DIE BATTLES AND ORDER STATISTICS

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ABSTRACT. Two players roll die with k sides, with each side equally likely of being rolled. Player one rolls m dice and player two rolls n dice. If player one's highest roll exceeds the highest roll of player two then player one wins, otherwise player two wins. We calculate the probability that player one wins, giving a concise summation and integral version, as well as estimating the probability that player one wins for many triples (m, n, k). The answer involves numerous useful techniques (adding zero, multiplying by one, telescoping series), as well as some beautiful formulas (formulas for sums of powers, the binomial theorem, order statistics, partial summation).

1. INTRODUCTION

Consider two people, player one and player two, competing in the following game. Player one rolls m dice, player two rolls n dice. Each die has k sides (labeled 1 through k), with each side equally likely of being rolled. Player one wins if the highest number he rolls is greater than the highest number player two rolls, and otherwise player two wins. We show that

Theorem 1.1. *Notation as above, the probability that player one wins is*

$$\text{Prob(Player one wins)} \ = \ \frac{1}{k^{m+n}} \sum_{a=2}^{k} \left[a^m - (a-1)^m\right] \cdot (a-1)^n. \tag{1}$$

We can further simplify this formula by using the binomial theorem to expand the difference (write a as $a-1$), and then use formulas for power sums. We leave this as an exercise for the reader, as this formula is already quite tractable for applications, and further to use these expansions requires coding the Bernoulli polynomials (of course, if one is programming in an environment where these are already defined, it is worthwhile to consider these additional simplifications).

Using partial summation, we may re-write the probability that player one wins, and find

Theorem 1.2. *Notation as above, with $[u]$ denoting the greatest integer less than or equal to u, we have*

$$\text{Prob(Player one wins)} \quad = \quad \frac{1}{k^{m+n}} \left[k^m \cdot (k-1)^n - \int_1^k [u]^m \cdot n(u-1)^{n-1} du\right]. \tag{2}$$

We prove Theorem 1.1 in §2 and Theorem 1.2 in §3, and conclude in §4 by estimating the probability that player one wins for various m, n and k. Specifically, we show that

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if k is large relative to m and n then

$$\text{Prob(Player one wins)} \ \approx \ \frac{m}{m+n} - \frac{m}{2(m+n-1)}\frac{n}{k}, \tag{3}$$

with an error of size $\frac{n^2}{k^2}$.

2. Proof of Theorem 1.1

To prove Theorem 1.1 we first introduce some notation, and then prove some useful lemmas. Let $X_1, \dots, X_m$ and $Y_1, \dots, Y_n$ denote the rolls of players one and two, respectively. To determine which player wins, what matters is not the individual rolls, but rather the maximum value rolled by each player. Thus we are interested[1] in

$$\begin{aligned} X_{[m]} &= \max_{1\le i\le m} X_i \\ Y_{[n]} &= \max_{1\le j\le n} Y_j. \end{aligned} \tag{4}$$

Lemma 2.1. *Notation as above, for* $a, b \in \{1, \dots, k\}$ *we have*

$$\begin{aligned} \text{Prob}(X_{[m]} = a) &= \frac{a^m - (a-1)^m}{k^m} \\ \text{Prob}(Y_{[n]} = b) &= \frac{b^n - (b-1)^n}{k^n}. \end{aligned} \tag{5}$$

Proof. By symmetry, it is enough to prove the formula for $X_{[m]}$. We take $a \in \{1, \dots, k\}$ as these are the possible rolls for a die with k sides. We calculate the probability that $X_{[m]}$ equals a. For this to be true, the largest roll must be a, and the remaining rolls must be a or less. Let ℓ be the number of rolls that are equal to a; note $\ell \in \{1, \dots, m\}$ (there are m rolls, and at least one equals a). Thus the remaining $m - \ell$ rolls must be $a - 1$ or smaller. Thus, in the set of m rolls, ℓ of the rolls are a (for example, if $m = 5$ and $\ell = 3$ then we could have rolls 1, 3 and 4 are a, or we could have rolls 2, 4 and 5 are a, and so on).

There are $\binom{m}{\ell}$ ways to choose ℓ rolls from m rolls to be a, and the probability that a roll is a is just $\frac{1}{k}$. Further there are $\binom{m-\ell}{m-\ell} = 1$ way to choose $m - \ell$ rolls to be not a. Each of these rolls is at most $a - 1$, and the probability that a roll is at most $a - 1$ is $\frac{a-1}{k}$. Note that if $a = 0$ the probability is 0 (which is good as it is impossible to roll a 0 with a die with sides 1 through k).

