MATH 187/487: FINAL: DECEMBER 2003
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Instructions: **Math 187:** Try and do two or three problems (at least); **Math 487:** Try and do three or four problems (at least), but you cannot take three problems from the first three; **All:** Do not work in groups (at least not for now). If you are stuck, I will provide hints later. If you cannot find enough interesting problems among the choices below, let me know and I'll add more.

Question 1 : Let $\pi(x)$ equal the number of primes less than or equal to x. Chebychev proved there exist constants A and B, $.8 < A < 1 < B < 1.2$, such that

$$\frac{Ax}{\log x} \ < \ \pi(x) \ < \ \frac{Bx}{\log x}.$$

Use Chebychev's result to prove Bertrand's Postulate: in any interval $[n, 2n]$, there exists at least one prime.

Question 2 : Recall the Binomial Theorem:

$$(x+y)^N \ = \ \sum_{k=0}^{N} \binom{N}{k} x^{N-k} y^k.$$

Find a similar formula for $(x+y+z)^N$; more generally, find a formula for $(x_1 + \cdots + x_m)^N$.

Question 3 : Unusual Suspects: The police rounded up Jim, Bud and Sam yesterday, because one of them was suspected of having robbed the local bank. The three suspects made the following statements under intensive questioning.

- Jim: I'm innocent.
- Bud: I'm innocent.
- Sam: Bud is the guilty one.

If only one of these statements turns out to be true, who robbed the bank?

Date: November 26, 2003.

Question 4 : Consider n ordered points in the plane, $P_i = (x_i, y_i)$. Connect P_1 to P_2, P_2 to $P_3, \dots,$ P_n to P_1. Find a simple formula for the area of the polygon in terms of the coordinates of the vertices.

Question 5 : Consider a 5×5 chessboard. Recall a queen can move horizontally, vertically, and diagonally. Place five queens on the board in such a way that there are three squares which cannot be attacked by any of the queens.

Question 6 : Five real numbers in $[0, 1]$ are chosen at random from the uniform probability distribution; this means the probability of observing a number in $[x, x + \Delta x]$ equals Δx. The player is shown each number in turn and asked whether he believes it to be the largest of the five. If the player guesses wrong, the game ends (player loses). If the player guesses correctly that the current number IS the largest, the player wins. If the player guesses correctly that the current number is NOT the largest, then the player is shown the next number and the game continues. Discuss the strategy the player should use.

Question 7 : Consider the equation

$$x_1 + \cdots + x_P \ = \ C,$$

where each x_i is a non-negative integer. We proved in class that for C a non-negative integer, the number of solutions of the above equation (ie, the number of tuples $(x_1, \dots, x_P)$ that work) is $\binom{C+P-1}{P-1}$. If you can, re-prove this. Also, use this to come up with a nice, simple expression for

$$\sum_{C=0}^{N} \binom{C+P-1}{P-1}.$$

Question 8 : Let $x \in [0, 1]$. Then x has a binary expansion:

$$x \ = \ \sum_{n=1}^{\infty} \frac{b_n}{2^n}, \quad b_n \in \{0, 1\}.$$

Let $S(m) \subset [0, 1]$ be the set of numbers with at most m ones in their binary expansion. Is $S(m)$ countable?

Let $S \subset [0, 1]$ be the set of numbers with finitely many ones in their binary expansion. Is S countable?

Question 9 : Recall α is an Algebraic number if there exists a polynomial with integer coefficients such that $f(\alpha) = 0$. We showed the set of Algebraic numbers is countable.

Define the Hyper-Algebraic numbers to be all complex numbers β such that there exists a polynomial with algebraic coefficients such that $g(\beta) = 0$. Prove the set of Hyper-Algebraic numbers are countable; you may use any polynomial of degree n has at most n complex roots.

Question 10 : More Chebychev: Prove the following by using Chebychev: for any integer N, there exists an even integer $2k$ such that there are at least N primes p with $p + 2k$ also prime. *$2k$ will unfortunately depend on N; if it didn't, we'd have solved the Twin Prime Problem, namely, there are infinitely many primes p such that $p+2$ is also prime.*

Question 11 : Consider a fair coin which is tossed N times. We record the results (heads, tails). It is very difficult to calculate the probability that the sequence of tosses has at least n consecutive heads somewhere in the sequence, $n \in \{0, 1, \dots, N\}$. What kind of bounds can you give for the probability that the sequence of N tosses has at least n heads? For example (no computers!), do you think there is at least a 50% chance of observing 5 consecutive heads in 100 tosses? If possible, let $n = n(N)$ (ie, n is a function of N, maybe $n = \sqrt{N}$ or $n = \frac{N}{20}$. Choose some nice function of N, and talk about some bounds.

Do your methods work for sub-sequences other than n heads, for example, $TTHTH$? I was asked this problem years ago by a molecular biology student, who was interested in the occurrences of sub-sequences in genetic material.