Statistical Theory of the Energy Levels of Complex Systems. II

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The distribution function of spacings S between nearest neighbors, in a long series of energy levels with average spacing D, is studied. The statistical properties of S are defined in terms of an ensemble of systems described in a previous paper. For large values of $t=(\pi S/2D)$, it is shown that the distribution of S can be deduced from the thermodynamical properties of a certain model. The model, which replaces the eigenvalue distribution by a continuous fluid, can be studied by the methods of classical electrostatics, potential theory, and thermodynamics. In this way the distribution function of spacings S is found to be asymptotically

$$Q(t)=At^{17/8}\exp[-\tfrac{1}{4}t^2-\tfrac{1}{2}t]$$

for large t. The numerical constant A can in principle not be determined from such a continuum model. Reasons are given for considering the remaining factors in the formula for $Q(t)$ to be reliable.

I. INTRODUCTION

In the first paper of the present series,[1] a general theory was developed with the purpose of describing the statistical behavior of energy levels in complex systems. In this paper the theory will be applied to one of the classical problems of energy-level analysis, the study of the theoretical distribution law of spacings between nearest-neighbor levels.

The theory of level spacings was begun by Wigner[2] with a famous conjecture—that in a long series of levels with average spacing D, the proportion of spacings which lie between S and $(S+dS)$ is given by

$$Q_W(t)dt, \quad t=(\pi S/2D). \tag{1}$$

$$Q_W(t)=(2t/\pi)\exp(-t^2/\pi). \tag{2}$$

This formula was supposed to apply to a series of levels having the same values of all identifiable quantum numbers such as angular momentum and parity. It is very well supported by experimental data,[3] and by numerical tests[4] with random matrices of high order. However, it is now known to be false. Mehta and Gaudin[5] have obtained an analytic expression for the correct distribution function $Q(t)$ and have computed $Q(t)$ numerically. They find that $Q_W(t)$ is not identical with $Q(t)$ but is astonishingly close to it, the difference $|Q_W-Q|$ being less than 0.0162 over the whole range of t. For practical purposes, Wigner's intuition has been abundantly justified.

The analytic expressions of Mehta and Gaudin provide in principle a complete solution to the level-spacing distribution problem. Unfortunately, their formula for $Q(t)$ is very complicated, being obtained as the Fredholm determinant of a certain integral equation in which t occurs as a parameter. The formula can be used for numerical computation when S is of the order of D (Gaudin[5] has computed $Q(t)$ for $t\leq 5$), and it provides a series expansion of $Q(t)$ in ascending powers of t which is useful for $t\lesssim 1$. However, it gives no precise information about the behavior of $Q(t)$ for large t. Further analytic study might well yield an asymptotic formula for $Q(t)$ valid in the limit $t\rightarrow\infty$, but this remains to be demonstrated.

The purpose of this paper is to attack the problem of the large level spacings, for which the Gaudin-Mehta analysis has not yet proved useful, by an entirely different approach. The behavior of $Q(t)$ for large t will be deduced from arguments of a mathematically non-rigorous kind, based upon thermodynamical considerations. The results are undoubtedly correct in their main features and fill the existing gap in our knowledge of the spacing distribution. If it should later turn out that a rigorous and more exact calculation of $Q(t)$ for large t can be extracted from the Mehta-Gaudin analysis, then the results of this paper would be interesting for a different reason. A comparison of the two calculations would then show what are the limits within which thermodynamical arguments may be trusted and beyond which such arguments may be misleading.

II. CONTINUUM MODEL

According to Sec. V of (I), the joint distribution function of the eigenvalues $[\exp(i\theta_j)]$, $j=1,\cdots,N$, of a random unitary $(N\times N)$ matrix is

$$Q_{N\beta}(\theta_1,\cdots,\theta_N)=C_{N\beta}\exp[-\beta W], \tag{3}$$

$$W=-\sum_{i<j}\ln|e^{i\theta_i}-e^{i\theta_j}|. \tag{4}$$

Here $C_{N\beta}$ is a normalization constant; β is a parameter which under normal circumstances takes the value 1, but may also be equal to 2 or 4 under certain conditions described in (I). Equation (3) is identical with the distribution function of N point charges on the unit

[1] Freeman J. Dyson, J. Math. Phys. **3**, 140 (1962). Quoted in what follows as (I).
[2] E. P. Wigner, Gatlinberg Conference on Neutron Physics, Oak Ridge National Laboratory Report ORNL 2309 (1957), p. 59.
[3] N. Rosenzweig and C. E. Porter, Phys. Rev. **120**, 1698 (1960). See, also, reference 4.
[4] C. E. Porter and N. Rosenzweig, Suomalaisen Tiedeakat. Toimituksia **A VI**, No. 44 (1960); see especially Figs. 19 and 20.
[5] M. L. Mehta, Nuclear Phys. **18**, 395 (1960); M. L. Mehta and M. Gaudin, *ibid.* **18**, 420 (1960); M. Gaudin, *ibid.* **25**, 447 (1961).

circle, repelling each other according to the laws of classical 2-dimensional electrostatics with the potential energy W, in thermodynamic equilibrium at temperature $T=\beta^{-1}$.

The main objective of this paper is to calculate the quantity $R_\beta(\alpha)$, defined as the probability that the angle $[-\alpha\leq\theta\leq\alpha]$ contains none of the θ_j. From Eq. (3) it follows that

$$R_\beta(\alpha)=[\Psi_{N\beta}(\alpha)/\Psi_{N\beta}(0)], \tag{5}$$

$$\Psi_{N\beta}(\alpha)=\int\cdots\int_\alpha^{2\pi-\alpha}\exp[-\beta W]d\theta_1\cdots d\theta_N. \tag{6}$$

Let now N become very large. Then $\Psi_{N\beta}(0)$ is the partition function of the Coulomb gas on the whole unit circle, while $\Psi_{N\beta}(\alpha)$ is the partition function of the same gas compressed into a circular arc of length $2(\pi-\alpha)$. In other words

$$R_\beta(\alpha)=\exp[\beta\{F_{N\beta}(0)-F_{N\beta}(\alpha)\}], \tag{7}$$

where $F_{N\beta}(\alpha)$ is the free energy of the Coulomb gas on the arc $2(\pi-\alpha)$.