Since the rolls are independent, the probability that exactly ℓ of m rolls are a and the remaining $m - \ell$ rolls are at most $a - 1$ is

$$\binom{m}{\ell}\frac{1}{k^\ell}\left(\frac{a-1}{k}\right)^{m-\ell} \ = \ \frac{1}{k^m}\binom{m}{\ell}(a-1)^{m-\ell}. \tag{6}$$

[1] The random variables introduced here, $X_{[m]}$ and $Y_{[n]}$, are examples of order statistics. Let $X_{[1]} \le X_{[2]} \le \cdots \le X_{[m]}$ be the X_i's arranged in increasing order. Order statistics have many useful properties. In this work we need very little, specifically all we need is the probability distributions of $X_{[m]}$ and $Y_{[n]}$.

As $\ell \in \{1, \dots, m\}$ we have

$$\begin{aligned} \text{Prob}(X_{[m]} = a) &= \sum_{\ell=1}^{m} \frac{1}{k^m} \binom{m}{\ell} (a-1)^{m-\ell} \\ &= \frac{1}{k^m} \left[\sum_{\ell=0}^{m} \binom{m}{\ell} 1^\ell \cdot (a-1)^{m-\ell} \right] - \frac{(a-1)^m}{k^m} \\ &= \frac{1}{k^m} \cdot (1 + (a-1))^m - \frac{(a-1)^m}{k^m}, \end{aligned} \tag{7}$$

where the last line follows from the Binomial Theorem:

$$(A+B)^m = \sum_{\ell=0}^{m} \binom{m}{\ell} A^\ell \cdot B^{m-\ell}. \tag{8}$$

Therefore

$$\text{Prob}(X_{[m]} = a) = \frac{a^m - (a-1)^m}{k^m}, \tag{9}$$

which completes the proof. □

Note in the above proof we added zero (by adding and subtracting the $\ell = 0$ term), multiplied by one (by writing 1^ℓ), and used the Binomial Theorem. These are very common techniques, useful for solving a variety of problems.

Lemma 2.2. *Notation as above, if player one's highest roll is $a \in \{1, \dots, k\}$, then the probability that player one wins is $\frac{(a-1)^n}{k^n}$.*

Proof. If $X_{[m]} = a = 1$ then player one loses, and by inspection the formula is true. Let us now assume $a \geq 2$. If $X_{[m]} = a$ then for player one to win, player two must have $Y_{[n]} \leq a-1$. Thus if $Y_{[n]} = b$, then b is any integer in $\{1, \dots, a-1\}$, and the probability of this happening is

$$\begin{aligned} \sum_{b=1}^{a-1} \text{Prob}(Y_{[n]} = b) &= \sum_{b=1}^{a-1} \frac{b^n - (b-1)^n}{k^n} \\ &= \frac{1}{k^n} \sum_{b=1}^{a-1} [b^n - (b-1)^n] \\ &= \frac{1}{k^n} \cdot (a-1)^n, \end{aligned} \tag{10}$$

where the last line follows from having a telescoping series. Writing it out, we see the sum is

$$[1^n - 0^n] + [2^n - 1^n] + [3^n - 2^n] + \cdots + [(a-1)^n - (a-2)^n], \tag{11}$$

and all terms cancel except 0^n and $(a-1)^n$, which completes the proof. □

Remark 2.3. Alternatively, we may prove Lemma 2.2 by noting that if player one's highest roll is $a \in \{1, \dots, k\}$, then player one wins if and only if all of player two's rolls are less than or equal to $a-1$. As each of the k possible rolls is equally likely, the probability that each of player two's rolls are one of the $a-1$ numbers less than a is just $\frac{a-1}{k}$ (note if $a = 1$ then this possibility is zero, which is consistent as player one never wins if his highest roll is a 1); thus the probability that all of player two's rolls are at most $a-1$ is just $\left(\frac{a-1}{k}\right)^n$.

We can now prove our main result:

Proof of Theorem 1.1. Note that if player one wins, then $X_{[m]} \geq 2$, so if we write $X_{[m]} = a$ then $a \geq 2$. We have

$$\begin{aligned} \text{Prob(Player one wins)} &= \sum_{a=2}^{k} \text{Prob}(X_{[m]} = a) \cdot \text{Prob}(Y_{[n]} \leq a-1) \\ &= \sum_{a=2}^{k} \text{Prob}(X_{[m]} = a) \cdot \frac{(a-1)^n}{k^n}, \end{aligned} \tag{12}$$

where the last equality follows from Lemma 2.2 (the probability that player two has $Y_{[n]} \leq a-1$, or in other words, the probability that player two's highest roll is at most $a-1$). We now use Lemma 2.1 to substitute for the probability that $X_{[m]} = a$ and obtain

$$\begin{aligned} \text{Prob(Player one wins)} &= \sum_{a=2}^{k} \frac{a^m - (a-1)^m}{k^m} \cdot \frac{(a-1)^n}{k^n} \\ &= \frac{1}{k^{m+n}} \sum_{a=2}^{k} \left[a^m - (a-1)^m\right] \cdot (a-1)^n, \end{aligned} \tag{13}$$

completing the proof. □

We use Theorem 1.1 to determine the probability that player one wins for some pairs of m and n and a six sided die (so $k = 6$). We list the results in Table 1.

