The Threefold Way. Algebraic Structure of Symmetry Groups and Ensembles in Quantum Mechanics

FREEMAN J. DYSON
Institute for Advanced Study, Princeton, New Jersey
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Using mathematical tools developed by Hermann Weyl, the Wigner classification of group-representations and co-representations is clarified and extended. The three types of representation, and the three types of co-representation, are shown to be directly related to the three types of division algebra with real coefficients, namely, the real numbers, complex numbers, and quaternions. The author's theory of matrix ensembles, in which again three possible types were found, is shown to be in exact correspondence with the Wigner classification of co-representations. In particular, it is proved that the most general kind of matrix ensemble, defined with a symmetry group which may be completely arbitrary, reduces to a direct product of independent irreducible ensembles each of which belongs to one of the three known types.

I. INTRODUCTION

THE purpose of this paper is to bring together and unify three trends of thought which have grown up independently during the last thirty years. These are (i) the classification by Wigner[1] of representations of groups which include time-inversion, (ii) Weyl's general theory of matric algebras and their commutator algebras,[2] and (iii) the study of ensembles of random matrices, begun by Wigner[3] and continued by various other physicists.[4] It will be shown that these three theories are all variations upon a single mathematical theme. It is not surprising that the three theories should turn out to be closely related, since they all took their origin from the work of the great algebraists Frobenius and Schur at the beginning of the twentieth century.[5]

Our way is threefold in another and deeper sense. In each of the three theories which we aim to unify, there appears a triple alternative, a choice between three mutually exclusive possibilities. (i) The irreducible representations of a group by unitary matrices fall into three classes, which are called potentially real, complex, and pseudoreal.[6] Another, and quite independent, threefold choice exists for representations of a group by unitary and anti-unitary matrices. Wigner[7] calls such representations co-representations, and he classifies them into types I, II, and III. (ii) The classical groups studied by Weyl are of three types, namely orthogonal, unitary, and symplectic. (iii) The present author[4] found three distinct kinds of ensembles of random matrices, to which he attached the same three names as are given to the classical groups. In the previous discussion of matrix ensembles,[4] the question whether all irreducible ensembles belong to one of these three types was not raised. This question will here be answered in the affirmative.

The recurrence of the threefold choice in all these contexts gave the first hint that a unified mathematical treatment of group representations, commutator algebras, and ensembles should be possible. It was Bargmann who pointed out to the author[8] that the root of the matter is to be found in the classical theorem of Frobenius.[9]

Frobenius' Theorem. Over the real number field

[1] E. P. Wigner, Nachr. Akad. Wiss. Göttingen, Math. physik. Kl., 546 (1932). See also, E. P. Wigner, *Group Theory and its Application to the Quantum Mechanics of Atomic Spectra* (Academic Press Inc., New York, 1959), English edition, Chaps 24 and 26.

[2] H. Weyl, *The Classical Groups, Their Invariants and Representations* (Princeton University Press, Princeton, New Jersey, 1939). Chapter 3 of this book contains the essential theorems on which all of our arguments hang. For Weyl's treatment of semilinear representations, see Duke Math. J. **3**, 200 (1937).

[3] E. P. Wigner, Ann. Math. **53**, 36 (1951); **62**, 548 (1955); **65**, 203 (1957); **67**, 325 (1958).

[4] F. J. Dyson, J. Math. Phys. **3**, 140, 157, and 166 (1962). This series of three papers includes references to earlier work by others in the same field. Paper IV in the series is being written in collaboration with Dr. M. L. Mehta and will be published later. The present paper should logically be considered to be number zero in the series, since it provides an improved mathematical and logical foundation for the rest of the series. Since Roman numerals contain no symbol for zero, we preferred to publish the present paper under a separate title.

[5] A sketch of the historical development is to be found in the section headed "Remembrance of Things Past" in Weyl's book (reference 2), p. 27.

