Probab. Theory Relat. Fields **111**, 323–332 (1998)

Probability Theory and Related Fields

A bounded N-tuplewise independent and identically distributed counterexample to the CLT

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Received: 17 May 1996 / In revised form: 28 January 1998

Summary. A sequence of random variables $X_1, X_2, X_3, \ldots$ is said to be N-tuplewise independent if $X_{i_1}, X_{i_2}, \ldots, X_{i_N}$ are independent whenever $(i_1, i_2, \ldots, i_N)$ is an N-tuple of distinct positive integers. For any fixed $N \in \mathbb{Z}^+$, we construct a sequence of bounded identically distributed N-tuplewise independent random variables which fail to satisfy the central limit theorem.

Mathematics Subject Classification (1991): 60F0S

Fix $N \in \mathbb{Z}^+$. A sequence $X_1, X_2, X_3, \ldots$ of random variables is said to be *N-tuplewise independent* if $X_{i_1}, X_{i_2}, \ldots, X_{i_N}$ are independent whenever $i_1, i_2, \ldots, i_N$ are distinct indices. Every sequence of random variables is 1-tuplewise independent. The concepts of 2-*tuplewise* and 4-*tuplewise* independence are usually referred to as *pairwise* and *quadruplewise* independence, respectively.

Janson [4] and Bradley [1] have constructed sequences of identically distributed bounded *pairwise independent* random variables which fail to satisfy the central limit theorem. Professor Dominik Szynal has asked the author whether *quadruplewise independent* random variables necessarily satisfy the central limit theorem. Some positive evidence might be provided by Szynal's result [6] that although the Hsu-Robbins law of large numbers [3] fails for pairwise independent random variables, it does hold for quadruplewise independent random vari-

Key words and phrases: Central limit theorem, N-tuplewise independence, quadruplewise independence

ables. However, as we shall show, there is no central limit theorem for quadruplewise independent random variables, and indeed for no choice of N is there a central limit theorem for N-tuplewise independent random variables.

Recall that a random variable Ξ is said to be *symmetric* if Ξ and $-\Xi$ both have the same distribution. Our main result is then as follows.

Theorem. *Fix $N \in \mathbb{Z}^+$ and let Ξ be an arbitrary symmetric random variable with a finite second moment. Assume $P(\Xi \neq 0) > 0$. Then there exists a sequence $X_1, X_2, X_3, \ldots$ of identically distributed N-tuplewise independent random variables which have the same distribution as Ξ but which are such that $n^{-1/2}(X_1 + \cdots + X_n)$ does* **not** *converge in distribution to a normal random variable, and hence $X_1, X_2, X_3, \ldots$ do* **not** *satisfy the central limit theorem.*

Moreover, we may choose this sequence so that $|X_1|, |X_2|, |X_3|, \ldots$ are independent.

The proof of the Theorem will give an explicit construction of the sequence $\{X_n\}$.

Letting Ξ be a variable taking on the values 1 and -1 with probability $\frac{1}{2}$ each, we can obtain a two-state bounded identically distributed N-tuplewise independent sequence of random variables failing to satisfy the central limit theorem.

Remark 1. Note that unlike the examples of Janson [4] and Bradley [1] which produced strictly stationary sequences, our example is not a strictly stationary sequence and the author does not know whether a strictly stationary N-tuplewise (or even quadruplewise or triplewise) independent counterexample to the central limit theorem is possible. One can adapt the counterexample given in the present paper to show that there is no Barry-Esseen type inequality for strictly stationary sequences of N-tuplewise independent random variables, and in fact that convergence rates in the central limit theorem in this setting can be arbitrarily slow. However, the author has not succeeded in constructing a strictly stationary and N-tuplewise independent sequence (for $N \geq 3$) which would actually fail to satisfy the central limit theorem.