TABLE 1. Probabilities for some pairs of m and n with a six sided die.

m	n	Prob(Player One Wins)
1	1	$5/12 \approx 41.6667\%$
2	1	$125/216 \approx 57.8704\%$
2	2	$505/1296 \approx 38.9660\%$
3	1	$95/144 \approx 65.9722\%$
3	2	$3667/7776 \approx 47.1579\%$
3	3	$5479/15552 \approx 35.2302\%$
4	1	$5501/7776 \approx 70.7433\%$
4	2	$24529/46656 \approx 52.5742\%$
4	3	$112751/279936 \approx 40.2774\%$
4	4	$529069/1679616 \approx 31.4994\%$

3. PROOF OF THEOREM 1.2

We may re-write the probability that player one wins by replacing the summation with an integral. From Theorem 1.1 we know that

$$\text{Prob(Player one wins)} \ = \ \frac{1}{k^{m+n}} \sum_{a=2}^{k} [a^m - (a-1)^m] \cdot (a-1)^n; \tag{14}$$

we apply the integral version of partial summation (for a proof, see, for example, [Rud], page 70), the discrete version of integration by parts. This transforms the summation to integration.

Lemma 3.1 (Partial Summation: Integral Version)**.** *Let $h(x)$ be a continuously differentiable function with $h(1) = 0$. Let $B(x) = \sum_{a \le x} b_a$. Then*

$$\sum_{a \le x} b_a h(a) \ = \ B(x)h(x) - \int_1^x B(u)h'(u)du. \tag{15}$$

Proof of Theorem 1.2. We may extend our sums down to $a = 1$ without any change, as $(1-1)^0 = 0$. For us, $b_a = a^m - (a-1)^m$. Thus $B(u) = [u]^m$, where $[u]$ is the greatest integer less than or equal to u. Further $h(u) = (u-1)^n$ so $h'(u) = n(u-1)^{n-1}$. Thus by the integral version of partial summation we have

$$\begin{aligned}\text{Prob(Player one wins)} &= \frac{1}{k^{m+n}}\left[B(k)h(k) - \int_1^k B(u)h'(u)du\right] \\ &= \frac{1}{k^{m+n}}\left[k^m \cdot (k-1)^n - \int_1^k [u]^m \cdot n(u-1)^{n-1}du\right],\end{aligned} \tag{16}$$

which completes the proof. □

4. Estimating probabilities for m, n and k large

We estimate the probability of player one winning as m, n and k tend to infinity. For simplicity, we consider the easiest case, when k is much, much larger than m and n (though m and n are still large). We replace $[u]^m$ with $(u - \frac{1}{2})^m$ with small error. Explicitly, its error will be of lower order, of size between zero and about mu^{m-1}. Since k is large, this will give a smaller contribution; we chose to use $-\frac{1}{2}$ to center the error. Thus we find

$$\begin{aligned}\text{Prob(Player one wins)} &\approx \frac{1}{k^{m+n}}\left[k^m \cdot (k-1)^n - \int_1^k \left(u - \frac{1}{2}\right)^m \cdot n(u-1)^{n-1}du\right] \\ &= \frac{1}{k^{m+n}}\left[k^m \cdot (k-1)^n - n\int_{1/2}^{k-1/2} v^m(v-1/2)^{n-1}dv\right],\end{aligned} \tag{17}$$

where we made the change of variables $v = u - 1/2$.

We estimate the integral by Taylor expanding $v^m(v-1/2)^{n-1}$, keeping only the first two terms:

$$v^m(v-1/2)^{n-1} \approx v^m\left(v^{n-1} - \frac{n-1}{2}v^{n-2}\right); \tag{18}$$

we leave it to the reader to keep track of the error from the other terms in the Taylor series expansion. As the power of v keeps decreasing in each additional term, the effect is that each term is (roughly) $\frac{n}{k}$ smaller than the previous (we get an extra factor of n in each additional Taylor series term, but the smaller power of v and the fact that we integrate up to k results in a loss of a factor of k from the previous term).

