Statistical Theory of the Energy Levels of Complex Systems. III

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A systematic method is developed for calculating the n-level correlation-function $R_n(x_1,\cdots,x_n)$, defined as the probability for finding n levels at positions $(x_1,\cdots,x_n)$, regardless of the positions of other levels. It is supposed that the levels of a complex system are statistically equivalent to the eigenvalues of a random symmetric unitary matrix of order $N\gg n$, according to the general theory described in an earlier paper. The 2-level correlation-function is found to be

$$R_2(x_1,x_2)=1-\{s(r)\}^2-\left\{\int_r^\infty s(t)dt\right\}\{ds(r)/dr\},$$
$$s(r)=[\sin(\pi r)/\pi r],\quad r=|x_1-x_2|,$$

the scale of energy being chosen so that the mean level-spacing is unity. It is shown how this result could in principle be used in order to determine the proportions of levels in two uncorrelated and superimposed series. An analytic expression for the distribution of nearest-neighbor level-spacings, discovered by Gaudin and Mehta, is rederived, and a similar expression is found for the distribution of spacings between next-nearest neighbors. An unexplained identity relates the nearest and next-nearest neighbor spacing distributions of a system invariant under time-reversal to the level-spacing distribution of a system without time-reversal invariance.

INTRODUCTION

THIS paper will be concerned with the study of the statistical properties of N points $[\exp(i\theta_j)]$, $j=1,\cdots,N$, distributed around the unit circle with the probability distribution function

$$P_N(\theta_1,\cdots,\theta_N)=C_N\prod_{i<j}|e^{i\theta_i}-e^{i\theta_j}|. \quad (1)$$

In paper I of the present series[1] it was shown that the distribution (1) holds for the eigenvalues of a symmetric unitary $(N\times N)$ matrix chosen at random o t of a certain ensemble, called the orthogonal ensemble. It was suggested that the series of angles $[\theta_1,\cdots,\theta_N]$ derived from this particular ensemble should provide a good model for the statistical behavior of the energy levels of a sufficiently complicated system. According to Eq. (I, 130), the probability distribution (1) is correctly normalized if the constant C_N has the value

$$C_N=[2^{2N}\pi^N\Gamma(1+\tfrac{1}{2}N)]^{-1}. \quad (2)$$

The main objective of the analysis is to calculate the *n-level correlation function*

$$R_n(\theta_1,\cdots,\theta_n)=[N!/(N-n)!]\int\cdots\int_0^{2\pi}P_N(\theta_1,\cdots,\theta_N)d\theta_{n+1}d\theta_{n+2}\cdots d\theta_N, \quad (3)$$

which measures the probability of finding a level (regardless of labeling) in each of the small intervals $[\theta_1,\ \theta_1+d\theta_1]$, $\cdots$, $[\theta_n,\theta_n+d\theta_n]$, the positions of the remaining $(N-n)$ levels being unobserved. In particular

$$R_1(\theta)=(N/2\pi) \quad (4)$$

is the constant over-all level density. Each function R_n for $n>1$ contains terms of various kinds, describing the grouping of n levels into various subgroups or clusters. For practical purposes it is convenient to work with the *n-level cluster function* defined by

$$T_n(\theta_1,\cdots,\theta_n)=\sum_G(-1)^{n-m}(m-1)!\times\prod_{j=1}^m R_{G_j}[\theta_k \text{ with } k \text{ in } G_j]. \quad (5)$$

Here G stands for any division of the indices $[1, 2, \cdots, n]$ into subgroups $[G_1, G_2, \cdots, G_m]$. Eqation (5) is a finite sum of products of R-functions, the first term in the sum being $[(-1)^{n-1}R_n(\theta_1,\cdots,\theta_n)]$, and the last being the constant

$$(n-1)!(N/2\pi)^n. \quad (6)$$

The inverse of Eq. (5) is

$$R_n(\theta_1,\cdots,\theta_n)=\sum_G(-1)^{n-m}\prod_{j=1}^m T_{G_j}[\theta_k \text{ with } k \text{ in } G_j]. \quad (7)$$

Thus each set of functions R_n and T_n is easily determined in terms of the other. The advantage of the cluster functions is that they have the property of vanishing when any one (or several) of the separations $|\theta_i-\theta_j|$ becomes large in comparison with the mean level spacing $(2\pi/N)$. The function T_n describes the correlation properties of a single cluster of n levels, isolated from the more trivial effects of lower-order correlations.

Of special interest for comparison with experiment are those features of the statistical model which tend to definite limits as $N\to\infty$. The cluster functions are convenient also from this point of view. In the limit $N\to\infty$, the angles θ_j must be replaced by real numbers

$$x_j=(N/2\pi)\theta_j, \quad (8)$$

[1] F. J. Dyson, J. Math. Phys. 3, 140 (1962); 3, 157 (1962), these two papers will be quoted as I and II.

each x_j being free to vary from $(-\infty)$ to $(+\infty)$, and the index j running from $(-\infty)$ to $(+\infty)$ also. The x_j then form a statistical model for an infinite series of energy-levels with mean spacing $D=1$. The cluster functions,

$$Y_n(x_1,\cdots,x_n)=\lim_{N\to\infty}(N/2\pi)^{-n}T_n(\theta_1,\cdots,\theta_n), \quad (9)$$

are well defined and finite everywhere. In particular

$$Y_1(x)=1, \quad (10)$$

while

$$Y_2(x_1,x_2)=Y_2(|x_1-x_2|) \quad (11)$$

defines the shape of the neutralizing charge-cloud induced by each particle around itself, when the model is interpreted as a classical Coulomb gas [see Sec. VI of I].

The cluster functions satisfy the identity

$$\int_{-\infty}^{\infty}Y_n(x_1,\cdots,x_n)dx_n=(n-1)Y_{n-1}(x_1,\cdots,x_{n-1}), \quad (12)$$

for $n>1$. This means that each Y_n is an integrable function of the $(n-1)$ variables $(x_1-x_n, x_2-x_n, \cdots, x_{n-1}-x_n)$, and has a Fourier transform

$$y_n(k_1,\cdots,k_{n-1})=\int\cdots\int_{-\infty}^{\infty}Y_n(x_1,\cdots,x_n)\exp\{2\pi i[k_1(x_1-x_n)+\cdots+k_{n-1}(x_{n-1}-x_n)]\}dx_1\cdots dx_{n-1}. \quad (13)$$

The limiting forms of the correlation functions R_n are not integrable, and their Fourier transforms involve products of δ functions.