Suppose $X_1, X_2, X_3, \ldots$ is a pairwise independent sequence of identically distributed random variables with finite second moments. It follows from [5, Corollary 1] that $X_1, X_2, X_3, \ldots$ satisfy the central limit theorem if and only if for all real t we have

$$n^{-1/2} \sum_{k=1}^{n} E\left[X_k e^{itn^{-1/2} S_{nk}}\right] \to 0 \quad \text{as} \quad n \to \infty \ , \tag{1}$$

where $S_{nk} = (\sum_{j=1}^{n} X_j) - X_k$. Thus, one approach to trying to prove the central limit theorem in the strictly stationary and quadruplewise independent case would be to try to prove that in that case (1) must hold. However, it is not at all clear how this could be done.

Remark 2. If in addition to *pairwise* independence one assumes that the random variables are *jointly symmetric*, i.e., that all finite-dimensional distributions remain unchanged whenever any subset of the random variables are multiplied by -1, then a central limit theorem does hold. In the identically distributed case this was shown by Hong [2], while Pruss and Szynal [5] have shown this in general under Lindeberg's condition. (Note that given joint symmetry, (1) is trivial since every term in the summation in it must vanish.)

Remark 3. Janson [4, p. 448, Remark 6] notes that if $\alpha_1, \alpha_2, \ldots, \alpha_N$ are independent random variables uniformly distributed on $[0, 1]$, and

$$X_n = \mathrm{e}^{2\pi i\left(\alpha_1+\alpha_2 n+\alpha_3 n^2 \cdots + \alpha_N n^{N-1}\right)} \ ,$$

then the $\{X_n\}_{n=1}^{\infty}$ are an N-tuplewise independent sequence. Partial sums of this sequence have been heavily studied in number theory (see, e.g., Vinogradov's book [7, 8]). It is apparently still an open question whether this sequence of X_n satisfies the central limit theorem, at least if $N \geq 3$. If $N = 2$, then the answer is negative [4]. To settle the question is of course equivalent to checking whether (1) holds for the above X_n if $N \geq 3$.

Remark 4. It is still not known whether a law of the iterated logarithm holds for N-tuplewise independent random variables, at least for $N \geq 3$. If $N = 2$, then the answer is negative as can be seen by using Janson's example [4].

The rest of the paper is devoted to the proof of our Theorem.

We require two crucial lemmas. Fix $N \geq 1$. Given $(x_1, \ldots, x_{N+1}) \in \mathbb{R}^{N+1}$, if one of $x_1, \ldots, x_{N+1}$ vanishes then let $s(x_1, \ldots, x_{N+1}) = 1$. Otherwise, put

$$s(x_1, \ldots, x_{N+1}) = (-1)^{\nu} \ ,$$

where

$$\nu = \mathrm{Card}\{i : x_i < 0,\ 1 \leq i \leq N+1\} \ .$$

Then the sequence

$$x_1, \ldots, x_N, s(x_1, \ldots, x_{N+1}) x_{N+1}$$

always either contains a zero or else has an even number of negative entries.

Lemma 1. *Fix $N \geq 2$. Let $Y_1, \ldots, Y_{N+1}$ be independent identically distributed normal random variables with mean zero and variance $\sigma^2 > 0$. Then,*

$$Y_1 + \cdots + Y_N + s(Y_1, \ldots, Y_{N+1})Y_{N+1}$$

does not have a normal distribution.

To prove this, we shall use the following lemma which also provides the basis of our construction of a sequence of N-tuplewise independent random variables.

Lemma 2. *Fix N and K in $\mathbb{Z}^+$. Let $\{Y_i\}_{i \in I}$ be a collection of independent identically distributed symmetric random variables, with* Card $I = (N+1)K$. *Let $A_1, \ldots, A_{N+1}$ be a partition of I into $N+1$ disjoint sets of K elements each. Put*

$$T_j = \sum_{i \in A_j} Y_i \ ,$$

and let

$$s = s(T_1, \ldots, T_{N+1}) \ .$$

For $i \in \bigcup_{j=1}^{N} A_j$, let $Y'_i = Y_i$, and for $i \in A_{N+1}$, let $Y'_i = sY_i$. Then, $\{Y'_i\}_{i \in I}$ is a collection of N-tuplewise independent and identically distributed random variables with the same distribution as the Y_i.

Proof of Lemma 2. For conciseness, write $[n] = \{1, 2, \ldots, n\}$. Without loss of generality, suppose that $I = [N+1] \times [K]$ and $A_i = \{i\} \times [K]$ for all $i \in [N+1]$. Use $\cong$ to denote equality in (joint) distribution (for vector valued random variables).