We now make three assumptions for which no rigorous mathematical justification exists.

(i) There is a macroscopic density function $\sigma_\alpha(\theta)$, a continuous function of θ on the arc $[\alpha<\theta<2\pi-\alpha]$, such that $\sigma_\alpha(\theta)d\theta$ is the number of θ_j in the range $[\theta<\theta_j<\theta+d\theta]$.

(ii) For a given density function $\sigma_\alpha(\theta)$, the free energy of the gas is composed of two parts

$$F=V+F_1, \tag{8}$$

where V is the macroscopic Coulomb energy

$$V=-\tfrac{1}{2}\int\int_\alpha^{2\pi-\alpha}\sigma_\alpha(\theta)\sigma_\alpha(\varphi)\ln|e^{i\theta}-e^{i\varphi}|d\theta d\varphi, \tag{9}$$

and F_1 is a sum of contributions from the individual arcs $[\theta,\theta+d\theta]$ of the gas,

$$F_1=\int_\alpha^{2\pi-\alpha}\sigma_\alpha(\theta)f_\beta[\sigma_\alpha(\theta)]d\theta, \tag{10}$$

where $f_\beta(\sigma)$ is the free energy per particle in a Coulomb gas having *uniform* density σ on the whole unit circle.

(iii) The overwhelmingly dominant contribution to the integral (6) comes from configurations not deviating significantly from a particular macroscopic density-distribution $\sigma_\alpha(\theta)$, namely that function $\sigma_\alpha(\theta)$ which makes F given by Eqs. (8)–(10) a minimum subject to

$$\int_\alpha^{2\pi-\alpha}\sigma_\alpha(\theta)d\theta=N. \tag{11}$$

These assumptions (i)–(iii) can be summarized in the single statement that *for large N the Coulomb gas obeys the laws of classical thermodynamics.* The assumption (10) means that the free energy (apart from the macroscopic Coulomb energy) is an extensive property of the system, the free-energy density at any point being a function of the local temperature and density alone. To a physicist these assumptions are so hallowed by custom that they hardly require justification. Every application of thermodynamics to systems of strongly interacting atoms or molecules rests on assumptions of this kind. We make no effort here to explore more deeply the mathematics of the problem.

The "continuum model" of the Coulomb gas on the arc $[\alpha<\theta<2\pi-\alpha]$ is defined to be a classical compressible fluid of density $\sigma_\alpha(\theta)$ per unit angle, obeying the laws of classical thermodynamics. The total free energy $F_{N\beta}(\alpha)$ of the continuum model is defined to be the minimum value of F given by Eqs. (8)–(10), and the function $\sigma_\alpha(\theta)$ is determined by requiring that F be a minimum subject to Eq. (11).

It remains to specify the function $f_\beta(\sigma)$. Let

$$f_\beta(\sigma)=U_\beta(\sigma)-\beta^{-1}S_\beta(\sigma), \tag{12}$$

where U_β is the energy and S_β the entropy per particle in a uniform gas of

$$N'=2\pi\sigma \tag{13}$$

charges on the whole unit circle. According to Eq. (I, 163), the energy per particle is

$$U_\beta(\sigma)=-\tfrac{1}{2}\ln N'+U(\beta), \tag{14}$$

since we are now talking about the total Coulomb energy including the ground-state energy $[-\frac{1}{2}N'\ln N']$. The physical meaning of Eq. (14) is made clearer by remembering that

$$U_\beta(\sigma)=+\tfrac{1}{2}\ln\Lambda, \tag{15}$$

where Λ is a Debye length representing the size of the neutralizing charge cloud around each particle. Since the Debye length must vary inversely with N', the dependence of $U_\beta(\sigma)$ on σ can only have the simple form (14).

According to the calculations of Sec. (IX) of I, the entropy $S_\beta(\sigma)$ should be independent of N' for large N'. However, we here run into an interesting example of Gibbs' paradox.[6] The entropy has been calculated in I for a classical gas of N distinguishable particles. Gibbs' Paradox lies in the fact that entropy so defined is not an extensive quantity. To obtain an extensive quantity, one must subtract $[\ln N!]$ from the classical entropy, which is equivalent to treating the particles as undistinguishable. This means that in Eq. (12) one should use

$$S_\beta(\sigma)=\ln(N/N')+S(\beta), \tag{16}$$

with $S(\beta)$ given by Eq. (154) of I.

[6] E. Schrödinger, *Statistical Thermodynamics* (Cambridge University Press, New York, 1952), pp. 58–62.

Assembling Eqs. (8)–(10), (12), (14), (16), we find

$$\beta F = G_2 + G_1 + G_0, \tag{17}$$

$$G_2 = -\tfrac{1}{2}\beta(N/2\pi)^2 \int\int_\alpha^{2\pi-\alpha} \rho_\alpha(\theta)\rho_\alpha(\varphi) \times \ln|e^{i\theta}-e^{i\varphi}|\,d\theta d\varphi, \tag{18}$$

$$G_1 = (1-\tfrac{1}{2}\beta)(N/2\pi)\int_\alpha^{2\pi-\alpha} \rho_\alpha(\theta)\ln(\rho_\alpha(\theta))d\theta, \tag{19}$$

$$G_0 = \beta N[F(\beta) - \tfrac{1}{2}\ln N], \tag{20}$$

with

$$\rho_\alpha(\theta) = (2\pi/N)\sigma_\alpha(\theta), \tag{21}$$

$$\int_\alpha^{2\pi-\alpha} \rho_\alpha(\theta)d\theta = 2\pi, \tag{22}$$

and $F(\beta)$ given by Eq. (151) of I. The fact that the energy and entropy both contribute to G_1 a term in $(\rho\ln\rho)$ is due to the special form of the Coulomb potential. When $\beta=2$ (the case of the unitary ensemble) the term G_1 is absent and the model becomes particularly simple.