Many important properties of the level-distribution depend only on the *two-level form-factor* defined by

$$b(k)=y_2(k)=\int_{-\infty}^{\infty}Y_2(r)e^{2\pi ikr}dr. \quad (14)$$

The normalization is chosen to make $b(0)=1$, by Eqs. (10) and (12).

In this paper it will be shown that all the cluster-functions Y_n are in principle calculable. The Y_n will be exhibited as coefficients in the expansion of a certain determinant. However, the elementary algebra that is required for the extraction of the higher Y_n is very tedious. Explicit evaluations will be made only for the two-level functions $Y_2(r)$ and $b(k)$.

The method of calculation is essentially copied from the work of Gaudin and Mehta,[2] who first discovered how to deal with integrals of the form of Eq. (3). All the serious difficulties are overcome by the Gaudin-Mehta method. The analysis differs from th t of Gaudin-Mehta in two respects. (i) We deal with distributions around a circle, while Gaudin and Mehta used a Gaussian distribution on a straight line. (ii) We are interested in a precise evaluation of the two-level correlation function, while they considered only the more difficult problem of the distribution of level-spacings.

II. GAUDIN-MEHTA METHOD ON A CIRCLE

To avoid minor complications, let $N=2m$ be even. Let $U(\theta)$ and $V(\theta)$ be any two functions defined on the unit circle $(-\pi\leq\theta\leq\pi)$. Consider the quantity H, defined as the expectation-value

$$H=\langle[\prod\nolimits_{\text{alt}}U(\theta_j)][\prod\nolimits'_{\text{alt}}V(\theta_j)]\rangle, \quad (15)$$

taken with respect to the probability-distribution (1). Here $\prod_{\text{alt}}$ means a product taken over a set of m alternate points θ_j as they lie on the unit circle, and $\prod'_{\text{alt}}$ means a product over the remaining m points θ_j. The alternating series may start anywhere on the circle, no special end-point being singled out.

By Eqs. (1) and (2), an explicit formula for H is

$$H=\frac{(2m)!}{2^{4m+1}\pi^m m!}\int_{-\pi}^{\pi}d\theta_1\int_{\theta_1}^{\pi}d\theta_2\cdots\int_{\theta_{2m-1}}^{\pi}d\theta_{2m}\times\prod_{i<j}[2\sin\tfrac{1}{2}(\theta_j-\theta_i)]\{\prod_{k=1}^{m}U(\theta_{2k-1})V(\theta_{2k})+\prod_{k=1}^{m}V(\theta_{2k-1})U(\theta_{2k})\}. \quad (16)$$

This may be transformed into

$$H=\frac{(2m)!}{2^{5m}\pi^m(m!)^2}\int\cdots\int_{-\pi}^{\pi}d\theta_1\cdots d\theta_{2m}\times\prod_{i<j}[2\sin\tfrac{1}{2}(\theta_j-\theta_i)]\prod_{k=1}^{m}\{V(\theta_{2k-1})U(\theta_{2k})\times\epsilon(\theta_{2k}-\theta_{2k-1})\}, \quad (17)$$

where $\epsilon(x)=(x/|x|)$. To deduce Eq. (17) from Eq. (16), let J be the integral on the right of Eq. (17). If any two of the θ_{2j} lie consecutively on the circle, the integrand of J is odd under interchange of these two variables, and that part of J vanishes. Similarly the part of J with any two of the θ_{2j-1} lying consecutively vanishes. The surviving part of J has the θ_{2j} and the θ_{2j-1} forming two alternate series, but not necessarily arranged in order as they appear in Eq. (16). Suppose that the θ_{2j} in J appear in a permutation P of the natural order, while the θ_{2j-1} appear in a permutation Q. The entire integral J then reduces to the integral

[2] M. L. Mehta, Nuclear Phys. 18, 395 (1960); M. L. Mehta and M. Gaudin, Nuclear Phys. 18, 420 (1960); M. Gaudin, Nuclear Phys. 25, 447 (1960).

appearing in Eq. (16), multiplied by the constant

$$C=\sum_{P,Q}\epsilon_P\epsilon_Q\prod_{k=1}^{m}\epsilon(2P_k-2Q_k+1). \tag{18}$$

But C is just $(m!)$ multiplied by an $(m\times m)$ determinant whose element number (i,j) is $(+1)$ when $i\geq j$, (-1) when $i<j$. The value of this determinant is 2^{m-1}, hence

$$C=2^{m-1}m!, \tag{19}$$

and Eq. (17) is proved.

The next step is to write

$$\prod_{i<j}[2\sin\tfrac{1}{2}(\theta_j-\theta_i)]=i^{-m}\det|\exp(ip\theta_j)|, \tag{20}$$

where the determinant is $(2m\times 2m)$, the column index j taking the values $(1,\cdots,2m)$ while the row-index p takes the values

$$p=-m+\tfrac{1}{2},\quad -m+\tfrac{3}{2},\quad \cdots,\quad m-\tfrac{1}{2}. \tag{21}$$

Squaring Eq. (17), using Eq. (20) and expanding the determinants, we obtain

$$H^2=\frac{[(2m)!]^2(-1)^m}{2^{12m}\pi^{2m}(m!)^4}\sum_{P,Q}\epsilon_P\epsilon_Q \times\prod_{k=1}^{m}\{g(P_{2k},P_{2k-1})g(Q_{2k},Q_{2k-1})\}, \tag{22}$$

with

$$g(p,q)=\int\int_{-\pi}^{\pi}d\theta d\varphi U(\theta)V(\varphi)\epsilon(\theta-\varphi) \times[\exp(ip\theta+iq\varphi)-\exp(iq\theta+ip\varphi)]. \tag{23}$$

P_j and Q_j are any two permutations taking values in the range (21) for $j=1,\cdots,2m$.

Let (p,q) be indices in the range (21). We say that (p,q) are "partners in P" if for some k we have $p=P_{2k}$, $q=P_{2k-1}$ or $p=P_{2k-1}$, $q=P_{2k}$. Similarly we define partners in Q. We can then construct a permutation R on $(-m+\frac{1}{2},\cdots,m-\frac{1}{2})$ such that $R_p=q$ if and only if (p,q) are partners in P or in Q. Such a permutation R must consist of a number L of cycles each of even order. When P and Q are given, the composition of the cycles in R is fixed, and only the sense of those cycles which have order greater than 2 is undetermined. Therefore, the number of distinct R associated with a given P and Q is

$$2^{L-h}, \tag{24}$$

where h is the number of cycles in R of order 2. This h is also the number of pairs (p,q) which are partners in both P and Q.

