HOMEWORK PROBLEMS FROM STRANG'S *LINEAR ALGEBRA AND ITS APPLICATIONS* (4TH EDITION)

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1. Chapter 1: Matrices and Gaussian Elimination

Page 9, # 3: Describe the intersection of the three planes $u+v+w+z=6$, $u+w+z=4$ and $u+w=2$ (all in four dimensional space). Is it a line, point, or an empty set? What is the intersection if the fourth plane $u=-1$ is included? Find a fourth equation so that there is no solution.

Page 11: # 22: If (a,b) is a multiple of (c,d) with $abcd \neq 0$, show that (a,c) is a multiple of (b,d). This leads to the observation that if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has dependent rows then it has dependent columns.

Page 16: # 11: Apply elimination (circle the pivots) and back substitution to solve

$$\begin{aligned} 2x - 3y + 0z &= 3 \\ 4x - 5y + 1z &= 7 \\ 2x - 1y - 3z &= 5. \end{aligned}$$

List the three operations involved: subtract ____ times row ____ from row ____.

Page 17, # 18: It is impossible for a system of linear equations to have exactly two solutions. *Explain why*. (a) If (x,y,z) and (X,Y,Z) are two solutions, what is another one? (b) if 25 planes meet at two points, where else do they meet?

Page 27, #10: True or false? Give a specific counterexample when false. (a) If columns 1 and 3 of B are the same, so are columns 1 and 3 of AB. (b) If rows 1 and 3 of B are the same, so are rows 1 and 3 of AB. (c) If rows 1 and 3 of A are the same, so are rows 1 and 3 of AB. (d) $(AB)^2 = A^2B^2$.

Page 27, #12: The product of two lower triangular matrices is lower triangular. Confirm this with a 3×3 example, and then explain how it follows from the laws of matrix multiplication.

Page 28, # 20: The matrix that rotates the $x - y$ plane by an angle θ is $A(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. Verify that $A(\theta_1)A(\theta_2) = A(\theta_1+\theta_2)$ from the identities for $\cos(\theta_1 + \theta_2)$ and $\sin(\theta_1 + \theta_2)$. What is $A(\theta)$ times $A(-\theta)$?

Page 40, #5: Factor A into LU and write down the upper triangular system $Ux = c$ which appears after elimination, where

$$Ax \ = \ \begin{pmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} \ = \ \begin{pmatrix} 2 \\ 2 \\ 5 \end{pmatrix}.$$

Page 42, # 25: When zero appears in a pivot position, $A = LU$ is not possible. Show why these are both impossible:

$$\begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \ell & 1 \end{pmatrix} \begin{pmatrix} d & e \\ 0 & f \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \ell & 1 & 0 \\ m & n & 1 \end{pmatrix} \begin{pmatrix} d & e & g \\ 0 & f & h \\ 0 & 0 & i \end{pmatrix}.$$

Page 43, # 29: Compute L and U for the symmetric matrix

$$A \ = \ \begin{pmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{pmatrix}.$$

Page 44, # 41: How many exchanges will permute $(5, 4, 3, 2, 1)$ back to $(1, 2, 3, 4, 5)$? How may exchanges to change $(6, 5, 4, 3, 2, 1)$ to $(1, 2, 3, 4, 5, 6)$? One is even and one is odd. For $(n, \dots, 1)$ to $(1, \dots, n)$, show that $n = 100$

and 101 are even and $n = 102$ and 103 are odd.

Page 44, # 42: If P_1 and P_2 are permutation matrices, so is $P_1 P_2$. This still has the rows of I in some order. Give examples with $P_1 P_2 \neq P_2 P_1$ and $P_3 P_4 = P_4 P_3$.

Page 45, # 45: If you take powers of a permutation, why is some P^k eventually equal to I? Find a 5×5 permutation P so that the smallest power equal to I is P^6. (This is challenge question. Combine a 2×2 block with a 3×3 block).

Page 45, #47: If P is any permutation, find a non-zero vector x so that $(I - P)x = 0$. (This will mean that $I - P$ has no inverse, and has determinant zero.)

Page 52, # 6: Use the Gauss-Jordan method to invert

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Page 53, # 11: Give examples of A and B such that (a) $A + B$ is not invertible although A and B are invertible; (b) $A + B$ is invertible although A and B are not invertible; (c) all of A, B and $A + B$ are invertible. In the last case use $A^{-1}(A + B)B^{-1} = B^{-1} + A^{-1}$ to show that $C = A^{-1} + B^{-1}$ is also invertible, and find a formula for C^{-1}.

Page 53, # 19: Compute the LDL^T factorization of

$$A = \begin{pmatrix} 1 & 3 & 5 \\ 3 & 12 & 18 \\ 5 & 18 & 30 \end{pmatrix}, \quad B = \begin{pmatrix} a & b \\ b & d \end{pmatrix}.$$

Page 55, # 40: True or false (with a counterexample if false and a reason if true): (a) A 4×4 matrix with a row of zeros is not invertible; (b) a matrix with 1's along the main diagonal is invertible; (c) if A is invertible then A^{-1} is invertible; (d) if A^T is invertible then A is invertible.