When $\alpha=0$, the equilibrium density is $\rho_0(\theta)=1$, which makes $G_2=G_1=0$. Therefore

$$\beta F_{N\beta}(0) = G_0 = \beta N[F(\beta)-\tfrac{1}{2}\ln N]. \tag{23}$$

Hence Eq. (7) gives

$$\ln R_\beta(\alpha) = -\min_\rho[G_2+G_1]. \tag{24}$$

The term G_0 is the only one in Eqs. (17)–(20) which depended on the detailed microstructure of the Coulomb gas, and it has disappeared from Eq. (24). Since our purpose in this paper is to compute $R_\beta(\alpha)$, we simply drop the constant G_0 and write

$$\beta F = G_2 + G_1, \tag{25}$$

both terms G_2 and G_1 being purely macroscopic in form.

The variational problem (24) is equivalent to the following set of equations.

$$\rho_\alpha(\theta) = A\exp[-\gamma V_\alpha(\theta)], \quad \alpha<\theta<2\pi-\alpha, \tag{26}$$

$$\gamma = [N\beta/\{\pi(2-\beta)\}], \tag{27}$$

$$V_\alpha(\theta) = -\int_\alpha^{2\pi-\alpha} \rho_\alpha(\varphi)\ln|e^{i\theta}-e^{i\varphi}|d\varphi. \tag{28}$$

This V_α is the electrostatic potential produced by the the charge-distribution ρ_α. Equation (26) is a "self-consistent field" type of equation, expressing the fact that the charge ρ_α is in thermal equilibrium in the potential V_α which it itself generates.

The only unexpected feature of these equations is that the "effective temperature" appearing in the exponent in Eq. (26) is not $T=\beta^{-1}$ but

$$T_e = \beta^{-1} - \tfrac{1}{2}. \tag{29}$$

When $\beta>2$, the effective temperature is negative, that is to say the charge has a statistical preference for regions of higher potential. The simple case of zero effective temperature occurs not at $\beta=\infty$ but at $\beta=2$. When $\beta=2$, we must replace Eq. (26) by

$$V_\alpha(\theta) = V_\alpha = \text{const}, \quad \alpha<\theta<2\pi-\alpha, \tag{30}$$

and the minimum problem reduces to a problem of classical electrostatics without any thermodynamics. In this case Eq. (24) gives

$$\ln R_2(\alpha) = (N^2/2\pi)V_\alpha. \tag{31}$$

III. SOLUTION FOR $\beta=2$

The continuum model for $\beta=2$ is defined by Eqs. (28) and (30). It consists of a charge-density $\rho_\alpha(\theta)$ distributed on a conducting wire which forms a circular arc of angle $2(\pi-\alpha)$. Since this is a standard problem of 2-dimensional potential theory, it can be immediately solved by the method of conformal mapping. Although the solution is well known, we reproduce the details of it here. The details will be needed in Sec. IV, when we go on to the more difficult case $\beta\neq 2$.

Let z be a complex variable representing points in the physical plane. The conducting wire consists of the curve

$$z = e^{i\theta}, \quad \alpha\leq\theta\leq 2\pi-\alpha, \tag{32}$$

lying in the z plane. The function

$$W_\alpha(z) = -\int_\alpha^{2\pi-\alpha} \rho_\alpha(\varphi)\ln(z-e^{i\varphi})d\varphi, \tag{33}$$

is analytic and many-valued in the whole z plane outside the wire. Its real part is one-valued, and by Eqs. (28), (30)

$$\text{Re}W_\alpha(z) = V_\alpha, \quad z=e^{i\theta}, \quad \alpha\leq\theta\leq 2\pi-\alpha. \tag{34}$$

By Eq. (22),

$$W_\alpha(z) \sim 2\pi\ln z \quad \text{as} \quad |z|\to\infty. \tag{35}$$

The potential is completely determined by the statement that $W_\alpha(z)$ is analytic and satisfies Eqs. (34) and (35).

The charge density is related to W_α by

$$\rho_\alpha^{\pm}(\theta) = \frac{1}{2\pi}[\lim_{z\to e^{i\theta}} |\partial W_\alpha/\partial z|]. \tag{36}$$

The limit $z\to e^{i\theta}$ may be taken from the outside of the unit circle, giving $\rho_\alpha^+(\theta)$, or from the inside, giving

$\rho_\alpha^-(\theta)$. The charge ρ_α^+ may be thought of as localized on the outer surface of the wire and ρ_α^- as localized on the inner surface. The total charge density is given by

$$\rho_\alpha(\theta)=\rho_\alpha^+(\theta)+\rho_\alpha^-(\theta), \tag{37}$$

$$\rho_\alpha^\pm(\theta)=\tfrac{1}{2}[\rho_\alpha(\theta)\pm 1]. \tag{38}$$

A convenient series of mappings is the following

$$\zeta=\frac{1-z}{1+z}, \quad z=\frac{1-\zeta}{1+\zeta}. \tag{39}$$

$$w=\frac{\zeta}{(\zeta^2+\delta^2)^{\frac{1}{2}}+\delta}, \quad \zeta=\frac{2w\delta}{1-w^2} \tag{40}$$