Conversely, if a permutation R containing only even cycles is given, this determines the pairing of partners in P and in Q with precisely the same degree of ambiguity 2^{L-h}. Given the pairings, the complete specification of P can be made in $2^m(m!)$ ways, and similarly for Q. Therefore every R can arise from

$$2^{L-h+2m}(m!)^2 \tag{25}$$

distinct pairs (P,Q).

The parity of the permutation R is

$$\epsilon_R=(-1)^L. \tag{26}$$

The combined parity $\epsilon_P\epsilon_Q$ is the parity of the permutation

$$S=\begin{bmatrix}P_1P_2\cdots P_{2m}\\ Q_1Q_2\cdots Q_{2m}\end{bmatrix}.$$

Now ϵ_S is unchanged if we interchange the pairs (Q_{2j-1},Q_{2j}) in blocks so as to make $P_1=Q_1$, $P_3=Q_3$, etc. The resulting permutation S is

$$\begin{bmatrix}P_2P_4\cdots P_{2m}\\ Q_2Q_4\cdots Q_{2m}\end{bmatrix}, \tag{27}$$

and this is obtained from R by taking just half the indices in each cycle. The parity of S is therefore

$$\epsilon_P\epsilon_Q=\epsilon_S=(-1)^\lambda, \tag{28}$$

and λ is the number of cycles in R whose length is divisible by 4. Since the sum of lengths of all the cycles in R is $2m$, we have

$$L-\lambda\equiv m\ (\text{mod. }2). \tag{29}$$

Equations (26), (28), and (29) give

$$\epsilon_P\epsilon_Q=(-1)^m\epsilon_R. \tag{30}$$

The sum (22) is now expressed in terms of the permutations R alone. Combining Eq. (22) with Eqs. (24), (25), (30), we find

$$H^2=\left[\frac{(2m)!}{2^{5m}\pi^m m!}\right]^2(-1)^m\sum_R\epsilon_R\prod_{p=-m+\frac{1}{2}}^{m-\frac{1}{2}}g(p,R_p). \tag{31}$$

The factor $(-1)^m$ reappears in Eq. (31) because we used the relation

$$g(p,q)=-g(q,p) \tag{32}$$

which follows from Eq. (23). The sum in Eq. (31) is over permutations R consisting of even cycles only. However, by virtue of Eq. (32), any permutation including an odd cycle would cancel in Eq. (31) against the same permutation with the odd cycle taken in the opposite sense. Therefore the sum over R may be extended to all permutations, and H^2 reduces to a determinant. The factor $(-1)^m$ may be again absorbed by changing $(q\rightarrow -q)$ in $g(p,q)$, which is equivalent to reversing the order of the $(2m)$ columns of $\det|g(p,q)|$. The constant factor in Eq. (31) may be absorbed by multiplying the pth row of the determinant by $[p/8\pi i]$.

In this way Eq. (31) reduces to

$$H^2=\det|f_{pq}|, \tag{33}$$

with

$$f_{pq}=\frac{p}{8\pi i}\int\int_{-\pi}^{\pi}d\theta d\varphi U(\theta)V(\varphi)\epsilon(\theta-\varphi)\times[\exp(ip\theta-iq\varphi)-\exp(ip\varphi-iq\theta)], \tag{34}$$

both p and q taking the values $[-m+\frac{1}{2},\cdots,m-\frac{1}{2}]$.

The result (33) corresponds to Eq. (10) in Mehta's paper.[3] Mehta does not make any use of his Eq. (10), and instead concentrates his attention on a determinantal expression for the first power of the integral corresponding to our H, namely his Eq. (14). The analogous formula for the first power of H is

$$H=\det|F_{pq}|\,;\ p,q=\tfrac{1}{2},\tfrac{3}{2},\cdots,m-\tfrac{1}{2}, \tag{35}$$

with

$$F_{pq}=\frac{p}{4\pi}\int\int_{-\pi}^{\pi}d\theta d\varphi U(\theta)V(\varphi)\epsilon(\theta-\varphi)\times[\cos p\varphi\,\sin q\theta-\cos p\theta\,\sin q\varphi]. \tag{36}$$

Equation (35) follows immediately from Eq. (33), provided that

$$U(\theta)V(\varphi)=U(-\theta)V(-\varphi), \tag{37}$$

which means in practice that U and V must either be both even functions or both odd functions on $[-\pi,+\pi]$.

Superficially, Eq. (35) appears simpler and more elegant than Eq. (33). However, the restriction (37) is highly inconvenient and makes it difficult to obtain directly from Eq. (35) any information about the cluster functions T_n. For our purposes Eq. (33), which holds without restriction on the functions U and V, is much more useful.

Another advantage of Eq. (33) is that it is independent of the arbitrary choice of the end points $[-\pi,\pi]$ on the circle. Since p and q are both half-odd-integers, one may write

$$f_{pq}=\frac{p}{8\pi i}\int\int d\theta d\varphi U(\theta)V(\varphi)\times[\exp(ip\theta-iq\varphi)-\exp(ip\varphi-iq\theta)], \tag{38}$$

the range of integration being limited only by

$$0<\theta-\varphi<2\pi. \tag{39}$$

A similar independence of the end point does not hold for Eq. (36).

When $U(\theta)=V(\theta)=1$, Eq. (34) gives

$$f_{pq}=\delta_{pq} \tag{40}$$

and so $H^2=1$, as it should be according to Eq. (15). This is a confirmation by direct calculation that the normalization of probability given by Eq. (2) is correct. In paper I this normalization was deduced independently, by a group-theoretical argument.

[3] The first paper in reference 2.