Define the vector-valued random variables $\mathbf{Y}_i = (Y_{(i,1)}, Y_{(i,2)}, \ldots, Y_{(i,K)})$ and $\mathbf{Y}'_i = (Y'_{(i,1)}, Y'_{(i,2)}, \ldots, Y'_{(i,K)})$. I claim that the sequence $\mathbf{Y}'_1, \ldots, \mathbf{Y}'_{N+1}$ is *exchangeable*, i.e.,

$$\left(\mathbf{Y}'_1, \ldots, \mathbf{Y}'_{N+1}\right) \cong \left(\mathbf{Y}'_{\pi(1)}, \ldots, \mathbf{Y}'_{\pi(N+1)}\right)$$

for all permutations π of $[N+1]$.

Assume for now that the exchangeability has been proved. Let $a_1, \ldots, a_N \in I$ be distinct indices. Then, since the A_i are disjoint, there exists a $k \in [N+1]$ such that none of the a_i (for $i \in [N]$) is in A_k. If $k = N+1$, then it follows that $(Y'_{a_1}, \ldots, Y'_{a_N})$ and $(Y_{a_1}, \ldots, Y_{a_N})$ coincide. Since the latter is a sequence of independent identically distributed random variables with the same distribution as the Y_i, the proof is then complete. Suppose now $k \neq N+1$. Let π be a permutation of $[N+1]$ exchanging k and $N+1$ and let $\rho(i,j) = (\pi(i), j)$. Then, the exchangeability of $\mathbf{Y}'_1, \ldots, \mathbf{Y}'_{N+1}$ implies that

$$\left(Y'_{a_1},\ldots,Y'_{a_N}\right) \cong \left(Y'_{\rho(a_1)},\ldots,Y'_{\rho(a_N)}\right) . \tag{2}$$

But by choice of ρ and k, we have $\{\rho(a_1),\ldots,\rho(a_N)\} \cap A_{N+1} = \emptyset$. Thus, $(Y'_{\rho(a_1)},\ldots,Y'_{\rho(a_N)})$ coincides with $(Y_{\rho(a_1)},\ldots,Y_{\rho(a_N)})$ by definition of the Y'_i. But $(Y_{\rho(a_1)},\ldots,Y_{\rho(a_N)})$ is a sequence of independent identically distributed random variables with the same distribution as the Y_i, and so the left hand side of (2) must also be such a sequence, which completes the proof.

It suffices to prove the exchangeability of $\mathbf{Y}'_1,\ldots,\mathbf{Y}'_{N+1}$. Since $\mathbf{Y}'_1,\ldots,\mathbf{Y}'_N$ is an exchangeable sequence (being independent and identically distributed, since it coincides with $\mathbf{Y}_1,\ldots,\mathbf{Y}_N$), it in fact suffices to show that

$$\left(\mathbf{Y}'_1,\ldots,\mathbf{Y}'_{N-1},\mathbf{Y}'_N,\mathbf{Y}'_{N+1}\right) \cong \left(\mathbf{Y}'_1,\ldots,\mathbf{Y}'_{N-1},\mathbf{Y}'_{N+1},\mathbf{Y}'_N\right) .$$

Fix Borel sets $U_1,\ldots,U_{N+1}$ in $\mathbb{R}^K$. Since $\mathbf{Y}'_i = \mathbf{Y}_i$ for $i < N+1$, what we must show is that

$$P\left(A,\mathbf{Y}_N \in U_N,\mathbf{Y}'_{N+1} \in U_{N+1}\right) = P\left(A,\mathbf{Y}'_{N+1} \in U_N,\mathbf{Y}_N \in U_{N+1}\right) , \tag{3}$$

where $A = \bigcap_{i=1}^{N-1}\{\mathbf{Y}_i \in U_i\}$ if $N > 1$ and where A is the whole probability space if $N = 1$. Write $t(x,y) = s(T_1,\ldots,T_{N-1},x,y)$ for real x and y. Note that $\mathbf{Y}'_{N+1} = t(T_N,T_{N+1})\mathbf{Y}_{N+1}$. We thus have