We use this to evaluate the v-integral. As we evaluate at $k - 1/2$ and not k, after evaluating we Taylor expand the result, again only keeping the terms of size k^{m+n-1} or higher. This means we keep two terms from the first integration and only one term from the second.

$$\begin{aligned} n\int_{1/2}^{k-1/2} v^m(v-1/2)^{n-1}dv &\approx n\int_{1/2}^{k-1/2}\left(v^{m+n-1}-\frac{n-1}{2}v^{m+n-2}\right)dv \\ &= \frac{n(k-1/2)^{m+n}}{m+n}-\frac{n(n-1)}{2}\frac{(k-1/2)^{m+n-1}}{m+n-1} \\ &\approx \left(\frac{n\cdot k^{m+n}}{m+n}-\frac{n\cdot k^{m+n-1}}{2}\right)-\frac{n(n-1)k^{m+n-1}}{2(m+n-1)} \\ &\approx \frac{n\cdot k^{m+n}}{m+n}-\frac{m+2(n-1)}{2(m+n-1)}\frac{n}{k}k^{m+n}, \qquad (19) \end{aligned}$$

where the last line follows from straightforward algebra.

As

$$k^m(k-1)^n \approx k^{m+n}-nk^{m+n-1}, \qquad (20)$$

(we are only keeping terms of size k^{m+n-1} in our approximations) substituting this and (19) into (17) yields

$$\begin{aligned} \text{Prob(Player one wins)} &\approx \frac{1}{k^{m+n}}\left[k^{m+n}-nk^{m+n-1}-\frac{n\cdot k^{m+n}}{m+n}+\frac{m+2(n-1)}{2(m+n-1)}\frac{n}{k}k^{m+n}\right] \\ &\approx 1-\frac{n}{k}-\frac{n}{m+n}+\frac{m+2(n-1)}{2(m+n-1)}\frac{n}{k} \\ &\approx \left(1-\frac{n}{m+n}\right)-\left(1-\frac{m+2(n-1)}{2(m+n-1)}\right)\frac{n}{k} \\ &\approx \frac{m}{m+n}-\frac{m}{2(m+n-1)}\frac{n}{k}, \qquad (21) \end{aligned}$$

where the next term in the expansion will be of size $\left(\frac{n}{k}\right)^2$ (and the one after that will be of size $\left(\frac{n}{k}\right)^3$, and so forth).

For m, n and k large, we give some comparisons of the above estimate to the actual probabilities in Table 2. We see that the approximation is fairly good, and provides a quick way to estimate the probability that player one wins for such triples. Notice how important it was to have an expansion more than just $\frac{m}{m+n}$ (i.e., keeping a second term). If we did not keep the second term, then if $m = n$ we would approximate the probability of player one winning as 50% instead of approximately $\frac{2k-n}{4k} = \frac{1}{2} - \frac{n}{4k}$. So if $m = n = 800$ and $k = 1000$ then the probability should be about 30% and not 50%.

Finally, we remark on the limit as $k \to \infty$ for fixed m and n. This may be interpreted as player one choosing m numbers uniformly in $[0, 1]$ and player two choosing n numbers uniformly in $[0, 1]$. By symmetry each choice is equally likely to be the largest number (and there is now zero probability that two numbers are equal). Thus the probability that player one wins is the probability that the largest of the $m + n$ numbers is one of his m numbers, or $\frac{m}{m+n}$; note this agrees beautifully with the $k \to \infty$ limit of (17).

5. Generalizations

This is the first of many problems one can ask about player one and player two. Other natural questions are

TABLE 2. Comparisons of actual and estimated probabilities of player one winning

m	n	k	Actual Prob	Estimated Prob
100	100	200	37.6902%	37.4372%
100	300	400	16.5094%	15.6015%
150	100	200	44.7862%	44.9398%
300	100	400	64.9785%	65.6015%
550	100	400	72.4317%	74.0222%
550	550	700	31.2905%	30.3393%
800	800	1000	30.9897%	29.9875%

(1) For fixed k, what is the probability that player one wins as m and n tend to infinity? Does it matter *how* m and n tend to infinity? For example, is the answer different if $m = n$ or $m = n^2$?

(2) What is the probability that player one's top two rolls exceed the top two rolls of player two? Or, more generally, consider the largest c rolls of player one and two. For $i \in \{1, \dots, c\}$, what is the probability that $X_{[m+1-i]} \geq Y_{[n+1-i]}$? What is the probability that exactly d of the $X_{[m+1-i]}$ exceed the corresponding $Y_{[n+1-i]}$ (with $i \in \{1, \dots, c\}$)? Such a calculation is useful in the board game RISK, where often the attacker uses three die and the defender two die.

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