$$u=\frac{1-ww_0}{w-w_0}, \quad w=\frac{1+uw_0}{u+w_0}. \tag{41}$$

Equation (39) maps the z plane onto the ζ plane with cuts along the imaginary axis from $(\pm i\delta)$ to $(\pm i\infty)$, where

$$\delta=\tan\tfrac{1}{2}\alpha. \tag{42}$$

Equation (40) maps the ζ plane onto the inside of the unit circle in the w plane, the point $z=\infty$, $\zeta=-1$ mapping onto $w=w_0$, with

$$w_0=-\tan\epsilon, \quad \epsilon=\tfrac{1}{4}(\pi-\alpha). \tag{43}$$

The end-points $z=\exp(\pm i\alpha)$ of the wire in the z-plane map onto the points $w=\pm i$. Equation (41) maps the inside of the unit circle in the w-plane onto the outside of the unit circle in the u plane. The point $z=\infty$ maps onto $u=\infty$ with the proportionality factor

$$u\sim z\sec(\tfrac{1}{2}\alpha), \quad |z|\to\infty. \tag{44}$$

The solution of the potential problem is simply

$$W_\alpha(z)=2\pi\ln[u\cos(\tfrac{1}{2}\alpha)]. \tag{45}$$

This satisfies Eqs. (34), (35) with

$$V_\alpha=2\pi\ln\cos(\tfrac{1}{2}\alpha). \tag{46}$$

The corresponding charge densities $\rho_\alpha^\pm(\theta)$ are given by Eq. (38) with

$$\rho_\alpha(\theta)=\sin(\tfrac{1}{2}\theta)[\sin^2(\tfrac{1}{2}\theta)-\sin^2(\tfrac{1}{2}\alpha)]^{-\frac{1}{2}}. \tag{47}$$

These results give immediately the asymptotic form of the spacing distribution in a series of energy levels with $\beta=2$. Let D be the average level-spacing, and

$$t=(\pi S/2D) \tag{48}$$

large compared with unity. The proportion of level spacings of size between t and $(t+dt)$ is $Q(t)dt$, where

$$Q(t)=\tfrac{1}{2}\pi d^2P/dt^2, \tag{49}$$

and $P(t)$ is the probability that a randomly chosen interval of length $(2tD/\pi)$ will be free of energy levels. By Eq. (5) we have

$$P(t)=R_2(\alpha), \quad t=\tfrac{1}{2}\alpha N. \tag{50}$$

Equations (31) and (46) then give in the limit as $N\to\infty$,

$$P(t)=\exp[N^2\ln\cos(t/N)]=\exp[-\tfrac{1}{2}t^2], \tag{51}$$

$$Q(t)\sim(\pi/2)t^2\exp[-\tfrac{1}{2}t^2], \quad \beta=2. \tag{52}$$

The continuum model predicts that the spacing distribution for $t\gg 1$ has the form (52). Unfortunately it is impossible to estimate by continuum-model calculations what the inherent errors of the model are likely to be. Therefore the degree of accuracy with which Eq. (52) holds is unknown. Clearly the discreteness of charge would make the distribution (47) wrong for angles θ within a range of about N^{-2} from the end point $\theta=\alpha$. Taking an optimistic view, one may conjecture that *the free energies of the continuum model and of a real Coulomb gas with discrete charges differ by an amount which remains bounded as* $t\to\infty$. One could hardly expect, in view of the unavoidable effect of discreteness at the end-points, that "remains bounded" could be replaced by "tends to zero" in this statement. The consequence of this conjecture is that, in all asymptotic formulas such as Eq. (52), the exponential factor and the power of t standing outside the exponential are probably correct, but the numerical coefficient is not to be taken seriously.

It is of some interest to compare the formula (52) with the result one would deduce from simple arguments of the kind which Wigner[2] used in making his conjecture Eq. (2). To derive Eq. (2), Wigner assumed

$$Q_W(t)=At\exp(-Bt^2), \tag{53}$$

and determined the constants A and B from the conditions

$$\int_0^\infty Q(t)dt=1, \tag{54}$$

$$\int_0^\infty Q(t)tdt=(\pi/2), \tag{55}$$

which must hold exactly for the correct distribution function $Q(t)$. The motivation for Eq. (53) came from three arguments: (i) "the repulsion of levels" is known to make the distribution linear in t for small t, (ii) the level repulsion should make the distribution approximately Gaussian for large t, and (iii) the formula should be as simple as possible. The resulting Eq. (2) was applicable to a level series with $\beta=1$, the case which normally occurs in experiments.

In the case $\beta=2$, which applies when time-reversal invariance is abandoned, the level-repulsion will make

$Q(t)$ quadratic in t for small t. The "Wigner conjecture" for this case would be

$$Q_W^2(t)=At^2\exp(-Bt^2), \qquad (56)$$

with A and B still determined by Eqs. (54) and (55). Putting in the numerical values, Eq. (56) becomes

$$Q_W^2(t)=[256t^2/\pi^6]\exp[-16t^2/\pi^3]. \qquad (57)$$

Comparing Eqs. (52) and (57), one sees that the Wigner conjecture is incorrect, but that the conjectured exponential factor differs from the true one only by the ratio $[32/\pi^3]$. It is probable that, as in the case $\beta=1$, the Wigner conjecture (57) lies numerically very close to the true distribution function over the whole range of t. An additional check on this point comes from the known form of the exact distribution function at small t. In Paper III of this series, we shall prove that

$$Q(t)\sim(8/3\pi)t^2,\quad t\ll 1,\quad \beta=2. \qquad (58)$$

The Wigner conjecture (57) then differs from the true $Q(t)$ at small t by the ratio $[96/\pi^4]$. It is remarkable that an incorrect formula can come as near as this to the truth.

