We now must be very careful. There is no problem in substituting for the standard normal's density; however, the exponential's density is only $e^{-(x-t)}$ if $x-t \geq 0$; otherwise it is zero. Thus, for each fixed x , the range of t is from $-\infty$ to x . This is slightly different from previous problems, where at the end of the day we frequently had $t \in [0, x]$ instead of the $t \in (-\infty, x]$ we have here. The reason for the difference is

that the first factor, $f_X(t)$, does not restrict t at all in this problem. Thus the density is

$$\begin{aligned}
 f_{X+Y}(t) &= \int_{-\infty}^x \frac{1}{2\pi} e^{-t^2/2} e^{-(x-t)} dt \\
 &= \frac{1}{2\pi} e^{-x} \int_{-\infty}^x e^{-t^2/2+t} dt \\
 &= \frac{1}{2\pi} e^{-x} \int_{-\infty}^x e^{-t^2/2+t-1/2+1/2} dt \\
 &= \frac{1}{2\pi} e^{-x} \int_{-\infty}^x e^{-\frac{1}{2}(t^2-2t+1)+\frac{1}{2}} dt \\
 &= \frac{1}{2\pi} e^{-x} e^{1/2} \int_{-\infty}^x e^{-(t-1)^2/2} dt;
 \end{aligned}$$

the key step above was completing the square, namely replacing $-\frac{t^2}{2} + t$ with $-\frac{t^2}{2} + t - \frac{1}{2} + \frac{1}{2}$. Changing variables by letting $u = t - 1$ gives

$$f_{X+Y}(t) = e^{-(x-\frac{1}{2})} \int_{-\infty}^{x-1} \frac{1}{2\pi} e^{-u^2/2} du = e^{-(x-\frac{1}{2})} F_X(x-1),$$

where F_X is the cumulative distribution function of the standard normal.

As the cumulative distribution function of the standard normal appears so frequently in the subject, it has been given a special name, and much effort has been spent tabulating its values and finding rapidly convergent series approximations. We define the error function, denoted erf , by

$$\text{erf}(x) = \int_0^x \frac{2}{\sqrt{\pi}} e^{-t^2} dt.$$

Note the integrand looks like the density of a normal with mean 0 and variance $1/2$; in fact, since the integrand is even we have

$$\text{erf}(x) = \int_{-x}^x \frac{1}{\sqrt{2\pi \cdot 1/2}} e^{-t^2/(2 \cdot \frac{1}{2})} dt,$$

which we may interpret as the area under the normal with mean 0 and variance $1/2$ between $-x$ and x .

As we use erf for areas involving a normal with mean 0 and variance $1/2$, we use Φ for the cumulative distribution function of the standard normal. Simple algebra gives

$$\Phi(x) = \frac{1}{2} + \frac{1}{2} \text{erf}\left(\frac{x}{\sqrt{2}}\right).$$

76. SHIN, GEA

Exercise 76.1. Let X_1, X_2 be independent random variables having the geometric distribution with parameter p . Using convolutions, find the density for $X_1 + X_2$.

Solution: As the X_i 's are geometric random variables, their density function is

$$f(x_i) = \begin{cases} (1-p)^{x_i-1}p & \text{if } x_i \geq 1 \\ 0 & \text{otherwise,} \end{cases}$$

with $0 < p \leq 1$. Let $Y = X_1 + X_2$. Since the density function of X_i is non-zero only when $x_i \geq 1$, the density function of Y is zero unless $y \geq 2$. Thus for $y \geq 2$ we have

$$\begin{aligned}
 g(y) &= (f_1 * f_2)(y) \\
 &= \sum_{t=1}^{y-1} f_1(t) f_2(y-t) \\
 &= \sum_{t=1}^{y-1} p(1-p)^{t-1} p(1-p)^{y-t-1} \\
 &= p^2 \sum_{t=1}^{y-1} (1-p)^{y-2} \\
 &= (y-1)p^2(1-p)^{y-2}.
 \end{aligned}$$

77. ZHANG, LIYANG

Exercise 77.1. Find the Fourier transform of $f(x) = e^{-a|x|}$ for any $a > 0$.

Solution: By definition of Fourier transform

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} e^{-a|x|} e^{-2\pi i x \xi} dx.$$

We now split the integration into two parts, $(-\infty, 0]$ and $[0, \infty)$ as our function f has a different definition in each region. In the arguments below we assume the complex exponential is integrated the same way as the standard exponential; in other words, the anti-derivative of e^{zx} with respect to x is just $z^{-1}e^{zx}$; we discuss how one would prove this below. We therefore have

$$\begin{aligned}
 \widehat{f}(\xi) &= \int_{-\infty}^0 e^{ax} e^{-2\pi i x \xi} dx + \int_0^{\infty} e^{-ax} e^{-2\pi i x \xi} dx \\
 &= \int_{-\infty}^0 e^{(a-2\pi i \xi)x} dx + \int_0^{\infty} e^{(-a-2\pi i \xi)x} dx \\
 &= \frac{1}{a-2\pi i \xi} (1-0) + \frac{1}{a+2\pi i \xi} (1-0) \\
 &= \frac{2a}{a^2 + 4\pi^2 \xi^2}.
 \end{aligned}$$

We now sketch how to prove our exponential claim. Consider

$$\int_{\alpha}^{\beta} e^{(a+bi)x} dx.$$

Using

$$e^{i\theta} = \cos \theta + i \sin \theta$$

and

$$e^{(a+bi)x} = e^{ax} e^{bix} = e^{ax} \cos(bx) + i e^{ax} \sin(bx),$$

we find

$$\int_{\alpha}^{\beta} e^{(a+bi)x} dx = \int_{\alpha}^{\beta} e^{ax} \cos(bx) dx + i \int_{\alpha}^{\beta} e^{ax} \sin(bx) dx.$$

We need to compute two classical integrals, that of $e^{ax} \cos(bx)$ and $e^{ax} \sin(bx)$. One way to do this is to integrate by parts twice and use the ‘bring it over’ method; the reason is setting $dv = e^{ax} dx$ leads to something easily integrated, while setting $u = \cos(bx)$ leads to a $\sin(bx)$ term, and then integrating by parts again gives a $\cos(bx)$ term.

Alternatively, we can use the method of divine inspiration (or a Table of Integrals) and note

$$\begin{aligned} \int e^{ax} \cos(bx) dx &= \frac{e^{ax}(a \cos(bx) + b \sin(bx))}{a^2 + b^2} \\ \int e^{ax} \sin(bx) dx &= \frac{e^{ax}(-b \cos(bx) + a \sin(bx))}{a^2 + b^2}. \end{aligned}$$

The proof can be completed by a painful but straightforward brute force analysis, where we evaluate these quantities at α and β and compare these to the claimed

$$\int_{\alpha}^{\beta} e^{(a+ib)x} dx = \left. \frac{e^{(a+ib)x}}{a+ib} \right|_{\alpha}^{\beta} = \frac{e^{(a+ib)\beta} - e^{(a+ib)\alpha}}{a+ib}.$$

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