Page 56, # 43: This matrix has a remarkable inverse. Find A^{-1} by elimination on $[AI]$. Extend to a 5×5 "alternating matrix" and guess its inverse:

$$A \ = \ \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Page 56, # 52: Let A be a matrix which is not identically zero. Show $A^2 = 0$ is possible but $A^T A = 0$ is not.

Page 58, # 65: A *group* of matrices includes AB and A^{-1} if it includes A and B. "Products and inverses stay in the group." Which of these sets are groups? Lower triangular matrices L with 1's on the diagonal, symmetric matrices S, positive matrices M, diagonal invertible matrices D, permutation matrices P. Invent two more matrix groups.

Page 65, #1.6: (a) There are 16 2×2 matrices whose entries are 1's and 0's. How many are invertible? (b) (Much harder) If you put 1's and 0's at random into the entries of a 10×10 matrix, is it more likely to be invertible or singular?

Page 66, # 1.20. The $n \times n$ permutation matrices are an important example of a "group". If you multiply them you stay inside the group; they have inverses in the group; the identity is in the group; and the law $P_1(P_2P_3) = (P_1P_2)P_3$ is true (because it is true for all matrices). (a) How many members belong to the groups of 4×4 and $n \times n$ permutation matrices? (b) Find a power k so that all 3×3 permutation matrices satisfy $P^k = I$.

2. CHAPTER 2: VECTOR SPACES

Page 73, # 1 Construct a subset of the $x - y$ plane in $\mathbb{R}$ that is (a) closed under vector addition and subtraction, but not scalar multiplication; (b) closed under scalar multiplication but not under vector addition. *Hint:* starting with u and v, add and subtract for (a). Try cu and cv for (b).

Page 74, #4: What is the smallest subspace of 3×3 matrices that contains all symmetric matrices *and* all lower triangular matrices? What is the largest subspace that is contained in both of those subspaces?

Page 74, #7: Which of the following are subsequences of $\mathbb{R}^\infty$: (a) all subsequences like $(1, 0, 1, 0, \dots)$ that include infinitely many zeros; (b) all sequences $(x_1, x_2, \dots)$ with $x_j = 0$ from some point onward; (c) all decreasing sequences: $x_{j+1} \leq x_j$ for each j; (d) al convergent sequences: the x_j have a limit as $j \to \infty$; (e) all arithmetic progressions: $x_{j+1} - x_j$ is the same for all j; (f) all geometric progressions $(x, kx, k^2x, \dots)$ allowing all x and k.

Page 76, #25: If we add an extra column b to a matrix A, then the column space gets larger unless ____. Give an example in which the column space gets larger and an example in which it doesn't. Why is $Ax = b$ solvable exactly when the column space *doesn't* get larger by including b?

Page 77, # 29: Construct a 3×3 matrix whose column space contains $(1, 1, 0)$ and $(1, 0, 1)$ but not $(1, 1, 1)$. Construct a 3×3 matrix whose column space is only a line.

Page 85, # 2: Reduce A and B to echelon form, to find their ranks. Which variables are free?

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

Page 87, # 24: Every column of AB is a combination of columns of A. Then the dimension of the column spaces give $\operatorname{rank}(AB) \leq \operatorname{rank}(A)$. Prove also that $\operatorname{rank}(AB) \leq \operatorname{rank}(B)$.

Page 87, #25: Suppose A and B are $n \times n$ matrices, and $AB = I$. Prove from $\text{rank}(AB) \leq \text{rank}(A)$ that the rank of A is n. So A is invertible and B must be its two-sided inverse. Therefore $BA = I$ (which is not so obvious!).

Page 87, #26: If A is 2×3 and C is 3×2, show from its rank that $CA \neq I$. Give an example in which $AC = I$. For $m < n$, a right inverse is not a left inverse.

Page 88, # 34: Under what conditions on b_1, b_2, b_3 is the following system solvable? Include b as a fourth column in $[Ab]$. Find all solutions when that condition holds:

$$\begin{aligned} 1x + 2y - 2z &= b_1 \\ 2x + 5y - 4z &= b_2 \\ 4x + 9y - 8z &= b_3. \end{aligned}$$

Page 89, #37: Why can't a 1×3 system have $x_p = (2, 4, 0)$ and x_n any multiple of $(1, 1, 1)$.

Page 89, # 44: Give examples of matrices A for which the number of solutions to $Ax = b$ is (a) 0 or 1, depending on b; (b) ∞, regardless of b; (c) 0 or ∞, depending on b; (d) 1, regardless of b.

Page 90, # 48: Reduce to $Ux = c$ (Gaussian elimination) and then $Rx = d$:

$$Ax = \begin{pmatrix} 1 & 0 & 2 & 3 \\ 1 & 3 & 2 & 0 \\ 2 & 0 & 4 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 10 \end{pmatrix} = b.$$

Page 90, #54: True or false (give a reason if true, or a counterexample if false): (a) a square matrix has no free variables; (b) an invertible matrix has no free variables; (c) an $m \times n$ matrix has no more than n pivot variables; (d) an $m \times n$ matrix has no more than m pivot variables.

Page 91, #61: Construct a matrix whose nullspace consists of all multiples of $(4, 3, 2, 1)$.

Page 91, #65: Construct a 2×2 matrix whose nullspace equals its column space.