$$\begin{aligned}
&P\left(A,\mathbf{Y}_N \in U_N,\mathbf{Y}'_{N+1} \in U_{N+1},t(T_N,T_{N+1}) = 1\right)\\
&\quad = P(A,\mathbf{Y}_N \in U_N,\mathbf{Y}_{N+1} \in U_{N+1},t(T_N,T_{N+1}) = 1)\\
&\quad = P(A,\mathbf{Y}_{N+1} \in U_N,\mathbf{Y}_N \in U_{N+1},t(T_N,T_{N+1}) = 1)\\
&\quad = P\left(A,\mathbf{Y}'_{N+1} \in U_N,\mathbf{Y}_N \in U_{N+1},t(T_N,T_{N+1}) = 1\right) ,
\end{aligned} \tag{4}$$

where the middle equality used the fact that $\mathbf{Y}_1,\ldots,\mathbf{Y}_{N+1}$ are exchangeable (being independent and identically distributed). On the other hand,

$$\begin{aligned}
&P\left(A,\mathbf{Y}_N \in U_N,\mathbf{Y}'_{N+1} \in U_{N+1},t(T_N,T_{N+1}) = -1\right)\\
&\quad = P(A,\mathbf{Y}_N \in U_N,-\mathbf{Y}_{N+1} \in U_{N+1},t(T_N,T_{N+1}) = -1)\\
&\quad = P(A,\mathbf{Y}_N \in U_N,\mathbf{Y}_{N+1} \in U_{N+1},t(T_N,-T_{N+1}) = -1)\\
&\quad = P(A,\mathbf{Y}_{N+1} \in U_N,\mathbf{Y}_N \in U_{N+1},t(T_{N+1},-T_N) = -1)\\
&\quad = P(A,\mathbf{Y}_{N+1} \in U_N,\mathbf{Y}_N \in U_{N+1},t(-T_{N+1},T_N) = -1)\\
&\quad = P(A,-\mathbf{Y}_{N+1} \in U_N,\mathbf{Y}_N \in U_{N+1},t(T_{N+1},T_N) = -1)\\
&\quad = P\left(A,\mathbf{Y}'_{N+1} \in U_N,\mathbf{Y}_N \in U_{N+1},t(T_N,T_{N+1}) = -1\right) .
\end{aligned} \tag{5}$$

Here, the first and last equalities used the identity $\mathbf{Y}'_{N+1} = t(T_N, T_{N+1})\mathbf{Y}_{N+1}$ (the last equality also used the fact that $t(x,y) = t(y,x)$); the second equality used the symmetry of the random variables which ensures that replacing the vector $\mathbf{Y}_{N+1}$ with $-\mathbf{Y}_{N+1}$ (while remembering to simultaneously replace its sum T_{N+1} with $-T_{N+1}$) will not change any joint distributions; the third equality used the fact that by independence and identical distribution, if we exchange the vectors $\mathbf{Y}_N$ and $\mathbf{Y}_{N+1}$ (while remembering to simultaneously exchange their sums T_N and T_{N+1}), then no joint distributions will change; the fourth equality used the identity $t(x,-y) = t(-x,y)$; the fifth equality once again used symmetry of distributions exactly as in the second equality.

Since $t(T_N, T_{N+1})$ takes on only the values 1 and -1, by adding up (4) and (5) we obtain (3), as desired. □

Proof of Lemma 1. Put $Y_i' = Y_i$ for $i = 1, \ldots, N$ and define

$$Y_{N+1}' = s(Y_1, \ldots, Y_{N+1})Y_{N+1} \ .$$

Let $T = Y_1 + \cdots + Y_{N+1}$ and set $T' = Y_1' + \cdots + Y_{N+1}'$. Put $I = \{1, \ldots, N+1\}$ and set $A_j = \{j\}$ for $1 \le j \le N+1$. By Lemma 2, the random variables $Y_1', \ldots, Y_{N+1}'$ are N-tuplewise independent and identically distributed with the same distribution as Y_1. Since in particular they are pairwise independent, we have

$$\operatorname{Var} T' = \sum_{i=1}^{N+1} \operatorname{Var} Y_i' = \sum_{i=1}^{N+1} \operatorname{Var} Y_i = \operatorname{Var} T \ .$$