III. TWO-LEVEL CORRELATION FUNCTIONS

Write $U(\theta)=V(\theta)=1+A(\theta)$ in Eq. (15), where $A(\theta)$ is still an arbitrary function. Then

$$H=\langle\prod[1+A(\theta_j)]\rangle, \tag{41}$$

the product now extending over all the $(2m)$ levels θ_j. Equations (33) and (34) now become

$$H^2=\det|\delta_{pq}+r_{pq}|, \tag{42}$$

$$r_{pq}=\frac{1}{2\pi}\left(1+\frac{p}{q}\right)\int_{-\pi}^{\pi}A(\theta)\exp[i(p-q)\theta]d\theta+\left(\frac{p}{4\pi i}\right)\int\int_{-\pi}^{\pi}A(\theta)A(\varphi)\epsilon(\theta-\varphi)\times\exp[i(p\theta-q\varphi)]d\theta d\varphi. \tag{43}$$

All the cluster-functions $T_n(\theta_1,\cdots,\theta_n)$ can, in principle, be determined by expanding the two sides of the identity (42) in powers of $A(\theta)$. According to Eqs. (3), (41), and (7),

$$H=\sum_{n=0}^{\infty}\frac{1}{n!}\int\cdots\int_0^{2\pi}R_n(\theta_1,\cdots,\theta_n)\times A(\theta_1)\cdots A(\theta_n)d\theta_1\cdots d\theta_n \tag{44}$$

$$=\exp\left\{\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n!}\int\cdots\int_0^{2\pi}T_n(\theta_1,\cdots,\theta_n)\times A(\theta_1)\cdots A(\theta_n)d\theta_1\cdots d\theta_n\right\}. \tag{45}$$

The determinant for H^2 can be expanded along its leading diagonal. The result is a series beginning with the terms

$$H^2=1+(2m/\pi)\int A(\theta)d\theta+\frac{m(2m-1)}{\pi^2}\left[\int A(\theta)d\theta\right]^2+\sum_p\left(\frac{p}{4\pi i}\right)\int\int A(\theta)A(\varphi)\epsilon(\theta-\varphi)\times\exp[ip(\theta-\varphi)]d\theta d\varphi-\frac{1}{8\pi^2}\sum_{p\neq q}\left(1+\frac{p}{q}\right)\left(1+\frac{q}{p}\right)\times\int\int A(\theta)A(\varphi)\exp[i(p-q)(\theta-\varphi)]d\theta d\varphi, \tag{46}$$

the remaining terms being of order A^3 and higher. Since the function $A(\theta)$ is arbitrary (and this is here essential), each T_n can be picked out as the coefficient

of $[A(\theta_1)\cdots A(\theta_n)]$ in the logarithm of the series (46).

In the case $n=2$, this procedure gives

$$T_2(\theta,\varphi)=-\sum_p (p/4\pi i)\epsilon(\theta-\varphi)\exp[ip(\theta-\varphi)]$$
$$+\frac{1}{8\pi^2}\sum_{p,q}\left(2+\frac{p}{q}+\frac{q}{p}\right)\exp[i(p-q)(\theta-\varphi)]$$
$$=+\tfrac{1}{2}\epsilon(\theta-\varphi)Ds_N(\theta-\varphi)-\{Is_N(\theta-\varphi)\}$$
$$\times\{Ds_N(\theta-\varphi)\}+\{s_N(\theta-\varphi)\}^2, \quad (47)$$

where we have written

$$s_N(\alpha)=\frac{1}{2\pi}\sum_p e^{ip\alpha}=\frac{\sin(m\alpha)}{2\pi\sin(\frac{1}{2}\alpha)}, \quad (48)$$

$$Df(\alpha)=(\partial/\partial\alpha)f(\alpha), \quad (49)$$

$$If(\alpha)=\int_0^\alpha f(\alpha')d\alpha'. \quad (50)$$

In the limit $N\to\infty$, Eq. (47) becomes

$$Y_2(x_1,x_2)=\{\tfrac{1}{2}-Is(r)\}\{Ds(r)\}+\{s(r)\}^2, \quad (51)$$
$$r=|x_1-x_2|,$$

$$s(r)=\lim_{N\to\infty}\{(2\pi/N)s_N(2\pi r/N)\}=\frac{\sin(\pi r)}{\pi r}. \quad (52)$$

Since

$$\int_0^\infty s(r)dr=\tfrac{1}{2}, \quad (53)$$

Eq. (51) is equivalent to the formula for R_2 stated in the abstract.

The behavior of Y_2 for small and large values of r is given by

$$Y_2(r)=1-\frac{1}{6}\pi^2r+\frac{1}{60}\pi^4r^3-\frac{1}{135}\pi^4r^4+\cdots, \quad (54)$$

$$Y_2(r)=\frac{1}{\pi^2r^2}-\frac{1+\cos^2(\pi r)}{\pi^4r^4}+\cdots. \quad (55)$$

The Fourier transform of Y_2 gives the two-level form factor according to Eq. (14),

$$b(k)=1-2|k|+|k|\ln(1+2|k|), \quad (|k|\le 1),$$
$$=-1+|k|\ln[(2|k|+1)/(2|k|-1)], \quad (|k|\ge 1). \quad (56)$$

This has the behavior

$$b(k)=1-2|k|+2k^2+\cdots \quad (57)$$

$$b(k)=\frac{1}{12k^2}+\frac{1}{80k^4}+\cdots \quad (58)$$

for small and large k. At the points ($k=\pm 1$) where the analytic form of $b(k)$ changes, not only $b(k)$ but also its first two derivatives are continuous. There is a discontinuity only in the third derivative. This is connected with the fact that the oscillating term in $Y_2(r)$ according to Eq. (55) is of order r^{-4} for large r.

The oscillating term in Eq. (55) is of considerable interest, as it indicates the presence of an incipient crystal-lattice structure or long-range order in a series of eigenvalues. Even at large separations, two eigenvalues feel the natural periodicity of the lattice, and have a slight preference for separations which are an integer multiple of the mean level spacing. Unfortunately, the r^{-4} dependence of this effect makes it unobservable in practice. To see the second maximum (at $r=1$) of the oscillatory term standing out from statistical fluctuations, one would need a well-observed series of more than 10 000 levels.

The Gaudin-Mehta method gives information not only about the total eigenvalue distribution but also about the separate distributions of odd-numbered or even-numbered levels. For example, one may take in Eq. (15) $U(\theta)=1$, $V(\theta)=1+A(\theta)$. Then

$$H=\langle\textstyle\prod_{\text{alt}}[1+A(\theta_j)]\rangle, \quad (59)$$

the product extending over m levels lying alternately on the unit circle. Equation (42) now holds with

$$r_{pq}=\frac{1}{4\pi}\left(1+\frac{p}{q}\right)\int_{-\pi}^{\pi}A(\theta)\exp[i(p-q)\theta]d\theta. \quad (60)$$

The expression of H involves correlation functions of the alternate eigenvalue series. The analysis proceeds as before, only the term in $\frac{1}{2}\epsilon(\theta-\varphi)$ is now missing from Eq. (47). The results are the following.