Page 98, #1: Show that $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ are independent but v_1, v_2, v_3 and $v_4 = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$ are dependent. Solve $c_1 v_1 + \cdots + c_4 v_4 = 0$ or $Ac = 0$. The v's go into the columns of A.

Page 99, #14: Choose $x = (x_1, x_2, x_3, x_4)$ in $\mathbb{R}^4$. It has 24 rearrangements like (x_2, x_1, x_3, x_4). Those 24 vectors, including x itself, span a subspace S. Find specific vectors x so that the dimension of S is (a) 0, (b) 1, (c) 3, (d) 4.

Page 102, #39: The cosine space F_3 contains all combinations $y(x) = A\cos x + B\cos 2x + C\cos 3x$. Find a basis for the subspace that has $y(0) = 0$.

Page 102, #40: Find a basis for the space of functions that satisfy (a) $dy/dx - 2y = 0$; (b) $dy/dx - y/x = 0$.

Page 102, #43: Write the 3×3 identity matrix as a combination of the other five permutation matrices! Then show that those five matrices are linearly independent. (Assume a combination gives zero, and check entries to prove each term is zero.) The five permutations are a basis for the subspace of 3×3 matrices with row and column sums all equal.

Page 110, #3: Find the dimension and a basis for the four fundamental subspaces for

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Page 110, #5: If the product AB is the zero matrix, show that the column space of B is contained in the nullspace of A. (Also the row space of A is in the left nullspace of B, since each row of A multiplies B to give a zero row.)

Page 111, # 16 (*A paradox*) Suppose A has a right-inverse B. Then $AB = I$ leads to $A^TAB = A^T$ or $B = (A^TA)^{-1}A^T$. But that satisfies $BA = I$; it is a *left*-inverse. Which step is not justified?

Page 112, # 20 (a) If a 7×9 matrix has rank 5, what are the dimensions of the four subspaces? What is the sum of the four dimensions? (b) If a 3×4 matrix has rank 3, what are its column space and left nullspace?

Page 112, # 27: A is an $m \times n$ matrix of rank r. Suppose there are right-hand sides b for which $Ax = b$ has *no solution*. (a) What inequalities ($<$ or $\le$) must be true between m, n and r? (b) How do you know that $A^Ty = 0$ has a non-zero solution?

Page 113, #37: True or false (with a reason or a counterexample): (a) A and A^T have the same number of pivots; (b) A and A^T have the same left nullspace; (c) if the row space equals the column space then $A = A^T$; (d) if $A^T = -A$ then the row space of A equals the column space.

Page 133, # 6: Which 3×3 matrices represent the transformations that (a) project every vector onto the $x - y$ plane; (b) reflect every vector through the $x - y$ plane; (c) rotate the $x - y$ plane through 90 degrees, leaving the z-axis alone; (d) rotate the $x - y$ plane, then the $x - z$ plane, then the $y - z$ plane, each through 90 degrees; (e) carry out the same rotations, but each through 180 degrees?

Page 133, #7: On the space P_3 of cubic polynomials, what matrix represents d^2/dt^2? Construct the 4×4 matrix from the standard basis $1, t, t^2, t^3$. Find its nullspace and column space. What do they mean in terms of polynomials?

Page 134, #14: Prove that T^2 is a linear transformation if T is linear (from $\mathbb{R}^3$ to $\mathbb{R}^3$).

Page 135, #36: (a) What matrix transforms $(1, 0)$ into $(2, 5)$ and transforms $(0, 1)$ to $(1, 3)$? (b) What matrix transforms $(2, 5)$ to $(1, 0)$ and $(1, 3)$ to $(0, 1)$? (c) Why does no matrix transform $(2, 6)$ to $(1, 0)$ and $(1, 3)$ to $(0, 1)$?

Page 136, #46: Show that the product ST of two reflections is a rotation. Multiply these reflection matrices to find the rotation angle:

$$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}, \quad \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}.$$

Page 136, #47: The 4×4 Hadamard matrix is entirely $+1$ and -1:

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1. \end{pmatrix}$$

Find H^{-1} and write $(7, 5, 3, 1)$ as a combination of the columns of H.

Page 137, #50: Suppose all vectors x in the unit square $0 \leq x_1, x_2 \leq 1$ are transformed to Ax (A is a 2×2 matrix). (a) What is the shape of the transformed region (all Ax)? (b) For which matrices A is the region a square? (c) For which A is it a line? (d) For which A is the new area still 1?

Page 138, #2.10: Invent a vector space that contains all linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$. You have to decide on a rule of addition. What is its dimension?

Page 139, # 2.23: How can you construct a matrix that transforms the coordinate vectors e_1, e_2, e_3 into three given vectors v_1, v_2, v_3. When will that matrix be invertible?

3. Chapter 3: Orthogonality

Page 148, # 2: Give an example in $\mathbb{R}^2$ of linearly independent vectors that are not orthogonal. Also, give an example of orthogonal vectors that are not independent.

Page 149, # 11: The fundamental theorem is often stated in the form of *Fredholm's alternative:* For any A and b one and only one of the following systems has a solution: (i) $Ax = b$; (ii) $A^T y = 0$ and $y^T b \neq 0$. Either b is in the column space $C(A)$ or there is a y in $N(A^T)$ such that $y^T b \neq 0$. Show that it is contradictory for (i) and (ii) to both hold.