IV. SOLUTION FOR $\beta\neq 2$

When $\beta\neq 2$, the continuum model leads to the nonlinear equations (26), (28), and an analytic solution is not to be expected. The problem becomes tractable only after introducing some kind of perturbation-theory approximations. Since the objective is an asymptotic formula valid for large values of $t=\frac{1}{2}\alpha N$, the perturbation theory should if possible represent an expansion in inverse powers of (αN). Fortunately, the formulation of the problem by the minimum principle Eq. (24) makes such an expansion possible. According to Eqs. (18) and (19), G_2 is of order $(\alpha N)^2$ while G_1 is of order (αN). The "unperturbed system" can be taken to be given by G_2 alone, the addition of G_1 being the "perturbation."

The unperturbed system is, apart from the constant factor β, identical with the case $\beta=2$ considered in Sec. III. So the unperturbed free energy is given by Eqs. (25), (31), and (46) and is

$$\beta F_0=-\tfrac{1}{2}\beta N^2\ln\cos(\tfrac{1}{2}\alpha). \qquad (59)$$

The unperturbed charge density will now be denoted by $\bar\rho_\alpha(\theta)$ and is given by Eq. (47).

Since Eq. (24) is a variation principle for the free energy, the first-order perturbation of βF is merely the value which G_1 takes with the unperturbed charge-distribution, namely

$$\beta F_1=(1-\tfrac{1}{2}\beta)(N/2\pi)\int_\alpha^{2\pi-\alpha}\bar\rho_\alpha(\theta)\ln\bar\rho_\alpha(\theta)d\theta. \qquad (60)$$

To evaluate Eq. (60) it is convenient to transform the integral into the u plane. Since charge is invariant in a conformal mapping,

$$\int\bar\rho_\alpha(\theta)d\theta=\int|du|. \qquad (61)$$

Hence Eqs. (47) and (60) give

$$\beta F_1=(1-\tfrac{1}{2}\beta)(N/2\pi)\times\int\ln\left|\frac{(u-\tan\epsilon)(u-\cot\epsilon)}{[u-\exp(\frac{1}{2}i\alpha)][u-\exp(-\frac{1}{2}i\alpha)]}\right||du|, \qquad (62)$$

the integral being taken around the unit circle. Now

$$\int(\ln|u-a|)|du|=2\pi\max[\ln|a|,0], \qquad (63)$$

this being the potential at the position a of a uniformly charged circle of radius 1. Therefore Eq. (62) gives

$$\beta F_1=(1-\tfrac{1}{2}\beta)N\ln\cot\epsilon=(1-\tfrac{1}{2}\beta)N\times\ln[\sec\tfrac{1}{2}\alpha+\tan\tfrac{1}{2}\alpha]. \qquad (64)$$

In the limit $N\to\infty$, Eqs. (59) and (64) give

$$\beta F_0+\beta F_1=\tfrac{1}{4}\beta t^2+(1-\tfrac{1}{2}\beta)t, \qquad (65)$$

as the first two terms of the desired expansion of the free energy in powers of t^{-1}.

The next term in the expansion will require second-order perturbation theory. The calculation becomes necessarily more complicated, but much of the pain can be avoided by working in the w plane defined by Eqs. (39), (40). On the unit circle of the w plane,

$$\bar\rho_\alpha(\theta)=\sec\tfrac{1}{2}\alpha|\sec\psi|,\quad w=\exp(i\psi), \qquad (66)$$

$$\int\bar\rho_\alpha(\theta)d\theta=\int m(\psi)d\psi, \qquad (67)$$

$$m(\psi)=|du/dw|=\sin\tfrac{1}{2}\alpha[1+\cos\tfrac{1}{2}\alpha\cos\psi]^{-1}. \qquad (68)$$

The perturbed charge density is written in the form

$$\rho_\alpha(\theta)=\bar\rho_\alpha(\theta)+q(\psi)h(\psi), \qquad (69)$$

$$q(\psi)=|dw/dz|=[\sec^2\tfrac{1}{2}\alpha-\cos^2\psi]/[2\tan\tfrac{1}{2}\alpha|\cos\psi|], \qquad (70)$$

where $h(\psi)$ is the unknown perturbation and is supposed to be a small quantity. According to Eq. (40), each point $z=\exp(i\theta)$ is mapped onto two points $w=\exp(i\psi)$, $w=\exp[i(\pi-\psi)]$. The outside surface of the arc $[\alpha<\theta<2\pi-\alpha]$ in the z plane is mapped onto the left half of the unit circle $[\cos\psi<0]$ in the w plane, while the inside surface of the arc in the z plane is mapped onto the right half of the unit circle $[\cos\psi>0]$ in the w plane. Therefore Eq. (69) may be written

$$\rho_\alpha(\theta)=q(\psi)[m(\psi)+m(\pi-\psi)+h(\psi)], \qquad (71)$$

and $h(\psi)$ must be regarded as an even function of $(\cos\psi)$. Since the total charge is not changed by the perturbation,

$$\int_0^{2\pi} h(\psi)d\psi=0. \tag{72}$$

It is now necessary to express the free energy (G_2+G_1) in terms of w-plane integrals. For G_1, Eqs. (19), (70), and (71) give

$$G_1=\tfrac{1}{2}(1-\tfrac{1}{2}\beta)(N/2\pi)\int_0^{2\pi}[M(\psi)+h(\psi)] \times\ln\{q(\psi)[M(\psi)+h(\psi)]\}d\psi, \tag{73}$$

$$M(\psi)=m(\psi)+m(\pi-\psi) =2\sin\tfrac{1}{2}\alpha[1-\cos^2(\tfrac{1}{2}\alpha)\cos^2\psi]^{-1}. \tag{74}$$

Expanding Eq. (73) to second order in $h(\psi)$, we find

$$G_1=\tfrac{1}{2}(1-\tfrac{1}{2}\beta)(N/2\pi)\int_0^{2\pi}[(M\ln\bar{\rho}_\alpha)+(h\ln\bar{\rho}_\alpha) +\tfrac{1}{2}M^{-1}h^2]d\psi. \tag{75}$$