[6] Chapter 24 of Wigner's book (reference 1). This classification was discovered by A. Loewy, Trans. Am. Math. Soc. **4**, 171 (1903). See also G. Frobenius and I. Schur, Sitzber. preuss. Akad. Wiss., Physik.-math. Kl. **186** (1906).

[7] Chapter 26 of Wigner's book (reference 1).

[8] V. Bargmann (private communication).

[9] G. Frobenius, J. reine u. angew. Math. **84**, 59 (1878); L. E. Dickson, *Linear Algebras* (Cambridge University Press, New York, 1914), p. 10.

there exist precisely three associative[10] *division algebras, namely the real numbers, the complex numbers, and the real quaternions.*

Once this is understood, the further development of the theory is extremely simple. All that is necessary is to apply the general theorems of Weyl[2] to the special case in which the ground field of the matric algebras is the field of real numbers.

Probably all these connections would have been clarified long ago, if quantum physicists had not been hampered by a prejudice in favor of complex and against real numbers. It has been generally believed that only the complex numbers could legitimately be used as the ground field in discussing quantum-mechanical operators. Over the complex field, Frobenius' theorem is of course not valid; the only division algebra over the complex field is formed by the complex numbers themselves. However, Frobenius' theorem is relevant precisely because the appropriate ground field for much of quantum mechanics is real rather than complex. Specifically, as soon as anti-unitary operators such as time inversion are included, it is simpler and more natural to work with a real ground field than to follow Weyl[2] in studying semilinear operators over the complex field.

Physicists have known for a long time that in practice, when invariance under time-inversion is in question, complex phases are no longer arbitrary and undetermined coefficients may be taken to be real. Physicists are, in fact, like M. Jourdain talking prose, using the real numbers for their ground field without knowing it. One purpose of this paper is to make the use of the real ground field in quantum mechanics official and undisguised.[11] No change in the physical content of the theory is thereby implied. Only it may be easier for students to understand what they are doing if the mathematical formalism is brought into closer correspondence with physical practice.

A final by-product of the work described in this paper is that it defines an area of quantum mechanics within which quaternions play a natural and essential role. Several attempts have been made in the past[12] to construct a radically new version of quantum mechanics in which complex numbers are from the beginning replaced by quaternions. Our analysis has nothing to do with these attempts. Proceeding in a modest and conservative spirit, we merely show that quaternions form the appropriate algebraic basis for a description of nature whenever we have to deal either with pseudoreal group representations or with co-representations of Wigner's type II. The context in which quaternions arose historically, in a study of the three-dimensional rotation group, can now be seen to be an extremely special case of this general principle. Every group which admits pseudoreal representations equally admits a natural description in terms of real quaternions.

II. GROUP ALGEBRA AND COMMUTATOR ALGEBRA

The starting point of our analysis is a group G which is supposed to be a symmetry-group for some quantum-mechanical system. For example, G could be a rotation group, or an isotopic-spin group, or a time-inversion group, or all of these in combination. The quantum-mechanical states belong to a linear vector space H_c of finite dimension n over the field C of complex numbers. An element g of G is represented in H_c by an operator $\Lambda(g)$ which is either unitary or antiunitary. Physically, the antiunitary $\Lambda(g)$ will correspond to operations g which involve time-inversion. We make the convention that the letter g may denote any element of G, the letter u denotes an element for which $\Lambda(u)$ is unitary, and the letter a denotes an element for which $\Lambda(a)$ is antiunitary. The set of u forms a subgroup G_1 of G. We assume that G contains some antiunitary elements a. Then G_1 is an invariant subgroup of G with index 2. The a form a set G_2 which is the unique co-set of G_1 in G.