Page 149, # 14: Show that $x - y$ is orthogonal to $x + y$ if and only if $||x|| = ||y||$.

Page 149, # 19: Why are these statements false? (a) If V is orthogonal to W then $V^\perp$ is orthogonal to $W^\perp$. (b) If V is orthogonal to W and W is orthogonal to Z then V is orthogonal to Z.

Page 150, # 32: Draw Figure 3.4 to show each subspace for

$$A \ = \ \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}, \quad B \ = \ \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix}.$$

Page 151, # 36: Extend Problem 35 to a p-dimensional subspace V and a q-dimensional subspace W of $\mathbb{R}^n$. What inequality on $p + q$ guarantees that V intersects W in a non-zero vector? These subspaces cannot be orthogonal.

Page 151, # 47: Construct a 3×3 matrix A with no zero entries whose columns are mutually perpendicular. Compute $A^T A$. Why is it a diagonal matrix.

Page 152, # 49. Why is each of these statements false? (a) $(1, 1, 1)$ is perpendicular to $(1, 1, -2)$ so the planes $x + y + z = 0$ and $x + y - 2z = 0$ are orthogonal subspaces. (b) The subspace spanned by $(1, 1, 0, 0, 0)$ and $(0, 0, 0, 1, 1)$ is the orthogonal complement of the subspace spanned by $(1, -1, 0, 0, 0)$ and $(2, -2, 3, 4, -4)$. (c) Two subspaces that meet only in the zero vector are orthogonal.

Page 152, # 51. Suppose that A is 3×4, B is 4×5 and $AB = 0$. Prove $\text{rank}(A) + \text{rank}(B) \leq 4$.

Page 157, # 1 (a) Given any two positive numbers x and y, choose the vector $b = (\sqrt{x}, \sqrt{y})$ and $a = (\sqrt{y}, \sqrt{x})$. Apply the Schwarz inequality to compare the arithmetic mean $(x+y)/2$ with the geometric mean $\sqrt{xy}$. (b) Suppose we start with a vector from the origin to the point x, and then add a vector of length $||y||$ connecting x to $x+y$. The third side of the triangle goes from the origin to $x+y$. *The triangle inequality asserts that this distance cannot be greater than the sum of the first two:* $||x+y|| \leq ||x|| + ||y||$. *After squaring both sides, and expanding* $(x+y)^T(x+y)$, *reduce this to the Schwarz inequality.*

Page 158, # 9: Square the matrix $P = aa^T/a^Ta$, which projects onto a line, and show that $P^2 = P$. (Note the number a^Ta in the middle of the matrix aa^Taa^T.)

Page 158, # 10: Is the projection matrix P invertible? Why or why not?

Page 158, # 12: Find the matrix that projects every point in the line plane onto the line $x + 2y = 0$.

Page 158, #13: Prove that the *trace* of $P = aa^T/a^Ta$ – which is the sum of its diagonal entries – always equals 1.

Page 159, # 15: Show that the length of Ax equals the length of A^Tx if $AA^T = A^TA$.

Page 159, #17: Project the vector b onto the line through a. Check that e is perpendicular to a: (a) $b = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ and $a = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$; (b) $b = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$ and $a = \begin{pmatrix} -1 \\ -3 \\ -1 \end{pmatrix}$.

Page 170, #6: Find the projection of b onto the column space of A:

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}. \tag{1}$$

Split b into $p + q$ with p in the column space and q perpendicular to that space. Which of the four subspaces contains q?

Page 171, #13: Find the best straight-line fit (least squares) to the measurements $b = 4$ at $t = -2$, $b = 3$ at $t = -1$, $b = 1$ at $t = 0$ and $b = 0$ at $t = 2$. Then find the projection of $b = (4, 3, 1, 0)^T$ onto the column space of

$$A = \begin{pmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{pmatrix}. \tag{2}$$

Page 185, # 2: Project $b = (0, 3, 0)$ onto each of the orthonormal vectors $a_1 = (2/3, 2/3, -1/3)$ and $a_2 = (-1/3, 2/3, 2/3)$, and then find its projection p onto the plane of a_1 and a_2.

Page 186, # 3: Find also the projection of $b = (0, 3, 0)$ onto $a_3 = (2/3, -1/3, 2/3)$, and add the three projections (see previous problem). Why is $P = a_1 a_1^T + a_2 a_2^T + a_3 a_3^T$ equal to I?

Page 186, # 9: If the vectors q_1, q_2, q_3 are orthonormal, what combination of q_1 and q_2 is closest to q_3?

Page 186, # 13: Apply the Gram-Schmidt process to $a = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $c = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, and write the result in the form $A = QR$.

Page 187, # 21: What is the closest function $a \cos x + b \sin x$ to the function $f(x) = \sin 2x$ on the interval from $-\pi$ to π? What is the closest straight line $c + dx$?

Page 187, # 24: Find the fourth Legendre polynomial. It is a cubic $x^3 + ax^2 + bx + c$ that is orthogonal to 1, x and $x^2 - \frac{1}{3}$ over the interval $-1 \leq x \leq 1$.

Page 188, # 31: True or false (give an example in either case): (a) Q^{-1} is an orthogonal matrix when Q is an orthogonal matrix; (b) If Q (a 3×2 matrix) has orthonormal columns then $||Qx||$ always equals $||x||$.