The term independent of h is the first-order free energy given by Eqs. (60), (64). Thus to second order in $h(\psi)$

$$G_1=\beta F_1+\tfrac{1}{2}(1-\tfrac{1}{2}\beta)(N/2\pi)\int_0^{2\pi}[h(\psi)\ln|\sec\psi| +\tfrac{1}{4}\csc\tfrac{1}{2}\alpha(1-\cos^2\tfrac{1}{2}\alpha\cos^2\psi)h^2(\psi)]d\psi. \tag{76}$$

The transformation of G_2 into the w plane can be made without any approximation. Equation (18) may be written

$$G_2=+\tfrac{1}{2}\beta(N/2\pi)^2\int_\alpha^{2\pi-\alpha}\rho_\alpha(\theta)\,\mathrm{Re}[W_\alpha(e^{i\theta})]d\theta, \tag{77}$$

where $W_\alpha(z)$ is given by Eq. (33). If $\rho_\alpha(\varphi)$ in Eq. (33) is taken to be the unperturbed charge density $\bar{\rho}_\alpha(\varphi)$, then $W_\alpha(z)$ has the value given by Eq. (45). However, $\rho_\alpha(\varphi)$ is now defined by Eq. (69), and therefore

$$W_\alpha(z)=2\pi\ln[u\cos(\tfrac{1}{2}\alpha)]+Y(z), \tag{78}$$

$$Y(z)=-\int_\alpha^{2\pi-\alpha}q(\psi)h(\psi)\ln(z-e^{i\varphi})d\varphi \tag{79}$$

$$=-\tfrac{1}{2}\int_0^{2\pi}h(\psi)\ln(z-e^{i\varphi})d\psi. \tag{80}$$

The factor $(\frac{1}{2})$ in Eq. (80) appears because the ψ integration corresponds to the arc $[\alpha<\varphi<2\pi-\alpha]$ taken twice. Now this function $Y(z)$ is analytic in z and tends to zero as $z\to\infty$ by virtue of Eq. (72). It is, therefore, analytic in w inside the unit circle. According to Eqs. (36), (70), and (79),

$$\frac{1}{2\pi}\lim_{w\to e^{i\psi}}[\partial Y/\partial w]=\frac{1}{2\pi q(\psi)}\lim_{z\to e^{i\varphi}}[\partial Y/\partial z]=\tfrac{1}{2}h(\psi), \tag{81}$$

the derivatives being taken in the radial direction. Therefore the function

$$-\int_0^{2\pi}h(\psi)\ln(w-e^{i\psi})d\psi \tag{82}$$

has the same normal derivative as $Y(z)$ at every point of the unit circle in the w plane, and can differ from $Y(z)$ only by a constant. By virtue of Eq. (72) this implies

$$\int_\alpha^{2\pi-\alpha}q(\psi)h(\psi)\,\mathrm{Re}[Y(e^{i\theta})]d\theta =-\tfrac{1}{2}\int\int_0^{2\pi}h(\psi)h(w)\ln|e^{i\psi}-e^{iw}|d\psi dw. \tag{83}$$

This expresses in a simple form in the w plane the part of G_2 which is quadratic in $h(\psi)$. The term linear in $h(\psi)$ vanishes since the unperturbed charge distribution $\bar{\rho}_\alpha(\theta)$ was chosen so as to make G_2 stationary. The term independent of $h(\psi)$ is just βF_0 given by Eq. (59). Therefore Eq. (77) reduces to

$$G_2=\beta F_0-\tfrac{1}{4}\beta(N/2\pi)^2\int\int_0^{2\pi}h(\psi)h(w) \times\ln|e^{i\psi}-e^{iw}|d\psi dw. \tag{84}$$

The total free energy to order $h^2(\psi)$ is

$$\beta F=\beta F_0+\beta F_1+\beta F_2, \tag{85}$$

where βF_2 is the sum of the terms involving $h(\psi)$ in Eqs. (76) and (84).

To determine $h(\psi)$, the quadratic form βF_2 must be minimized. It is convenient to expand $h(\psi)$ in a Fourier series

$$h(\psi)=\sum_{n=1}^{\infty}h_n\cos(2n\psi). \tag{86}$$

The constant term is zero by Eq. (72), and the odd terms are zero since $h(\psi)$ is even in $(\cos\psi)$. Substituting Eq. (86) into Eqs. (76) and (84), and taking the limit $N\to\infty$, we find

$$\beta F_2=(1-\tfrac{1}{2}\beta)[\tfrac{1}{4}\sum n^{-1}u_n+(32t)^{-1}\sum(u_n^2+u_nu_{n+1})] +(\beta/32)\sum n^{-1}u_n^2, \tag{87}$$

with

$$u_n=(-1)^nNh_n. \tag{88}$$

For large values of t, the u_n which makes Eq. (87) a minimum will be a slowly varying function of n. With negligible error we may replace the term $(u_n u_{n+1})$ by (u_n^2), and the minimization then becomes trivial. The result is

$$\beta F_2 = -\tfrac{1}{2}\beta^{-1}(1-\tfrac{1}{2}\beta)^2 \sum_{n=1}^{\infty} n^{-1}[1+(2n/\beta t) \times (1-\tfrac{1}{2}\beta)]^{-1}. \quad (89)$$

The series is convergent and gives for large t the asymptotic expression

$$\beta F_2 = -\tfrac{1}{2}\beta^{-1}(1-\tfrac{1}{2}\beta)^2\{\ln[\beta t/2(1-\tfrac{1}{2}\beta)]+\gamma\}, \quad (90)$$

where γ is Euler's constant.