In an infinite eigenvalue series with mean spacing $D=1$, let

$$\tfrac{1}{2}[1-Y_2^e(x_1,x_2)]dx_1dx_2 \quad (61)$$

be the probability for finding two eigenvalues in the intervals $[x_1, x_1+dx_1]$, $[x_2, x_2+dx_2]$, both belonging to the same alternate series. Then

$$Y_2^e(x_1,x_2)=\{s(r)\}^2-\{Is(r)\}\{Ds(r)\}, \quad (62)$$

with the same notations as in Eq. (51). For large and small r we find

$$Y_2^e(r)=1-\frac{1}{135}\pi^4r^4+\cdots, \quad (63)$$

$$Y_2^e(r)=-\frac{\cos(\pi r)}{2r}+\frac{1+\frac{1}{2}\pi\sin(\pi r)}{\pi^2r^2}-\frac{1+\cos^2(\pi r)}{\pi^4r^4}$$
$$+\cdots. \quad (64)$$

The corresponding two-level form factor is

$$b^e(k)=2-2|k|+|k|\ln(|2|k|-1|), \quad (|k|<1),$$
$$=0, \quad (|k|>1). \quad (65)$$

The long-range order of the eigenvalue series appears much more strongly in Eq. (64) than in Eq. (55), and it shows clearest of all in the singularity of the Fourier transform $b^e(k)$ at $k=\frac{1}{2}$. This behavior of the alternate eigenvalues proves that the long-range crystalline structure of the level-series is real. In a one-dimensional gas, the operation of merely picking out alternate atoms for examination could not create long-range order if long-range order had not been present to start with.

In an observed series of levels of practical length, say a few hundred levels, the first few oscillations of Eq. (64) should be distinguishable. But for this test of the theory to be meaningful, it is necessary to be sure that all the observed levels belong to a single series and that none have been missed.

A very intriguing possibility suggests itself in the situation, very frequent in practice, in which an observed set of levels is a superposition of two uncorrelated series. This situation arises, for example, when slow neutrons are captured by an odd-A nucleus into levels with two possible spin values. In general it is difficult to say which levels belong to which series, and even the proportion of levels in the two series is a matter of conjecture.[4]

Let a level-series be a mixture of two uncorrelated series 1 and 2, in the proportions f, $(1-f)$. The 2-level cluster function $\bar{Y}_2$ of the combined series is then

$$\bar{Y}_2(r)=f^2Y_2(fr)+(1-f)^2Y_2[(1-f)r], \tag{66}$$

and the form factor is

$$\bar{b}(k)=fb(k/f)+(1-f)b[k/(1-f)]. \tag{67}$$

The function $\bar{b}(k)$ can be measured by Fourier analysis of the observed 2-level correlations. In principle, if the series were long enough, one could find the discontinuities in the third derivative of $\bar{b}(k)$ and so determine f directly. In practice this will not be possible, because the function $b(k)$ is too smooth. The stronger discontinuity of $b^e(k)$ is also of no help, since there is no way to pick out alternate levels from two superimposed series. Practical methods for determining f in such cases will be discussed in paper IV.

IV. REGULARITY OF EIGENVALUES AROUND THE CIRCLE

As a simple application of the theory of Sec. III, we calculate the mean-square deviation of the eigenvalues $[\exp(i\theta_j)]$ from a regular arrangement of N points on the unit circle. This mean-square deviation is

$$\Delta^2=\left\langle \frac{1}{N}\min_{\alpha}\sum_{j=1}^{N}\left|\exp(i\theta_j)-\exp\left(\frac{2\pi ij}{N}+i\alpha\right)\right|^2\right\rangle. \tag{68}$$

The minimization over the angle α is elementary, and gives

$$\Delta^2=1-(\zeta/N^2), \tag{69}$$

$$\zeta=\langle|\sum_{j=1}^{N}\exp[i\theta_j-(2\pi ij/N)]|^2\rangle. \tag{70}$$

The ensemble average can be expressed in terms of two-level correlations only, and Eq. (70) becomes

$$\zeta=\pi^2\csc^2(\pi/N)\left[1-(\pi/N^2)\times\int_{-\pi}^{\pi}T_2(\varphi)\{2\pi|\varphi|-\varphi^2\}d\varphi\right]. \tag{71}$$

By use of the Fourier expansion

$$2\pi|\varphi|-\varphi^2=\frac{8}{3}\pi^2-4\sum_{1}^{\infty}k^{-2}\cos(k\varphi), \tag{72}$$

Eq. (71) can be reduced to the form

$$\zeta=\frac{\pi^2}{\sin^2(\pi/N)}\left\{1+\frac{2}{N}\sum_{1}^{\infty}k^{-2}[b(k/N)-1]\right\}. \tag{73}$$

Now let $N\to\infty$. The sum in (73) may be split into two parts, $1\leq k<\eta N$ and $k>\eta N$, where η is small compared with unity. In the first sum we approximate $b(k/N)$ by $[1-(2k/N)]$ according to Eq. (56). The second sum reduces to the integral

$$\int_{\eta}^{\infty}[b(x)-1]x^{-2}dx=2[(\ln\eta)-1]+\tfrac{1}{4}\pi^2. \tag{74}$$

Equations (69) and (73) then give

$$\Delta^2=\frac{4}{N^2}\left[(\ln N)+\gamma+1-\frac{5\pi^2}{24}\right]. \tag{75}$$

Since the mean level spacing is $D=(2\pi/N)$, we have for large N

$$(\Delta^2/D^2)\sim(1/\pi^2)\ln N. \tag{76}$$

Thus the deviations of the θ_j from a regular polygonal arrangement are very small, on the average, even when N is as large as 10^3 or 10^6.

V. COMPARISON WITH THE UNITARY ENSEMBLE

It is of some interest to compare the results hitherto obtained with the corresponding results for an eigen-

[4] The general belief among nuclear theorists is that, when a nucleus of spin J captures a slow neutron, the compound states of spin $(J-\frac{1}{2})$ and $(J+\frac{1}{2})$ will occur roughly in the proportion J to $(J+1)$. E.g., T. D. Newton, Can. J. Phys. **34**, 804 (1956). The experimental evidence for or against this belief is quite meager. See for example J. A. Harvey, D. J. Hughes, R. S. Carter, and V. E. Pilcher, Phys. Rev. **99**, 10 (1955).

value series with the probability distribution

$$P_{Nu}(\theta_1,\cdots,\theta_N)=(1/N!)(2\pi)^{-N}\prod_{i<j}|e^{i\theta_i}-e^{i\theta_j}|^2, \quad (77)$$

instead of Eq. (1). The distribution P_{Nu} would be correct for the eigenvalues of a random unitary matrix in the unitary ensemble defined in paper I. This would be a model for the energy levels of a complex system without invariance under time reversal.