4. Chapter 4: Determinants

Page 206, # 1: If a 4×4 matrix has $\det A = 1/2$, find $\det(2A)$, $\det(-A)$, $\det(A^2)$ and $\det(A^{-1})$.

Page 206, # 4: By applying row operations to produce an upper triangular U, compute

$$\left|\begin{array}{cccc} 1 & 2 & -2 & 0 \\ 2 & 3 & -4 & 1 \\ -1 & -2 & 0 & 2 \\ 0 & 2 & 5 & 3 \end{array}\right|, \quad \left|\begin{array}{cccc} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & -2 \end{array}\right|.$$

Page 207, # 10: If Q is an orthogonal matrix, so that $Q^TQ = I$, prove that $\det Q = \pm 1$. What kind of box is formed from the rows (or columns) of Q?

Page 208, # 12: Use row operations to verify that the 3×3 "Vandermonde determinant" is

$$\left|\begin{array}{ccc} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{array}\right| = (b-a)(c-a)(c-b).$$

Page 208, # 14: True or false, with a reason if true and a counterexample if false. (a) If A and B are identical except that $b_{11} = 2a_{11}$, then $\det B = 2 \det A$. (b) The determinant is the product of the pivots. (c) If A is invertible and B is singular, then $A+B$ is invertible. (d) If A is invertible and B is singular then AB is singular. (e) The determinant $AB - BA$ is zero.

Page 208, #19: Suppose that $CD = -DC$, and find the flaw in the following argument: Taking determinants gives $\det C \det D = -\det D \det C$, so either $\det C = 0$ or $\det D = 0$. Thus $CD = -DC$ is possible only if C or D is singular.

Page 209, # 26: If a_{ij} is $i \cdot j$, show that $\det A = 0$ (except when $A = (1)$).

Page 210, # 30: Show that the partial derivatives of $\ln(\det a)$ give A^{-1}: if $f(a,b,c,d) = \ln(ad - bc)$ then

$$A^{-1} = \left(\begin{array}{cc} \partial f/\partial a & \partial f/\partial c \\ \partial f/\partial b & \partial f/\partial d \end{array}\right).$$

Page 215, # 3: True or false? (a) The determinant of $S^{-1}AS$ equals the determinant of A. (b) If $\det A = 0$ then at least one of the cofactors must be zero. (c) A matrix whose entries are 0's and 1's has determinant 0, 1 or -1.

Page 215, # 4 (a) Find the LU factorization, the pivots and the determinant of the 4×4 matrix whose entries are $a_{ij} = \min(i,j)$. (Write out the matrix.) (b) Find the determinant if $a_{ij} = \min(n_i, n_j)$, where $n_1 = 2$, $n_2 = 6$, $n_3 = 8$ and $n_4 = 10$. Can you give a general rule for any $n_1 \leq n_2 \leq n_3 \leq n_4$?

Page 215, # 6: Suppose A_n is the $n \times n$ tridiagonal matrix with 1's on the three diagonals:

$$A_1 = (1), \quad A_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \ldots.$$

Let $D_n = \det A_n$; we want to find it. (a) Expand in cofactors along the first row to show that $D_n = D_{n-1} - D_{n-2}$. (b) Starting from $D_1 = 1$ and $D_2 = 0$, find $D_3, \ldots, D_8$. By noticing how these numbers cycle around (with what period?), find D_{1000}.

Page 216, #9: How many multiplications to find an $n \times n$ determinant from (a) the big formula (6): $\det A = \sum_{\text{all } P's} (a_{1\alpha} a_{2\beta} \cdots a_{n\nu}) \det P$; (b) the cofactor formula (10), building from the count for $n-1$: $\det A = a_{i1}C_{i1} + \cdots + a_{in}C_{in}$, where the cofactor C_{ij} is the determinant of M_{ij} with the correct sign (delete row i and column j: $C_{ij} = (-1)^{i+j} \det M_{ij}$); (c) the product of pivots formula (including the elimination steps).

Page 217, # 18: Place the smallest number of zeros in a 4×4 matrix that will guarantee $\det A = 0$. Place as many zeros as possible while still allowing $\det A \neq 0$.

Page 218, #28: The $n \times n$ determinant C_n has 1's above and below the main diagonal:

$$C_1 = |0|, \quad C_2 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \quad C_3 = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}.$$

(a) What are the determinants C_1, C_2, C_3, C_4? (b) By cofactors find the relation between C_n and C_{n-1} and C_{n-2}. Find C_{10}.

Page 219, # 34: With 2×2 blocks, you cannot always use block determinants!

$$\begin{vmatrix} A & B \\ 0 & D \end{vmatrix} = |A||D|, \quad \text{but} \quad \begin{vmatrix} A & B \\ 0 & D \end{vmatrix} \neq |A||D| - |C||B|.$$

(a) Why is the first statement true? Somehow B does not enter. (b) Show by example that the first equality fails (as shown) when C enters. (c) Show by example that the answer $\det(AD - CB)$ is also wrong.