The $\Lambda(a)$ are not matrices over the field of complex numbers. The notion of group representation can be enlarged, following Weyl[2] and Wigner,[7] so as to include such semilinear operations. However, we find it simpler and more fruitful to represent

[10] The restriction to associative algebras is forced by the fact that the rule of matrix multiplication is associative. In all applications of group theory to quantum mechanics we identify the operation of multiplication with ordinary matrix multiplication. It is well-known that a fourth division algebra over the real number field exists, namely the algebra of octonions, if multiplication is allowed to be nonassociative. It is interesting to speculate upon possible physical interpretations of the octonion algebra [see A. Pais, Phys. Rev. Letters 7, 291, 1961]. We have tried, and failed, to find a natural way to fit octonions into the mathematical framework developed in this paper.

[11] The general formalism of quantum mechanics over a real ground field has been worked out by E. C. G. Stueckelberg, Helv. Phys. Acta **32**, 254 (1959); **33**, 727 (1960). Two further papers by Stueckelberg and collaborators have been circulated as preprints and will appear in Helv. Phys. Acta. These papers have many points of contact with the present work. For a brief summary of Stueckelberg's conclusions, see also the paper of Finkelstein *et al.* (reference 12).

[12] G. Birkhoff and J. von Neumann, Ann. Math. **37**, 823 (1936). E. J. Schremp, Phys. Rev. **99**, 1603 (1955); **113**, 936 (1959). D. Finkelstein, J. M. Jauch, S. Schiminovich, and D. Speiser, J. Math. Phys. **3**, 207 (1962).

the $\Lambda(a)$ by true matrices over the field R of real numbers. We define the correspondence

$$\Lambda(g) \leftrightarrow M(g) \tag{1}$$

in the following way. $M(g)$ is a $[2n \times 2n]$ matrix with real elements. Each (2×2) block in $M(u)$ is derived from a single element of the $[n \times n]$ complex matrix $\Lambda(u)$ by the replacement

$$\alpha + i\beta \leftrightarrow \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}. \tag{2}$$

Each $\Lambda(a)$ is of the form

$$\Lambda(a) = U(a)j, \tag{3}$$

where $U(a)$ is unitary and j is the operation of complex conjugation. The $M(a)$ are defined by making the substitution (2) in $U(a)$ together with the replacement

$$j = I_n \times \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{4}$$

The space H_R in which the matrices $M(g)$ operate is a real $2n$-dimensional vector space. Each vector in H_R is composed of the real and imaginary parts of the components of the corresponding vector in H_c. It is convenient to consider the symbol

$$i = I_n \times \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{5}$$

also as a matrix operating in H_R.

The $M(g)$ now form a true $2n$-dimensional representation of the group G over the field R. The distinction between unitary and antiunitary elements of G is provided by the commutation rules

$$M(u)i = iM(u), \tag{6}$$

$$M(a)i = -iM(a). \tag{7}$$

All the $M(g)$ are orthogonal matrices.

A matric algebra over R is defined as a set of matrices which is closed under the three operations of addition, matrix multiplication, and multiplication by scalar coefficients in R. Three such algebras will now be introduced:

A generated by the $M(u)$, (8)

B generated by the $M(u)$ and i, (9)

D generated by the $M(u)$, $M(a)$, and i. (10)

The commutator algebra of a given algebra K is defined as the set of matrices which commute with all matrices in K. The commutator algebra is itself a matric algebra over R. In particular we define

$$X = \text{commutator algebra of } A, \tag{11}$$

$$Y = \text{commutator algebra of } B, \tag{12}$$

$$Z = \text{commutator algebra of } D. \tag{13}$$

The inclusion relations

$$A \subset B \subset D \tag{14}$$

immediately imply

$$X \supset Y \supset Z. \tag{15}$$

The algebra A is given the name "group algebra of G_1 over R." In an obvious sense, B is identical with the group algebra of G_1 over C. The algebra D is not a group algebra over C in the ordinary sense, but it may be considered to be the group algebra of G over C. However, it is important that we have defined each of A, B, D as algebras with coefficients in R.