Moreover $E[T'] = E[T] = 0$. Evidently T is a normal random variable. Hence, if T' is also a normal random variable, it must have the same distribution as T since it has the same variance and mean. But I claim that

$$E\left[(T')^{N+1}\right] \neq E\left[T^{N+1}\right] \ . \tag{6}$$

From this claim it follows that T' does not have the same distribution as T and hence is not normal. To prove the claim, proceed as follows. We may write

$$(x_1 + \cdots + x_{N+1})^{N+1} = \phi(x_1, \ldots, x_{N+1}) + (N+1)! \cdot \prod_{i=1}^{N+1} x_i \ ,$$

for any real numbers $x_1, \ldots, x_{N+1}$, where ϕ is a symmetric polynomial with the property that each term of $\phi(x_1, \ldots, x_{N+1})$ depends on at most N of the variables $x_1, \ldots, x_{N+1}$. But

$$E[\phi(Y_1, \ldots, Y_{N+1})] = E\left[\phi\left(Y_1', \ldots, Y_{N+1}'\right)\right]$$

since $Y_1', \ldots, Y_{N+1}'$ are N-tuplewise independent and identically distributed so that the joint distributions of any k of the variables $Y_1', \ldots, Y_{N+1}'$ for $k \le N$ are the same as the joint distributions of any k

of the variables $Y_1, \ldots, Y_{N+1}$, and since each term in the polynomial ϕ depends on at most N of the variables. Thus,

$$E\left[(T')^{N+1} - T^{N+1}\right] = (N+1)! \cdot E\left[\prod_{i=1}^{N+1} Y_i' - \prod_{i=1}^{N+1} Y_i\right] \ . \tag{7}$$

But $s(Y_1, \ldots, Y_N)$ was chosen in such a way that the list

$$Y_1, \ldots, Y_N, s(Y_1, \ldots, Y_N) Y_{N+1}$$

either contains a zero or else has an even number of negative entries, so that

$$\prod_{i=1}^{N+1} Y_i' \geq 0 \ .$$

Since $|Y_i'| = |Y_i|$ for all i, it follows that

$$E\left[\prod_{i=1}^{N+1} Y_i'\right] = E\left[\left|\prod_{i=1}^{N+1} Y_i\right|\right] > 0 \ . \tag{8}$$

On the other hand,

$$E\left[\prod_{i=1}^{N+1} Y_i\right] = \prod_{i=1}^{N+1} E[Y_i] = 0 \ , \tag{9}$$

by independence. From (7), (8) and (9) we immediately obtain (6) as desired. □

We may now prove our Theorem.

Proof of Theorem. Fix $N \in \mathbb{Z}^+$. Replacing N by $\max(N, 2)$, we may assume that $N \geq 2$. Let a_k be a sequence of positive integers such that

$$\lim_{k \to \infty} \frac{a_1 + \cdots + a_k}{a_{k+1}} = 0 \ . \tag{10}$$

For instance, one could let $a_k = 2^{2^k}$.

Set

$$b_k = 1 + (N+1) \sum_{j=1}^{k-1} a_k \ ,$$

for $k \in \mathbb{Z}^+$. Note that $b_{k+1} - b_k = (N+1)a_k$. Let $\Xi_1, \Xi_2, \ldots$ be a sequence of independent identically distributed random variables with the same distribution as Ξ. Let

$$I_k = \{b_k, b_k + 1, \ldots, b_{k+1} - 1\} \ .$$

We shall now define X_n when $n \in I_k$. Fix $k \in \mathbb{Z}^+$. Let $A_{k,1}, \ldots, A_{k,N+1}$ be a partition of I_k into $N+1$ disjoint sets of a_k elements each. Put

$$T_{k,j} = \sum_{i \in A_{k,j}} \Xi_i \ ,$$

for $1 \le j \le N + 1$. Let

$$s_k = s\big(T_{k,1}, \ldots, T_{k,N+1}\big) \ ,$$

in the notation of the lemmas. Fix n such that $b_k \le n < b_{k+1}$. Define

$$X_n = \begin{cases} s_k \Xi_n & \text{if } n \in A_{k,N+1} \\ \Xi_n & \text{otherwise} \end{cases} \ .$$