In the unitary ensemble the integrals (3) can be performed without difficulty, giving the n-level correlation functions

$$R_{nu}(\theta_1,\cdots,\theta_n)=\det|s_N(\theta_j-\theta_k)|, \quad (78)$$

the determinant being $(n\times n)$ and having every diagonal element equal to $(N/2\pi)$. It is easy then to verify from Eq. (7) that the cluster functions are

$$T_{nu}(\theta_1,\cdots,\theta_n)=\sum\prod[s_N(\theta_j-\theta_k)]. \quad (79)$$

Here $s_N(\alpha)$ is given by Eq. (48), and $\sum\prod$ means a sum over $[(n-1)!]$ products, of which one is

$$s_N(\theta_1-\theta_2)s_N(\theta_2-\theta_3)\cdots s_N(\theta_n-\theta_1), \quad (80)$$

the others being obtained from it by cyclic permutations of $[1,2,\cdots,n]$. In particular, when $n=2$,

$$T_{2u}(\theta_1,\theta_2)=\{s_N(\theta_1-\theta_2)\}^2, \quad (81)$$

and therefore

$$\begin{aligned} b_u(k)&=1-|k|, \quad (|k|<1),\\ &=0, \quad (|k|>1). \end{aligned} \quad (82)$$

The analogs to Eqs. (54), (55) are

$$Y_{2u}(r)=1-\frac{1}{3}\pi^2r^2+\frac{2}{45}\pi^4r^4+\cdots, \quad (83)$$

$$Y_{2u}(r)=\frac{\sin^2(\pi r)}{\pi^2r^2}. \quad (84)$$

Equation (84) shows that the long-range order is much more marked in the unitary case than it was in Eq. (55). This was to be expected, since the unitary case is equivalent to a Coulomb gas at temperature $T=\frac{1}{2}$ instead of $T=1$.

The analog of Eq. (75) is

$$\Delta_u^2=(2/N^2)[(\ln N)+\gamma-(\pi^2/6)]. \quad (85)$$

Thus the mean-square displacement of the eigenvalues from a regular lattice is only half as great as before.

It is an interesting problem, which we have not been able to solve, to calculate the correlation-functions for the symplectic ensemble of paper I. In that case the square in Eq. (77) is replaced by a fourth power. The effects of long-range order must then be even stronger than in the unitary case.

VI. ENERGY AND SPECIFIC HEAT OF THE EQUIVALENT COULOMB GAS

We now return to the study of the probability distribution (1) of the eigenvalues of symmetric unitary matrices. In Sec. VI of I it was shown that this is also the distribution function of a classical Coulomb gas at temperature $T=1$. A measure of the energy of the gas is provided by the quantity

$$W=-\sum_{i<j}\ln|e^{i\theta_i}-e^{i\theta_j}|+\tfrac{1}{2}N\ln N. \quad (86)$$

This W is a convenient statistic by which to compare observed eigenvalue distributions with the theory, since we saw in Sec. IX (C) of I that the theoretical expectation-value and variance of W are both calculable. In fact, for large N,

$$\langle W\rangle=NU, \quad U=1-\tfrac{1}{2}\gamma-\tfrac{1}{2}\ln 2=0.365, \quad (87)$$

$$\langle(W-\langle W\rangle)^2\rangle=NC, \quad C=\frac{3}{2}-\frac{\pi^2}{8}=0.266. \quad (88)$$

Here U is the mean energy, and C the specific heat per particle, in the Coulomb gas at temperature 1. The values given in Eqs. (87), (88) were obtained from the conjectured analytic form of the partition function of the gas.

The value of $\langle W\rangle$ can also be expressed in terms of the 2-level cluster functions. A little manipulation of Eq. (86) gives

$$\begin{aligned} \langle W\rangle&=\tfrac{1}{2}\int\int T_2(\theta,\varphi)\ln|e^{i\theta}-e^{i\varphi}|\,d\theta\,d\varphi+\tfrac{1}{2}N\ln N\\ &=\tfrac{1}{2}N\int_{-\infty}^{\infty}Y_2(x)\ln|2\pi x|\,dx\\ &=-\tfrac{1}{2}N\lim_{\eta\to 0}\left\{\int_\eta^\infty\frac{b(k)}{k}dk+(\ln\eta)+\gamma\right\}. \end{aligned} \quad (89)$$

The integral (89) is elementary, and we obtain thereby a direct check of Eq. (87).

Suppose next that the observed levels are a mixture of two uncorrelated series in the proportions f, $(1-f)$. Equation (89) still holds, with the form factor $\bar{b}(k)$ given by Eq. (67). The expectation-value of W for the mixed series is therefore

$$\langle W\rangle=NU-\tfrac{1}{2}N\{f\ln f+(1-f)\ln(1-f)\}. \quad (90)$$

The expectation value of W^2 will bring in cluster functions of 2, 3, and 4 levels. After some algebra we find

$$\langle(W-\langle W\rangle)^2\rangle$$

$$=-\tfrac{1}{4}N\int\int\int Y_4(x,y,z,w)\ln(2\pi|x-y|)$$

$$\times\ln(2\pi|z-w|)dydzdw+N\int\int Y_3(x,y,z)$$

$$\times\ln(2\pi|x-y|)\ln(2\pi|x-z|)dydz-\tfrac{1}{2}N\int Y_2(x,y)$$

$$\times\{\ln(2\pi|x-y|)\}^2dy+\tfrac{1}{2}N\int\int\int[Y_2(x,z)$$

$$-\delta(x-z)][Y_2(y,w)-\delta(y-w)]\ln(2\pi|x-y|)$$

$$\times\ln(2\pi|z-w|)dydzdw. \quad (91)$$

The last term of Eq. (91) may be simply expressed in terms of the 2-level formfactor $b(k)$; it is in fact

$$\tfrac{1}{4}N\int_0^\infty[b(k)-1]^2k^{-2}dk. \quad (92)$$

The integrals in Eq. (91) could all in principle be evaluated, using the methods of Sec. III to determine the functions Y_3 and Y_4. However, this would be a tremendous labor; even to write down the explicit form of Y_3 takes many lines of print. It is very fortunate that the sum of all the integrals is known independently, being given by Eq. (88).

In the case of the unitary probability-distribution, the Y_n are simple functions given by Eq. (79), and the integrals (89) and (91) can be performed without too much trouble. In this case we obtain Eqs. (87) and (88) with

$$U=\tfrac{1}{2}(1-\gamma), \quad C=2-(\pi^2/6), \quad (93)$$

in agreement with the results of Sec. IX of I. This serves as a double check, verifying both the conjectured partition-function of paper I and the algebra leading to Eq. (91).

