Math 187 / 487: Binomial Coefficient Handout[*]

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Abstract

We give a quick introduction to binomial coefficients, and prove the binomial theorem.

1 Definitions

Recall the factorial function:

$$n! \;=\; n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1. \tag{1}$$

For the factorial function, we assume n is a non-negative integer (ie, either zero or a positive integer). We define $0!$ to be 1. This is a convenient choice, and simplifies many formulas. The factorial function can be analytically continued to a new function which agrees with the old on the non-negative integers, but makes sense for all complex numbers!

We define the following combinatorial quantities:

$$\binom{n}{r} \;=\; \frac{n!}{r!(n-r)!}, 0 \leq r \leq n \tag{2}$$

and

$$P(n,r) \;=\; \frac{n!}{(n-r)!},\; 0 \leq r \leq n. \tag{3}$$

In some books, one often encounters $C(n,r)$ for $\binom{n}{r}$. The C stands for Combinations (order doesn't count), and the P stands for Permutations (order counts).

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2 Permutations: $P(n, r)$

$P(n, r)$ is the number of ways of choosing r objects from n, when order matters. Obviously, we should have $1 \leq r \leq n$ – we can't choose more objects than there are objects!

How many ways are there to choose zero objects? There is just one way to choose nothing, and note $P(n, 0) = \frac{n!}{n!} = 1$.

How many ways are there to choose one object, when order matters? There are n objects, once we choose 1 object we are done. Thus, $P(n, r) = 1$. Note $\frac{n!}{(n-1)!} = n$.

How many ways are there to choose two objects, when order matters? There are n choices for the first object, and then there are $n - 1$ people objects available for the next slot. We now choose one of them. Thus, there are $n(n - 1)$ ways to choose two objects so that order matters. Note again that this is $\frac{n!}{(n-2)!}$.

Continuing in this way, what if we want to choose r objects, with order counting? For example, the first person chosen is the president, the second is the vice president, the third the treasurer, and so on. There are n choices for the first position. We now have $n - 1$ people left, and choose one of these people for the second position – there are $n - 1$ ways of doing this. We now have $n - 2$ people left, and we choose one for the third position – there are $n - 2$ ways of doing this. Finally, we come to the last choice. There are $n - (r - 1)$ people left (we have chosen $r - 1$ people), and there are $n - (r - 1)$ ways of choosing someone from $n - (r - 1)$ people.

Thus, we find

$$P(n, r) \;=\; n \cdot (n - 1) \cdots (n - (r - 1)). \tag{4}$$

Multiply the above by $\frac{(n-r)!}{(n-r)!}$. The numerator becomes $n!$, the denominator $(n - r)!$.

We have therefore proved

Lemma 2.1. $P(n, r) = \frac{n!}{(n-r)!}$ *is the number of ways to choose* r *people from* n *people, when order matters.*

3 Combinations: $\binom{n}{r}$

We now ask a harder question – how many ways are there to choose r people from n people, where it doesn't matter what *order* we choose the r people. All that matters is who is chosen, and who isn't.

This is similar to forming committees. Often everyone on a committee has an equal vote.

We first consider an easier problem: let us assume we have r different positions (president, vice president, treasurer, et cetera), so that we now have an *ordered* problem. How many ways can we choose r people from n people to sit on this *ordered* committee?

This is exactly what $P(n,r)$ is! Thus, there are $P(n,r)$ ordered committees of r people chosen from n people.

Given an ordered committee with people $A_1, \ldots, A_r$, how many ways can I re-order the committee? For example, initially A_1 might be president, and A_2 the vice president, and so on.

We can think of this as there being r boxes, labelled president, vice president, treasurer, and so on. We then need to put a person in each box (each person can be used just once). Well, there are r choices of a person to put in the first box (president box), then there are $r-1$ choices of a person to be in the second box (vice president box), all the way down to there is just 1 person left for the last box.