We next introduce some convenient notations, following Weyl.[2] If K is any algebra and m a positive integer, we denote by mK the algebra of matrices consisting of m identical blocks,

$$\begin{bmatrix} M & O & O & \cdots \\ O & M & O & \\ \vdots & & \ddots & \\ O & \cdots & & M \end{bmatrix}, \tag{16}$$

with M in K. Symbolically, we may write this as an outer product,

$$mK = I_m \times K. \tag{17}$$

We denote by $[K]_m$ the algebra of all matrices consisting of m^2 blocks,

$$\begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1m} \\ M_{21} & M_{22} & \cdots & M_{2m} \\ \vdots & \vdots & & \vdots \\ M_{m1} & M_{m2} & \cdots & M_{mm} \end{bmatrix}, \tag{18}$$

with each M_{ij} independently a matrix in K. In particular, when $K = R$ is the algebra of scalars, R_m is the algebra of all real matrices of degree m.

Two algebras A, A' are said to be equivalent $(A \sim A')$ if there exists a fixed nonsingular matrix N such that every matrix M of A is related to the corresponding M' in A' by

$$M = NM'N^{-1}. \tag{19}$$

If A is any algebra, the inverse algebra $\bar{A}$ is obtained from A by inverting the order of factors in all products, thus

$$\bar{M} = \bar{M}_1 \bar{M}_2 \tag{20}$$

if and only if

$$M = M_2 M_1. \tag{21}$$

Finally, a division algebra is defined to be an algebra in which every nonzero element M has a reciprocal M^{-1}.

With these notations and definitions, we are in a position to state the main theorem of Weyl.[13]

Weyl's Theorem. Let K be any group algebra over R, and L its commutator algebra. Then K and L are simultaneously equivalent to the canonical forms

$$K \sim \sum_j s_j [E_j]_{t_j}, \qquad L \sim \sum_j t_j [\bar{E}_j]_{s_j}. \tag{22}$$

The summations here represent direct sums over diagonal blocks of matrices. Each value of j corresponds to one inequivalent irreducible representation of the group Γ which generates K over R. For each j, E_j is a division algebra, and s_j, t_j are positive integers. The matrix block corresponding to index j has degree

$$d_j = s_j t_j e_j, \tag{23}$$

where e_j is the degree of E_j.

The following remarks may be made concerning this theorem.

Remark 1. The relation between the algebras K and L is symmetrical. Thus K is also the commutator algebra of L.

Remark 2. When the sums (22) reduce to a single term, the algebras K and L are called simple. In this case the suffixes j may be dropped.

Remark 3. When K is generated by an irreducible representation of Γ, K is simple and the integer s is equal to unity. In this case

$$K \sim E_t, \qquad L \sim t\bar{E}. \tag{24}$$

Remark 4. By Frobenius' theorem (see Sec. I), the possible division algebras over R are three in number, and are denoted by R, C, and Q. R has degree 1, and is generated by the scalar $I_1 = 1$. C has degree 2 and is generated by

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad e_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{25}$$

Note that this e_2 is not necessarily identical with the original imaginary unit i defined by Eq. (5). Since C is commutative, $C = \bar{C}$. The quaternion division algebra Q has degree 4 and is generated by

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \tau_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad \tau_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \tau_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}. \tag{26}$$

The inverse algebra $\bar{Q}$ is then generated by

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \tau_1' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \tau_2' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad \tau_3' = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}. \tag{27}$$

This particular representation of Q and $\bar{Q}$ by real

[13] This is theorem (3.5B) on p. 95 of Weyl's book (reference 2), combined with the theorem that every group ring is fully reducible (p. 101 of the same book).

matrices is called the regular representation. It has the property that all matrices in Q commute with all matrices in $\tilde{Q}$. Thus Q and $\tilde{Q}$ are commutator algebras of each other, as required by the theorem.

Remark 5. When K is a simple algebra, the division algebra E is uniquely fixed and must be either R, C, or Q. In these three cases we say that the representation of Γ by K is of type R, type C, or type Q, respectively.