When the variance of W is computed for a mixture of two level-series, it turns out that only the last term of Eq. (91) is affected. After much cancellation of terms, we find

$$\langle(W-\langle W\rangle)^2\rangle=NC+\tfrac{1}{2}N\int_0^\infty[b(fk)-1]$$

$$\times[b((1-f)k)-1]k^{-2}dk. \quad (94)$$

The integral in Eq. (94) does not seem to be expressible in terms of elementary functions. In the simplest case $f=\frac{1}{2}$, the added term (94) is just equal to the term (92) already appearing in Eq. (91), but even in this case no analytic integration seems possible.

If $b(k)$ given by Eq. (56) is plotted numerically, it is seen that

$$b(k)=\exp(-2|k|) \quad (95)$$

is quite an accurate approximation except for large $|k|$. Large values of $|k|$ are unimportant in Eq. (94), and so Eq. (95) should give a useful approximation. Substitution of Eq. (95) into (94) gives the result

$$\langle(W-\langle W\rangle)^2\rangle=NC-N\{f\ln f+(1-f)\ln(1-f)\}, \quad (96)$$

which should be accurate within 10% for all f. Note that Eq. (90), although of similar appearance to Eq. (96), is an exact formula not resting upon the approximation (95).

Suppose that a series of N eigenvalues around the unit circle is known to be a mixture of two series having separate probability distributions of the form (1), only the proportions f, $(1-f)$ of the two series being unknown. Then a single measurement of W for the combined series will yield a value for f by Eq. (90). The variance of this measurement of f is by Eq. (96)

$$\langle(\Delta f)^2\rangle=\frac{4}{N}\frac{C-f\ln f-(1-f)\ln(1-f)}{\ln f-\ln(1-f)}. \quad (97)$$

Unless f happens to be very close to $\frac{1}{2}$, the expected error in the measurement of f is of the order of $N^{-\frac{1}{2}}$.

This method of measuring f is obviously far more precise than the method discussed in Sec. III. However, it is not yet a practical method, since it requires observation of the eigenvalues round an entire circle. In paper (IV) we shall show how the method can be adapted to the practical situation in which we observe eigenvalues only on a small part of the circle.

VII. LEVEL-SPACING DISTRIBUTION

Until now we have studied only the probability-distributions $R_n(\theta_1,\cdots,\theta_n)$ for finding n levels at a given set of positions, irrespective of the remaining levels. Gaudin and Mehta,[2] following Wigner, were mainly interested in the level-spacing distribution function $S(\theta)$. This is defined by the statement that $S(\theta_2-\theta_1)d\theta_1d\theta_2$ is the probability for finding 2 levels in the intervals $[\theta_1,\theta_1+d\theta_1]$ and $[\theta_2,\theta_2+d\theta_2]$, and no levels in the interval $[\theta_1+d\theta_1,\theta_2]$. Also

$$S(\theta)=[d^2R/d\theta^2], \quad (98)$$

where $R(\theta)$ is the probability that a randomly chosen interval of length θ is empty of eigenvalues.

The connection between $R(\theta)$ and the cluster-functions T_n is

$$R(2\alpha)=\exp\Big\{-\sum_{n=1}^{\infty}\frac{1}{n!}\int\cdots\int_{-\alpha}^{\alpha}T_n(\theta_1,\cdots,\theta_n)$$

$$\times d\theta_1\cdots d\theta_n\Big\}. \quad (99)$$

In fact $R(2\alpha)$ is precisely the expression H given by

Eqs. (41) and (45), if we choose for $U(\theta)=V(\theta)=1+A(\theta)$ the function

$$U(\theta)=1(-\pi+\alpha<\theta<\pi-\alpha)$$
$$=0(\pi-\alpha<\theta<\pi+\alpha). \quad (100)$$

It is convenient to choose the center of the excluded interval to be at $\theta=\pi$, so that $U(\theta)$ is an even function in $(-\pi, +\pi)$. Then Eqs. (35), (36) can be used, and we deduce

$$R(2\alpha)=\det\left|\delta_{pq}-\frac{\sin[(p-q)\alpha]}{\pi(p-q)}-\frac{\sin[(p+q)\alpha]}{\pi(p+q)}\right|, \quad (101)$$

p and q taking the values $(\frac{1}{2}, \cdots, m-\frac{1}{2})$. This is the analog, for the circle, of Gaudin's Eq. (12),[5] which gives the corresponding expression for the Gaussian model with finite N. The analogy becomes even closer if we write Eq. (101) as

$$R(2\alpha)=\det\left|\delta_{pq}-\frac{1}{\pi}\int_{-\alpha}^{\alpha}\cos(p\theta)\cos(q\theta)d\theta\right|. \quad (102)$$

When $N\to\infty$, the determinant (102) becomes the Fredholm determinant of an integral equation, and our results coincide with those of Gaudin. We find that, for an infinite eigenvalue series with mean spacing $D=1$, the probability $E(x)$ that a random interval of length x be empty of eigenvalues is

$$E(x)=\prod_{j=1}^{\infty}(1-x\lambda_j^2), \quad (103)$$

where the λ_j are the eigenvalues of the integral equation

$$\lambda F(y)=\int_0^1\cos(\tfrac{1}{2}\pi xyz)F(z)dz. \quad (104)$$

Gaudin[3] has shown how to use Eq. (103) for the practical computation of $E(x)$.

A different application of the Gaudin-Mehta method is made by choosing, instead of Eq. (100),

$$U(\theta)=V(\theta)=1(-\pi+\alpha<\theta<\pi-\alpha)$$
$$U(\theta)=0, \quad V(\theta)=2(\pi-\alpha<\theta<\pi+\alpha). \quad (105)$$

Let $R'(2\alpha)$ denote the expression H resulting from this choice. $R'(\theta)$ is then the probability that a randomly chosen interval of length θ will contain *not more than one eigenvalue*. The function

$$S'(\theta)=[d^2(R+R')/d\theta^2] \quad (106)$$

is the probability distribution for spacings between pairs of *next-nearest neighbors*. Equations (35) and (36) now give

$$R'(2\alpha)=\det\left|\delta_{pq}-\frac{\sin(p-q)\alpha}{\pi(p-q)}+\frac{\sin(p+q)\alpha}{\pi(p+q)}\right| \quad (107)$$

$$=\det\left|\delta_{pq}-\frac{1}{\pi}\int_{-\alpha}^{\alpha}\sin(p\theta)\sin(q\theta)d\theta\right|. \quad (108)$$

The symmetry between Eqs. (102) and (108) is remarkable, and we do not understand why it exists.