We now prove that $X_1, X_2, \ldots$ is an N-tuplewise independent and identically distributed sequence. Let $\mathscr{G}_k$ be the σ-field generated by $\{X_i\}_{i \in I_k}$. Then $\mathscr{G}_1, \mathscr{G}_2, \ldots$ are independent σ-fields. Therefore, it suffices for us to prove that the $\{X_i\}_{i \in I_k}$ are N-tuplewise independent and identically distributed with the same distribution as Ξ for each fixed k. But to see this, we only have to let $I = I_k$, define $Y_i = \Xi_i$ for $i \in I$, set $A_j = A_{k,j}$ for $j = 1, \ldots, N + 1$, and apply Lemma 2.

Therefore $X_1, X_2, \ldots$ is indeed an N-tuplewise independent and identically distributed sequence. We must now show that the normalized partial sums of the X_i do not converge in distribution to a normal random variable. To see this, let

$$Z_k = \frac{1}{\sqrt{(N+1)a_k}} \sum_{i=b_k}^{b_{k+1}-1} X_i \ .$$

and

$$W_k = \frac{1}{\sqrt{b_{k+1} - 1}} \sum_{i=1}^{b_{k+1}-1} X_i \ .$$

Now,

$$E\left[\left(\sum_{i=1}^{b_k - 1} X_i\right)^2\right] = E\left[\sum_{i=1}^{b_k - 1} X_i^2\right] = (b_k - 1)\sigma^2 \ , \qquad (11)$$

because the X_i are pairwise independent (as they are N-tuplewise independent for some $N \ge 2$) with mean zero, and where $\sigma^2 = \operatorname{Var} \Xi$. It follows that

$$\frac{1}{\sqrt{b_{k+1} - 1}} \sum_{i=1}^{b_k - 1} X_i$$

converges to zero in L^2 as $k \to \infty$ since $b_k/(b_{k+1} - 1) \to 0$ by (10). Moreover, by (11) we have W_k uniformly bounded in L^2-norm while (10) implies that $(N+1)a_k/b_{k+1} \to 1$. Therefore,

$$Z_k - W_k \to 0$$

in L^2. Thus, if W_k converges in distribution to a normal random variable, so does Z_k. Therefore it suffices for us to show that Z_k fails to converge to a normal random variable.

Now,

$$Z_k = \frac{1}{\sqrt{(N+1)a_k}} \left(s_k T_{k,N+1} + \sum_{j=1}^{N} T_{k,j} \right) .$$

But $T_{k,j}$ is a sum of a_k independent copies of Ξ_1 and hence, by the classical central limit theorem for independent identically distributed random variables, $a_k^{-1/2} T_{k,j}$ converges in distribution to a normal random variable with mean zero and variance σ^2 as $k \to \infty$ for each fixed $j \in \{1, \ldots, N+1\}$. Let $Y_1, \ldots, Y_{N+1}$ be independent normal random variables with mean zero and variance σ^2. As can be easily verified,

$$\frac{1}{\sqrt{a_k}} \left(s_k T_{k,N+1} + \sum_{j=1}^{N} T_{k,N} \right)$$

then converges in distribution to

$$s(Y_1, \ldots, Y_{N+1}) Y_{N+1} + \sum_{j=1}^{N} Y_j \ , \tag{12}$$

as $k \to \infty$. But by Lemma 1, the distribution of (12) is not normal, and hence Z_k cannot converge to a normal distribution as $k \to \infty$.

Moreover, we have $|X_n| = |\Xi_n|$ almost surely for each n, and since the Ξ_n are independent it follows that $|X_1|, |X_2|, |X_3|, \ldots$ are independent. □

Acknowledgments. The research was largely done while the author was at the University of British Columbia and was partially supported by Professor J. J. F. Fournier's NSERC Grant #4822. The author is most grateful to an anonymous referee for a very careful reading and for pointing out some errors and gaps in a previous version of the proof of Lemma 2.

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