As we observed in Sec. III, the continuum model cannot be expected to give the constant term in the free energy correctly as $t\rightarrow\infty$. The constant term in Eq. (90) is probably meaningless. Therefore we drop the constant term and obtain the final expression for the free energy

$$\beta F = \tfrac{1}{4}\beta t^2 + (1-\tfrac{1}{2}\beta)t - \tfrac{1}{2}\beta^{-1}(1-\tfrac{1}{2}\beta)^2 \ln t, \quad (91)$$

with an error which should be bounded as $t\rightarrow\infty$. The term in $(\ln t)$ is probably reliable. At the very worst, the second-order perturbation calculation, being based on a variation principle, shows that the error in Eq. (91) cannot be of greater order than $(\ln t)$.

Equations (5) and (24) give the result

$$P_\beta(t) \sim A t^{f(\beta)} \exp[-\tfrac{1}{4}\beta t^2 - (1-\tfrac{1}{2}\beta)t], \quad f(\beta) = (1-\tfrac{1}{2}\beta)^2/2\beta \quad (92)$$

for the probability that a randomly-chosen interval of length $(2tD/\pi)$ be empty of levels in a series with mean spacing D. According to Eq. (49), this gives for the distribution-function of large spacings

$$Q_\beta(t) \sim A t^{2+f(\beta)} \exp[-\tfrac{1}{4}\beta t^2 - (1-\tfrac{1}{2}\beta)t]. \quad (93)$$

In the case $\beta=2$, these expressions reduce to Eqs. (51) and (52). In applying the theory to nuclear level distributions, in which there is invariance under rotations and under time reversal, the case $\beta=1$ is relevant, and we find for $t\rightarrow\infty$

$$Q_1(t) \sim A t^{17/8} \exp[-\tfrac{1}{4}t^2 - \tfrac{1}{2}t]. \quad (94)$$

This is the case to which Wigner's conjecture Eq. (2) applied. We conclude that Wigner's conjecture underestimates the frequency of large spacings by a factor which tends to infinity as $t\rightarrow\infty$. Needless to say, the range of t for which Wigner's conjecture is seriously in error includes so few level-spacings that it is for practical purposes completely unimportant.

For systems with odd spin, invariance under time reversal, and no rotational symmetry, we showed in Sec. III of I that the case $\beta=4$ applies. The level-spacing distribution is then asymptotically

$$Q_4(t) = A t^{17/8} \exp[-t^2 + t]. \quad (95)$$

V. ADDITIONAL REMARKS

a. Accuracy of the Perturbation-Theory Calculation

The calculations of this paper are subject to two kinds of errors, (i) the inherent inaccuracy of the continuum model, and (ii) the inaccuracy of the treatment of the continuum model by perturbation theory in Sec. IV. We believe that the magnitude of type (i) errors in the free energy is bounded as $t\rightarrow\infty$; this belief is based only on physical intuition and cannot be checked by calculations within the continuum model. The magnitude of type (ii) errors can in principle be checked by pushing the perturbation calculations further.

We here examine the magnitude of the perturbation term $h(\psi)$ in Eq. (71) in comparison with the unperturbed term $M(\psi)$ given by Eq. (74). The explicit form of u_n obtained by minimizing Eq. (87) is

$$u_n = -2t[n+(\beta/(2-\beta))t]^{-1}. \quad (96)$$

With Eqs. (86) and (88), this gives

$$h(\psi) = -\alpha \sum_{1}^{\infty} \{n+[\beta/(2-\beta)]t\}^{-1}(-1)^n \cos 2n\psi. \quad (97)$$

The order of magnitude of $h(\psi)$ is

$$h(\psi) \sim \alpha \ln|t\cos\psi|, \quad |\cos\psi| < t^{-1}, \quad (98)$$

$$h(\psi) \sim \alpha |t\cos\psi|^{-2}, \quad t^{-1} < |\cos\psi| < t^{-\frac{1}{2}}, \quad (99)$$

$$h(\psi) \sim \alpha t^{-1}, \quad t^{-\frac{1}{2}} < |\cos\psi|. \quad (100)$$

The comparison term $M(\psi)$ is by Eq. (74) always at least of the order α. We have then

$$h(\psi) \ll M(\psi) \quad (101)$$

except in the range (98). So the perturbation theory is reliable except in the range of angles ψ within t^{-1} of $(\pm\tfrac{1}{2}\pi)$. The excluded range is mapped in the z plane onto the range

$$\alpha < |\theta| < \alpha + t^{-2}\alpha, \quad (102)$$

at the extreme tips of the arc $[\alpha<\theta<2\pi-\alpha]$. Equation (47) then shows that the total amount of the unperturbed charge in the excluded region is approximately one unit. Therefore the perturbation theory breaks down just in the space occupied by a single charge at the tips of the arc, where the continuum model is anyhow meaningless.

The foregoing argument indicates that the series (86) for $h(\psi)$ has a meaning up to frequencies n of the order of t, while the terms with $n>t$ are meaningless. The same conclusion holds for the series (89) giving the second-order contribution βF_2 to the free energy. The terms in βF_2 up to $n\sim t$ give the part of Eq. (90)

proportional to ($\ln t$), while the terms with $n>t$ affect only the constant in Eq. (90). We thus arrive at the following general conclusions concerning the accuracy of the calculations.

(a) Errors of type (ii) are of the order of unity in the free energy and do not affect the logarithmic term in Eq. (91).

(b) Errors of type (i) probably appear only where the perturbation theory breaks down, and therefore the perturbation theory makes physical sense as far as we have carried it.