Page 227, # 16: *Quick proof of Cramer's rule.* The determinant is a linear function of column 1. It is zero if two columns are equal. When $b = Ax = x_1a_1 + x_2a_2 + x_3a_3$ goes into column 1 to produce B_1, the determinant is

$$|ba_2a_3| \ = \ |x_1a_1 + x_2a_2 + x_3a_3a_2a_3| \ = \ x_1|a_1a_2a_3| \ = \ x_1 \det A.$$

(a) What formula for x_1 comes from the left side equals the right side? (b) What steps lead to the middle equation?

Page 228, # 32: If the columns of a 4×4 matrix have lengths L_1, L_2, L_3 and L_4, what is the largest possible value of the determinant (based on volume)? If all entries are 1 or -1, what are those lengths and the maximum determinant?

Page 230, #4.5: If the entries of A and A^{-1} are integers, how do you know that both determinants are 1 or -1? *Hint:* What is $\det A$ times $\det A^{-1}$?

Page 230, # 4.9: If P_1 is an even permutation matrix and P_2 is odd, deduce from $P_1 + P_2 = P_1(P_1^T + P_2^T)P_2$ that $\det(P_1 + P_2) = 0$.

Page 231, # 4.16: The circular shift permutes $(1, 2, \dots, n)$ into $(2, 3, \dots, 1)$. What is the corresponding permutation matrix P and (depending on n) what is its determinant?

5. Chapter 5: Eigenvalues and Eigenvectors

Page 240, # 1: Find the eigenvalues and eigenvectors of $A = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$. Verify that the trace equals the sum of the eigenvalues and the determinant equals their product.

Page 241, # 4: Solve $du/dt = Pu$, when P is a projection:

$$\frac{du}{dt} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} u, \quad u(0) = \begin{pmatrix} 5 \\ 3 \end{pmatrix}.$$

Part of $u(0)$ increases exponentially while the nullspace part stays fixed.

Page 241, # 6: Give an example to show that the eigenvalues can be changed when a multiple of one row is subtracted from another. Why is a zero eigenvalue *not* changed by the steps of elimination?

Page 242, # 14: Find the rank and all four eigenvalues for both the matrix of ones and the checkboard matrix:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

Which eigenvectors correspond to nonzero eigenvalues?

Page 242, # 18: Suppose A has eigenvalues 0, 3 and 5 with corresponding independent eigenvectors u, v, and w. (a) Give a basis for the nullspace and a basis for the column space. (b) Find a particular solution to $Ax = v + w$. Find all solutions. (c) Show that $Ax = u$ has no solution. (If it had a solution, then _____ would be in the column space.)

Page 243, # 26: Solve $\det(Q - \lambda I) = 0$ by the quadratic formula to reach $\lambda = \cos\theta \pm i \sin\theta$:

$$Q = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

rotates the xy-plane by the angle θ. Find the eigenvectors of Q by solving $(Q - \lambda I)x = 0$. Use $i^2 = -1$.

Page 244, # 39: *Is there a real* 2×2 *matrix* (other than I) *with* $A^3 = I$? Its eigenvalues must satisfy $\lambda^3 = I$. They can be $e^{2\pi i/3}$ and $e^{-2\pi i/3}$. What trace and determinant would give this? Construct A.

Page 244, # 40: There are six 3×3 permutation matrices P. What numbers can be the *determinant* of P? What numbers can be *pivots*? What numbers can be the *trace* of P? What *four numbers* can be eigenvalues of P?

Page 250, #1: Factor the following matrices into $S\Lambda S^{-1}$:

$$A \;=\; \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad A \;=\; \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}.$$

Page 250, #7: If $A = \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}$, find A^{100} by diagonalizing A.

Page 251, # 9: Show by direct calculation that AB and BA have the same trace when

$$A \;=\; \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad B \;=\; \begin{pmatrix} q & r \\ s & t \end{pmatrix}.$$

Deduce that $AB - BA = I$ is impossible (except in infinite dimensions).

Page 251, #19: True or false: if the n columns of S (eigenvectors of A) are independent, then (a) A is invertible; (b) A is diagonalizable; (c) S is invertible; (d) S is diagonalizable.

Page 252, #25: True or false: if the eigenvalues of A are 2, 2, 5 then the matrix is certainly (a) invertible; (b) diagonalizable; (c) not diagonalizable.

Page 254, #44: If A is 5×5, then $AB - BA$ is the zero matrix gives 25 equations for the 25 entries in B. Show the 25×25 matrix is singular by noticing a simple non-zero solution B.

Page 263, #8: Suppose there is an epidemic in which every month half of those who are well become sick, and a quarter of those who are sick become dead. Find the steady state solution to the corresponding Markov process

$$\begin{pmatrix} d_{k+1} \\ s_{s+1} \\ w_{k+1} \end{pmatrix} = \begin{pmatrix} 1 & 1/4 & 0 \\ 0 & 3/4 & 1/2 \\ 0 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} d_k \\ s_s \\ w_k \end{pmatrix}.$$

Page 264, # 15: If A is a Markov matrix, show that the sum of the components of Ax equals the sum of the components of x. Deduce that if $Ax = \lambda x$ with $\lambda \neq 1$, the components of the eigenvector add to zero.

Page 265, # 19: Multiplying term by term, check that $(I-A)(I+A+A^2+\cdots) = I$. This series represents $(I-A)^{-1}$. It is nonnegative when A is nonnegative, provided it has a finite sum; the condition for that is $\lambda_{\max} < 1$. Add up the infinite series, and confirm that it equals $(I-A)^{-1}$, for the consumption matrix

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

which has $\lambda_{\max} < 1$.