Remark 6. We shall apply Weyl's theorem to the algebras A, B, and D defined at the beginning of this section. In the case of A, the group Γ is identical with G_1. In the case of B, the group Γ is the direct product of G_1 with the Abelian group Γ_4 generated by (I, i). In the case of D, the group Γ is the product of G with Γ_4, the commutation rules between G and Γ_4 being given by Eqs. (6) and (7). Each of A, B, D is thus a group algebra over R in the ordinary sense, although only B is a group algebra over C.

The following lemma[14] is important in determining the structural relations between the algebras B and D.

Lemma. Let $M_1(g)$, $M_2(g)$ be two inequivalent irreducible components of the algebra D. Then the subalgebras $M_1(u)$, $M_2(u)$ in B are inequivalent, and no irreducible component of $M_1(u)$ can be equivalent to any irreducible component of $M_2(u)$.

To prove the lemma, we assume that $M_1(g)$ and $M_2(g)$ are inequivalent and that $M_1(u)$ and $M_2(u)$ have two equivalent irreducible components. There then exists a matrix P in the algebra Y, linking the two inequivalent blocks M_1 and M_2 of the algebra D, but commuting with the algebra B. This P satisfies

$$Pi = iP, \qquad PM_1(a_1a_2^{-1}) = M_2(a_1a_2^{-1})P, \tag{28}$$

for any two antiunitary elements a_1, a_2 in G. Therefore,

$$[M_2(a_1)]^{-1}PM_1(a_1) = [M_2(a_2)]^{-1}PM_1(a_2) = W, \tag{29}$$

where W is a matrix independent of a_1, a_2. Hence

$$PM_1(a) = M_2(a)W \tag{30}$$

for all a in G. Since Eq. (30) also holds with a replaced by a^{-1}, we have

$$M_2(a)P = WM_1(a). \tag{31}$$

Therefore,

$$(P + W)M_1(a) = M_2(a)(P + W) \tag{32}$$

for all a in G, and this implies

$$(P + W)M_1(g) = M_2(g)(P + W). \tag{33}$$

Since $M_1(g)$ and $M_2(g)$ are supposed irreducible and inequivalent, Schur's lemma[15] now implies

$$P + W = 0. \tag{34}$$

But then Eq. (30) becomes

$$PM_1(a) = -M_2(a)P. \tag{35}$$

Equations (28) and (35) together give

$$iPM_1(g) = M_2(g)iP \tag{36}$$

for all g in G, and therefore by Schur's lemma again

$$iP = 0.$$

Thus the operator P cannot exist, and the lemma is proved.

Remark 7. An equivalent statement of this lemma is as follows. Let the algebras Y and Z be written in the canonical form of Weyl's theorem as direct sums of diagonal blocks,

$$Y = \sum_k Y_k, \qquad Z = \sum_j Z_j, \tag{37}$$

where the Z_j are inequivalent simple algebras and likewise the Y_k. The lemma states that each Y_k is confined to a single block containing precisely one Z_j. This means that the structural relation between Y and Z is completely determined by considering the separate blocks Z_j.

III. WIGNER'S CLASSIFICATION OF IRREDUCIBLE REPRESENTATIONS

In this section we shall establish the connection between Weyl's theory of group algebras (Sec. II) and the classification of group representations by Wigner.[1]

A few preliminary observations must first be made. The equivalence relations in Eq. (22) refer to a transformation to canonical forms by a similarity relation (19) in which N may be an arbitrary real nonsingular matrix. According to a standard argument,[16] since the algebra K is generated by an orthogonal group representation, the transformation matrix N may be chosen to be orthogonal. Next we show that, when the algebra K is one of the trio A, B, or D, the matrix N may be chosen so as to commute with i. The operator i belongs to B and D,

[14] This lemma could probably be deduced as a special case from the general theorems of A. H. Clifford, Ann. Math. 38, 533 (1937), concerning the connections between representations of groups and subgroups. However, it seemed simpler to give a direct and elementary proof of the lemma without appeal to Clifford's work.

[15] See Wigner (reference 1), p. 75, Theorem 2.