(c) It would make no sense to carry the perturbation theory to higher orders, since any higher-order terms would be of the same order of magnitude as the type (i) errors.

b. Gaussian Model

The calculations of this paper were based on the circular ensembles defined in (I). The same thermodynamic methods could just as well have been applied directly to the Gaussian ensemble[7] which has been the starting-point for the other workers[2,4,5] in this field. In the Gaussian ensemble, the angles $[\theta_1,\cdots,\theta_N]$ are replaced by real numbers $[\epsilon_1,\cdots,\epsilon_N]$ free to vary from $(-\infty)$ to $(+\infty)$. The potential energy W is given by

$$W=-\sum_{i<j}\ln|\epsilon_i-\epsilon_j|+(4a^2)^{-1}\sum_j \epsilon_j^2, \qquad (103)$$

instead of by Eq. (4). The continuum model is then a classical charged fluid, confined to a straight conducting wire and attracted to a fixed point 0 on the wire by a harmonic potential.

The analysis of the Gaussian continuum model proceeds almost as easily as for the circular model. There is only one essential complication. The conducting wire cannot be allowed to be infinite, because the attractive potential would then bring in charge from large distances in indefinite amounts. Negative charge-density is allowed by classical electrostatics but not by the conditions of this model. The appropriate model is a conducting wire of finite length, the length being chosen so that the charge-density shall be positive everywhere on the wire but zero at the end-points. When the model with a gap is introduced, the length of the wire must be adjusted so that the condition of zero charge density at the ends is maintained. Once this is done, the calculations proceed as before, and the final results are identical with those we have obtained in Secs. III and IV.

[7] E. P. Wigner, Proc. 4th Can. Math. Congress, p. 174 (Toronto, 1959), has in fact used this method to determine the over-all eigenvalue distribution of the Gaussian ensemble. For the over-all distribution, in contrast to the distribution of level spacings, a zero-temperature approximation is sufficient. Wigner was therefore able to derive the "semi-circle law" for the eigenvalue distribution, using a purely electrostatic model without any thermodynamics.

c. Case of Negative t

The partition function $\Psi_{N\beta}$ given by Eq. (6) has a well-defined meaning when the angle α is replaced by $(-\alpha)$. The integration with respect to each variable θ_j is then to be taken from 0 to 2π, with the interval from $(-\alpha)$ to $(+\alpha)$ counted twice. The ratio $R_\beta(-\alpha)$ is the expectation value of 2^k, where k is the number of the θ_j lying in the range $|\theta|<\alpha$. The function $P_\beta(-t)$ is the expectation value of 2^k, where k is the number of energy levels, in a series with mean spacing D, which happen to lie in a randomly chosen interval of length $(2tD/\pi)$. The expectation values are to be taken from the usual ensemble at temperature β^{-1}.

At infinite temperature ($\beta=0$) the value of $P_\beta(-t)$ is

$$P_0(-t)=[1+(\alpha/\pi)]^N=\exp[2t/\pi]. \qquad (104)$$

At any temperature we have

$$P_\beta(-t)=\langle 2^k\rangle \geq 2^{\langle k\rangle}=\exp[2t\ln 2/\pi]. \qquad (105)$$

In fact $P_\beta(-t)$ is a decreasing function of β and always lies between the limits (104) and (105).

The behavior of $P_\beta(-t)$ for large t can be determined from a continuum model. Instead of having a gap from $\theta=-\alpha$ to $\theta=+\alpha$, the model is now a complete circular wire with a potential

$$U=-\beta^{-1}\ln 2 \qquad (106)$$

applied to the interval ($|\theta|<\alpha$). This adds a term

$$G_1'=-(N/2\pi)\ln 2\int_{-\alpha}^{\alpha}\rho_\alpha(\theta)d\theta \qquad (107)$$

to the free energy given by Eqs. (18), (19), the other integrals now all running from 0 to 2π.

We can calculate the free energy as before by perturbation theory, using G_2 alone for the unperturbed system. The calculation is much simpler than for positive t. The unperturbed charge density is $\bar{\rho}_\alpha(\theta)=1$, and the unperturbed free energy is zero. The first-order perturbation produced by Eq. (107) is

$$\beta F_1=-(2t/\pi)\ln 2. \qquad (108)$$

Second-order perturbation theory adds to this a contribution

$$\beta F_2=-\beta^{-1}[(\ln 2)/\pi]^2\ln t. \qquad (109)$$

The asymptotic behavior of $P_\beta(-t)$ at large t is thus

$$P_\beta(-t)\sim At^{g(\beta)}\exp[(2\ln 2/\pi)t],$$
$$g(\beta)=(\ln 2)^2/\pi^2\beta \qquad (110)$$

and the asymptotic behavior of $Q_\beta(-t)$ is the same.

These results for negative t are not of any practical importance. Their chief interest is that they impose necessary conditions which any exact analytic formula for $Q(t)$ must satisfy. In particular, even the elementary inequality (105) is not satisfied by Wigner's conjectured Eq. (2), and this provides the shortest proof that Wigner's conjecture cannot be exactly correct.

d. Magnitude of Fluctuations in Level Density

One further consequence emerges from Eq. (110). Let an interval of length (MD) be chosen at random in a long series of energy levels with average spacing D. Let k be the number of levels lying in the interval. Then Eq. (110) may be written

$$\langle 2^k \rangle = AM^{a(\beta)} 2^M. \tag{111}$$

This implies that the variable k is distributed about its mean value $\langle k \rangle = M$ with a mean-square fluctuation

$$\langle (k-M)^2 \rangle = [(2/\pi^2\beta) \ln M] + R, \tag{112}$$

the remainder term R being bounded for large M. Equation (112) shows that the fluctuations in k are enormously less than they would be for an uncorrelated series of levels, which would give

$$\langle (k-M)^2 \rangle = M. \tag{113}$$

The difference between Eqs. (112) and (113) is a measure of the power of the long-range level repulsion in suppressing large fluctuations of level density. For the case $\beta = 1$ which applies to observed level-series, a more precise result than Eq. (112) will be proved in Paper IV.