Page 266, # 29: The powers A^k approach zero if all $|\lambda_i| < 1$, and they blow up if any $|\lambda_i| > 1$. Peter Lax gives four striking examples in his book *Linear Algebra*: $A = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$, $B = 32{-}5{-}3$, $C = \begin{pmatrix} 5 & 7 \\ -3 & -4 \end{pmatrix}$, $D = \begin{pmatrix} 5 & 6.9 \\ -3 & -4 \end{pmatrix}$, where $||A^{1024}|| > 10^{700}$, $B^{1024} = I$, $C^{1-24} = -C$, $||D^{1024}|| < 10^{-78}$. Find the eigenvalues $\lambda = e^{i\theta}$ of B and C to show that $B^4 = I$ and $C^3 = -I$.

Page 275, # 1: Following the first example in this section, find the eigenvalues and eigenvectors, and the exponential e^{At}, for$A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$.

Page 276, #4: If P is a projection matrix, show from the infinite series that $e^P \approx I + 2.718P$.

Page 276, # 6: The higher order equation $y'' + y = 0$ can be written as a first-order system by introducing the velocity y' as another unknown:

$$\frac{d}{dt}\begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} y' \\ y'' \end{pmatrix} = \begin{pmatrix} y' \\ -y \end{pmatrix}.$$

If this is $du/dt = Au$, what is the 2×2 matrix A? Find its eigenvalues and eigenvectors, and compute the solution that starts from $y(0) = 2$, $y'(0) = 0$.

Page 279, # 38. Generally $e^A e^B$ is different from $e^B e^A$. They are both different from e^{A+B} (in general). Check this with $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$.

Page 280, # 41: Give two reasons why the matrix exponential e^{At} is never singular: (a) write its inverse; (b) write its eigenvalues: if $Ax = \lambda x$ then $e^{At}x = \underline{\quad\quad} x$.

Page 290, #12: Give a reason if true or a counterexample if false: (a) if A is Hermitian then $A + iI$ is invertible; (b) if Q is orthogonal then $Q + \frac{1}{2}I$ is invertible; (c) if A is real then $A + iI$ is invertible.

Page 291, # 21: Describe all 3×3 matrices that are simultaneously Hermitian, unitary and diagonal. How many are there?

Page 292, # 41: If $A = R + iS$ is a Hermitian matrix, are the real matrices R and S symmetric?

Page 292, # 44: How are the eigenvalues of A^H (square matrix) related to the eigenvalues of A?

Page 304, #24a: Show by direct multiplication that every triangular matrix T, say 3×3, satisfies its own characteristic equation: $(T - \lambda_1 I)(T - \lambda_2 I)(T - \lambda_3 I) = 0$.

Page 304, # 30: Show A and B are similar by finding M so that $B = M^{-1}AM$: (a) $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$; (b) $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$; (c) $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}$.

Page 304, # 32: There are 16 2×2 matrices whose entries are 0's and 1's. Similar matrices go into the same family. How many families? How many matrices (total of 16) in each family?

Page 304, # 34: If A and B have exactly the same eigenvalues and eigenvectors, does $A = B$? With n independent eigenvectors, we do have $A = B$. Find $A \neq B$ when $\lambda = 0, 0$ (repeated), but there is only one line of eigenvectors $(x, 0)$.

Page 305, # 42: Prove AB and BA have the same eigenvalues.

Page 307, # 5.5: Does there exist a matrix A such that the entire family $A + cI$ is invertible for all complex numbers c? Find a real matrix A with $A + rI$ invertible for all real r.

Page 308,# 5.11: If P is the matrix that projects $\mathbb{R}^n$ onto a subspace S, explain why every vector in S is an eigenvector, and so is every vector in $S^\perp$. What are the eigenvalues? (Note the connection to $P^2 = P$, which means that $\lambda^2 = \lambda$.)

Page 308, # 5.12: Show that every matrix of order greater than 1 is the sum of two singular matrices.

Page 309, # 5.25: (a) Find a non-zero matrix N such that $N^3 = 0$. (b) If $Nx = \lambda x$ show that $\lambda = 0$. (c) Prove that N (called a nilpotent matrix) cannot be symmetric.

Page 309, # 5.30: What is the limit as $k \to \infty$ (the Markov steady state) of $\begin{pmatrix} .4 & .3 \\ .6 & .7 \end{pmatrix}^k \begin{pmatrix} a \\ b \end{pmatrix}$?

6. Chapter 6: Positive Definite Matrices

Page 316, #1: The quadratic $f(x,y) = x^2 + 4xy + 2y^2$ has a saddle point at the origin, despite the fact that its coefficients are positive. Write $f(x,y)$ as the difference of two squares.

Page 316, #4: Decide between a minimum, maximum or saddle point for the following functions: (a) $F(x,y) = -1 + 4(e^x - x) - 5x\sin y + 6y^2$ at the point $x = y = 0$; (b) $F(x,y) = (x^2 - 2x)\cos y$, with stationary point at $x = 1$ and $y = \pi$.

Page 316, #8: If $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ is positive definite, test $A^{-1} = \begin{pmatrix} p & q \\ q & r \end{pmatrix}$ for positive definiteness.