[16] See Wigner (reference 1), p. 78.

and to the commutator algebras X and Y of A and B. So in each of the three cases, i belongs either to K or to L. When the transition to canonical forms is made, i is transformed into some matrix i' which has nonzero elements only within the blocks where the canonical form of K or L exists. The transformed i' still satisfies

$$[i']^2 = -I. \tag{38}$$

It is therefore possible to transform i' back into the standard form i by a real orthogonal transformation working within each diagonal block separately. As a result, we have an orthogonal matrix N which transforms K and L into the canonical forms (22) and transforms i into i. This N then commutes with i.

When N is chosen to be orthogonal and to commute with i, N is identical with a unitary transformation of the original complex vector space H_c. Thus the canonical forms (22) are obtained by a change in the representation of state vectors, according to the usual terminology of quantum mechanics. It is convenient for us to choose N to be a transformation of this special kind. When this is done, the division algebras C and Q will not in general appear in the particular representations (25) and (26). For the quantum-mechanical applications it is useful to have i in the standard form (5), whereas there is no strong reason to prefer the representations (25), (26) of C and Q to other equivalent representations.

Let now G_1 be a group composed of unitary operators only. For the moment we are not concerned with the antiunitary part of G, and so we consider the algebras A, B, X, Y only. Suppose that the operators $\Lambda(u)$ form an irreducible representation of G_1 over C. Since C is the only division algebra over C, the forms of the algebras B and Y are completely determined by Weyl's theorem

$$B = (C)_n, \qquad Y = nC. \tag{39}$$

Equation (39) is in fact merely a statement of Schur's lemma.[15] Also, it follows from the definitions that every matrix in X which commutes with i belongs to Y.

The order (number of linearly independent elements) of the algebra B is $2n^2$. According to Eqs. (8) and (9), the order of A is $2n^2$ if i belongs to A, n^2 if i does not belong to A. Weyl's theorem then gives precisely three possible canonical forms for the algebras A, X, as follows:

$$A = 2R_n, \qquad X = nR_2, \tag{40}$$

$$A = C_n, \qquad X = nC, \tag{41}$$

$$A = Q_m, \qquad X = m\bar{Q}, \tag{42}$$

where we have written $m = \frac{1}{2}n$. In all three cases the operator i belongs to X, and the order of X is 2 or 4.

Wigner's classification of irreducible representations $\Lambda(u)$ is the following. Let $\Lambda^*(u)$ be the representation formed by taking the complex conjugate in each element of $\Lambda(u)$. If

$$\Lambda^*(u) = M\Lambda(u)M^{-1}, \quad \text{all} \quad u, \tag{43}$$

with M unitary and symmetric, then $\Lambda(u)$ is "potentially real." If Eq. (43) holds with M unitary and antisymmetric, then $\Lambda(u)$ is "pseudoreal." If $\Lambda^*(u)$ is not equivalent to $\Lambda(u)$, then $\Lambda(u)$ is "complex." We write M as usual as a $(2n \times 2n)$ real matrix, and define

$$P = jM \tag{44}$$

with j given by Eq. (4). Then Eq. (43) holds if and only if the matrix P belongs to the commutator algebra X. Therefore an equivalent statement of Wigner's classification is this. If X contains an antiunitary operator P with

$$P^2 = I, \tag{45}$$

then $\Lambda(g)$ is potentially real. If X contains an antiunitary operator P with

$$P^2 = -I, \tag{46}$$

then $\Lambda(g)$ is pseudoreal. If X contains no antiunitary operator P, then $\Lambda(g)$ is complex. An inspection of the canonical forms (40), (41), (42) then yields the following theorem.

Equivalence Theorem I. Let $\Lambda(u)$ be an irreducible representation over C of a unitary group G_1. Let the algebra A be defined by Eq. (8) with real coefficients. Then

(i) *If A is of type R, $\Lambda(u)$ is potentially real;*
(ii) *If A is of type C, $\Lambda(u)$ is complex;*
(iii) *If A is of type Q, $\Lambda(u)$ is pseudoreal.*

In each case the converse is also true.