Thus, given an ordered committee of r people, we find there are $r!$ re-orderings or re-arrangements of that committee. As the people are distinct, no two of these ordered committees will be the same.

Thus, let $\binom{n}{r}$ denote the number of ways of choosing r people from n, when order *doesn't* count. Each choice of r people can then be made into $r!$ ordered committees. All these ordered committees are different. Further, two different choices of r people (where order doesn't count) cannot give rise to the same ordered committee – as order doesn't count, there must be at least one person in one set that is not in another.

For example, we could choose two different unordered committees containing people $\{A, B, D\}$ in one case, and $\{A, C, D\}$ in the other. As person B is in the first unordered committee, all ordered committees from this set will have B serving; since B is not in the second unordered committee, none of those ordered committees will contain B.

Claim 3.1.

$$\binom{n}{r} \cdot r! \;=\; P(n,r). \tag{5}$$

Proof. We know there are $P(n,r)$ ordered committees. We can enumerate all the ordered committees as follows: first, choose the r people who will serve on the committee *without* specifying who is serving in which position. By definition, there are $\binom{n}{r}$ ways to do this. *Then*, for each unordered committee of r people, there are $r!$ different ordered committees. Thus, the total number of ordered committees is also $\binom{n}{r} \cdot r!$, as claimed. $\square$

We can now solve for $\binom{n}{r}$:

$$\begin{aligned} \binom{n}{r} \cdot r! &= P(n,r) \\ \binom{n}{r} &= \frac{P(n,r)}{r!} \\ &= \frac{n!}{r!(n-r)!}. \end{aligned} \tag{6}$$

Exercise 3.2. *Prove* $\binom{n}{r} = \binom{n}{n-r}$*. Interpret this as the number of ways of choosing* r *people to serve on a committee (order doesn't count) is equivalent to choosing* $n-r$ *people* not *to serve on a committee (where, obviously, it doesn't matter what order you choose people* not *to serve).*

4 Binomial Theorem

Theorem 4.1. *For* $n \geq 0$ *and an integer, we have*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k. \tag{7}$$

Exercise 4.2. *Using Exercise 3.2, show we may also write the binomial theorem in the form*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}. \tag{8}$$

Proof. When we expand $(x+y)^n$, we have n factors. From each, we *either* chose to take an x *or* we choose to take a y; obviously, we cannot do both. As there are n terms, and we have a binary choice each time, there are 2^n terms when we expand. Many, however, will have the same powers of x and y, and we can amalgamate.

The reason is multiplication is commutative, and $xxyxyy = yyxxyx = x^3y^3$. Clearly, we can never see a power of x greater than n or less than 0, and similarly for y. Moreover, however many powers of y we choose, say k, determines the power of x. There are n factors. If we choose y from exactly k factors, then we must choose x from the remaining $n-k$ factors Thus, all terms are of the form $x^{n-k}y^k$, and we have, for some coefficients $a(k,n)$ (depending on n and k), that

$$(x+y)^n \;=\; \sum_{k=0}^{n} a(k,n)x^{n-k}y^k. \tag{9}$$

The proof is completed by showing $a(k,n) = \binom{n}{k}$. The factor in front of $x^{n-k}y^k$ is simply the number of ways of choosing a factor of y exactly k times and a factor of x exactly $n-k$ times. Thus, of the n factors, we need to know how many ways can we choose k factors (to give us ys). This is an unordered counting problem – we don't care *what* order we chose the k factors, we just care how many ways can we choose k factors. This is just $\binom{n}{k}$, which completes the proof.

□

Remark 4.3. *Note that choosing k factors to be y is equivalent to choosing $n-k$ factors to be x; using $\binom{n}{k} = \binom{n}{n-k}$, one finds the same expansion.*

Exercise 4.4. *Let k, n be non-negative integers with $k \le n$. Prove $\binom{n}{k} \le 2^n$. If n is odd, one can improve on this and show $\binom{n}{k} \le 2^{n-1}$.*