When $N\to\infty$, the limit of $R'(2\alpha)$ is $E'(x)$, the probability for an interval x to contain not more than one eigenvalue in a series with mean spacing $D=1$. Equation (108) gives the result

$$E'(x)=\prod_{j=1}^{\infty}(1-x\mu_j^2), \quad (109)$$

where the μ_j are the eigenvalues of the integral equation

$$\mu F(y)=\int_0^1\sin(\tfrac{1}{2}\pi xyz)F(z)dz. \quad (110)$$

Gaudin's method would allow one to compute $E'(x)$, and hence the next-nearest neighbor spacing distribution, numerically.

Now comes a still more peculiar coincidence. Let $R_u(2\alpha)$ be the probability for an interval (2α) to be empty, in an eigenvalue series taken from the unitary probability-distribution (77). Then Eq. (78) gives

$$R_u(2\alpha)-\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int\cdots\int_{-\alpha}^{\alpha}\det|s_N(\theta_j-\theta_k)|\times d\theta_1\cdots d\theta_n. \quad (111)$$

Using Eq. (48) and standard theorems from the algebra of determinants, Eq. (111) becomes

$$R_u(2\alpha)=\det\left|\delta_{pq}-\frac{\sin(p-q)\alpha}{\pi(p-q)}\right|$$
$$=\det\left|\delta_{pq}-\frac{1}{2\pi}\int_{-\alpha}^{\alpha}e^{i(p-q)\theta}d\theta\right|. \quad (112)$$

The determinant here is $(N\times N)$, the indices p and q taking the values $[-m+\frac{1}{2}, -m+\frac{3}{2}, \cdots, m-\frac{1}{2}]$. The similarity to Eqs. (102) and (108) is again striking. If we now add and subtract the rows and columns of the determinant (112) with indices $(\pm p)$, $(\pm q)$, we find the identity

$$R_u(2\alpha)=R(2\alpha)R'(2\alpha). \quad (113)$$

When $N\to\infty$, the limit of $R_u(2\alpha)$ is $E_u(x)$, the probability for an interval x to be empty in an infinite eigenvalue series with mean spacing $D=1$ in the unitary ensemble. Equation (112) then reduces to the

[5] Page 450 of the third paper in reference 2.

Fredholm determinant

$$E_u(x)=\prod_{j=1}^{\infty}(1-\epsilon_j x\nu_j^2), \tag{114}$$

where the ν_j are the eigenvalues of the integral equation

$$\nu F(y)=\tfrac{1}{2}\int_{-1}^{1}\exp(\tfrac{1}{2}\pi ixyz)F(z)dz, \tag{115}$$

and the ϵ_j are $(+1)$ or (-1) according as the corresponding eigenfunction is even or odd. The even ν_j are identical with the λ_j satisfying Eq. (104), while the odd ν_j are equal to $(i\mu_j)$ with μ_j satisfying Eq. (110). Therefore Eqs. (103), (109), and (114) satisfy the identity

$$E_u(x)=E(x)E'(x), \tag{116}$$

which is just the limit of Eq. (113) as $N\to\infty$.

The meaning of Eq. (113) can be illustrated in a concrete way as follows. Let $\sum_u$ be an eigenvalue series of order $N=2m$, taken from the unitary probability distribution (77). Let $\sum_M$ be another eigenvalue series of order N, constructed according to the following recipe: take two independent eigenvalue series $\sum_1$ and $\sum_2$, each of order N and belonging to the usual probability-distribution (1), superimpose the two series, and then pick out alternate eigenvalues from the mixed series. In view of a certain biological analogy, the suffix M may here be considered to stand for the word "meiosis." Now the product $R(2\alpha)R'(2\alpha)$ is just the probability that a random interval of length (2α) contains no term of the series $\sum_M$. Therefore Eq. (113) has the following meaning: *the distributions of level-spacings in the series $\sum_u$ and $\sum_M$ are identical.*

This property of the series $\sum_M$ suggests that we also examine its 2-level cluster function $T_{2M}(\alpha)$, which can be derived quite easily from the results of Sec. III. The construction of $\sum_M$ gives the formula

$$T_{2M}=\left(\frac{N}{2\pi}\right)^2-(\sum R_{2j})(\sum P_{2k+1})$$
$$-(\sum R_{2j+1})(\sum P_{2k})-2(\sum Q_{2j})(\sum Q_{2k+1}). \tag{117}$$

Here $P_j(\alpha)$ is defined to be the probability, for the separate eigenvalue series $\sum_1$ or $\sum_2$, that an interval α shall contain precisely j eigenvalues. $Q_j(\alpha)$ is the probability that an interval α, with one end point at an eigenvalue of $\sum_1$, shall contain j additional eigenvalues. $R_j(\alpha)$ is the probability that an interval α has eigenvalues of $\sum_1$ at both end points and j additional eigenvalues in its interior. From Eqs. (47) and (62) it is easy to compute

$$\sum_0^{m-1}R_{2j}=\tfrac{1}{2}(k^2-s_N{}^2+Is_NDs_N-Ds_N),$$
$$k=(N/2\pi), \tag{118}$$

$$\sum_0^{m-2}R_{2j+1}=\tfrac{1}{2}(k^2-s_N{}^2+Is_NDs_N), \tag{119}$$

$$\sum_0^{m-1}Q_{2j}=\tfrac{1}{2}(k+s_N), \tag{120}$$

$$\sum_0^{m-1}Q_{2j+1}=\tfrac{1}{2}(k-s_N), \tag{121}$$

$$\sum_0^{m}P_{2j}=1-Is_N, \tag{122}$$

$$\sum_0^{m-1}P_{2j+1}=Is_N, \tag{123}$$

the notations being defined by Eqs. (48)–(50). Substituting these expressions into Eq. (117) and using (81), we find

$$T_{2M}(\alpha)=s_N{}^2=T_{2u}(\alpha). \tag{124}$$

So the series $\sum_M$ and $\sum_u$ have identical 2-level correlations. This fact is additional and distinct from the identity of their spacing distributions.

In view of the foregoing, we make the general conjecture that *all statistical properties of the eigenvalue series $\sum_M$ and $\sum_u$ are identical.* We can find no general argument to explain why this conjecture should be true; but, if it were false, the identities (113) and (124) would become even more mysterious than they already are.

Note added in proof: This conjecture has subsequently been proved by Dr. J. Gunson of the University of Birmingham, England.