Page 326, #1: For what range of numbers a and b are the matrices A and B positive definite:

$$A \ = \ \begin{pmatrix} a & 2 & 2 \\ 2 & a & 2 \\ 2 & 2 & a \end{pmatrix}, \quad B \ = \ \begin{pmatrix} 1 & 2 & 4 \\ 2 & b & 8 \\ 4 & 8 & 7 \end{pmatrix}. \tag{3}$$

Page 329, #26: Draw the titled ellipse $x^2 + xy + y^2 = 1$ and find the half-lengths of its axes from the eigenvalues of the corresponding A.

Page 330, #37: If C is non-singular, show that A and C^TAC have the same rank. Thus they have the same number of zero eigenvalues.

Page 330: #43: A group of non-singular matrices includes AB and A^{-1} if it includes A and B (products and inverses stay in the group). Which of these sets are groups: (a) positive definite symmetric matrices A; (b) orthogonal matrices Q; (c) all exponentials e^{tA} of a fixed matrix A; (d) matrices P with positive eigenvalues; (e) matrices D with determinant 1. Invent a group containing only positive definite matrices.

Page 345, #5: For any symmetric matrix A, compute the ratio $R(x)$ for the special choice $x = (1, 1, \dots, 1)^T$. How is the sum of all entries a_{ij} related to λ_1 and λ_n?

Page 345, #7: If B is positive definite, show from the Rayleigh quotient that the smallest eigenvalue of $A+B$ is larger than the smallest eigenvalue of A.

Page 345, #8: If λ_1 and μ_1 are the smallest eigenvalues of A and B, show that the smallest eigenvalue θ_1 of $A+B$ is at least as large as $\lambda_1+\mu_1$. (Try the corresponding eigenvector x in the Rayleigh quotients.)

Note: Problems 7 and 8 are perhaps the most typical and most important results that come easily from Rayleigh's principle, but only with great difficulty from the eigenvalue equations themselves.

Page 346, #16 (recommended): Let A be the $n\times n$ matrix such that the upper $(n-1)\times(n-1)$ block is all zeros, the last row is $1,2,\ldots,n$ and the last column is $1,2,\ldots,n$. Decide the signs of the n eigenvalues.

7. CHAPTER 7: COMPUTATIONS WITH MATRICES

Page 357, #3: Explain why $||ABx|| \leq ||A||\cdot||B||\cdot||x||$, and deduce from $||A|| = \max_{x\neq 0}||Ax||/||x||$ that $||AB|| \leq ||A||\cdot||B||$. Show that this also implies $c(AB)\leq c(A)c(B)$.

Page 357, #4: For the positive definite $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, compute $||A^{-1}|| = 1/\lambda_1$, $||A|| = \lambda_2$ and $c(A)=\lambda_2/\lambda_1$. Find a right-hand side b and a perturbation ∂b such that the error is worst possible, $||\partial x||/||x|| = c||\partial b||/||b||$.

Page 357, #9: Show that $\max|\lambda|$ is not a true norm by finding a 2×2 counterexample to $\lambda_{\max}(A+B)\leq\lambda_{\max}(A)+\lambda_{\max}(B)$ and $\lambda_{\max}(AB)\leq \lambda_{\max}(A)\lambda_{\max}(B)$.

Page 358, #18: The ℓ^1 norm is $||x||_1 = |x_1|+|x_2|+\cdots+|x_n|$. The ℓ^∞ norm is $||x||_\infty = \max|x_i|$. Compute $||x||, ||x||_1, ||x||_\infty$ for the vectors $(1,1,1,1,1)$ and $(.1,.7,.3,.4,.5)$.

Page 358, #19: Prove that $||x||_\infty \leq ||x|| \leq ||x||_1$. Show from the Cauchy-Schwarz inequality that the ratios $||x||/||x||_\infty$ and $||x||_1/||x||$ are never larger than $\sqrt{n}$. Which vector $(x_1,\ldots,x_n)$ gives ratios equal to $\sqrt{n}$?

Page 358, #20: All vector norms must satisfy the triangle inequality. Prove that $||x+y||_\infty \le ||x||_\infty + ||y||_\infty$ and $||x+y||_1 \le ||x||_1 + ||y||_1$.

Page 365, #1: For the matrix $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ with eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 3$, apply the power method $u+k+1 = Au_k$ three times to the initial guess $u_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. What is the limiting vector u_∞?

Page 366, #2: For the same A and initial guess $u_0 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, compare three inverse power steps to one shifted step with $\alpha = u_0^T A u_0 / u_0^T u_0$:

$$u_{k+1} = A^{-1} u_k \ = \ \frac{1}{3} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} u_k \quad \text{or} \quad u \ = \ (A - \alpha I)^{-1} u_0. \tag{4}$$

The limiting vector u_∞ is now a multiple of the other eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Page 366, #4: The Markov matrix $A = \begin{pmatrix} .9 & .3 \\ .1 & .7 \end{pmatrix}$ has $\lambda = 1$ and $.6$, and the power method $u_k = A^k u_0$ converges to $\begin{pmatrix} .75 \\ .25 \end{pmatrix}$. Find the eigenvectors of A^{-1}. What does the inverse power method $u_{-k} = A^{-k} u_0$ converge to (after you multiply by $.6^k$)?