The following remarks are corollaries of Theorem I.

Remark 1. The matrices $M(u)$ form a real representation of the group G_1. This representation is irreducible over R when A is of type C or Q. It reduces to two equivalent irreducible components when A is of type R.

Remark 2. It is well known[6] that $\Lambda(u)$ is potentially real if and only if it is equivalent to a representation composed entirely of real matrices. We now can make another statement of the same kind. The irreducible representation $\Lambda(u)$ is pseudoreal if and only if it is equivalent to a representation composed entirely of matrices whose elements are quaternions with real coefficients.

Remark 3. It is well-known (see Wigner's book,[1] p. 289) that the irreducible representations of the 3-dimensional rotation group are potentially real for integer spin, pseudoreal for half-integer spin. From remark 2 it then follows that the integer-spin representations may be taken to be real, and the half-integer-spin representations may be written in terms of real quaternion matrices.

We now turn our attention to the full group G including antiunitary operators. We shall be concerned with the algebras B, D and their commutators Y, Z. An irreducible co-representation of G is a set of matrices $M(u)$, $M(a)$ such that the algebra D is irreducible over R. According to Weyl's theorem there are then three possibilities for the canonical forms of D and Z.

$$D = R_{2n}, \qquad Z = (2n)R, \tag{47}$$

$$D = C_n, \qquad Z = nC, \tag{48}$$

$$D = Q_m, \qquad Z = m\bar{Q}, \quad m = \tfrac{1}{2}n. \tag{49}$$

The algebra B may now be reducible, but its irreducible components must be of the form sC_t. Also, by Eqs. (9) and (10), the order of D must be exactly twice that of B. Equations (47), (48), and (49) then imply that the order of B is $2n^2$, n^2, $\frac{1}{2}n^2$ in the three cases. The only possibilities are

$$B = C_n, \qquad Y = nC, \tag{50}$$

$$B = C_m + C_m, \qquad Y = mC + mC, \tag{51}$$

$$B = 2C_m, \qquad Y = mC_2, \tag{52}$$

and these correspond precisely to the three alternatives (47) to (49).

Wigner's classification of irreducible co-representations[7] is the following. The co-representation is type I if its unitary part is irreducible. It is type II if its unitary part reduces to two equivalent irreducible components. It is type III if its unitary part reduces to two inequivalent irreducible components. Now when the co-representation generates the algebra D, the unitary part of it generates the algebra B. An inspection of Eqs. (50) to (52) shows that these three alternatives correspond to the Wigner types I, III, II, respectively.

Equivalence Theorem II. Let $\Lambda(g)$ be an irreducible co-representation over C of a group G including antiunitary operations. Let the algebra D be defined by Eq. (10) with real coefficients. Then

(i) *If D is of type R, $\Lambda(g)$ is of Wigner type I,*
(ii) *If D is of type C, $\Lambda(g)$ is of Wigner type III,*
(iii) *If D is of type Q, $\Lambda(g)$ is of Wigner type II.*

In each case the converse is also true.

Remark 4. If follows from this theorem that an irreducible co-representation is of type II if and only if it can be expressed in terms of matrices whose elements are real quaternions.

Remark 5. According to Eqs. (47) to (52), the algebra Y has always precisely double the order of the algebra Z. Also, it is known that Y contains the matrix i, which commutes with Z but does not belong to Z. Therefore, in the case here considered (D being irreducible and Z a simple algebra), Y is precisely the direct product of Z with the algebra generated by $(1, i)$.

Remark 6. The statement that Y is the direct product of Z with $(1, i)$ has been established for the case of Z simple. However, by virtue of the lemma of Sec. II (see remark 7 following the lemma) the same relation between Y and Z holds in the general case.

Remark 7. The lemma of Sec. II can be stated very concisely as a statement about co-representations: inequivalent irreducible co-representations of G contain inequivalent irreducible representations of G_1.

IV. FURTHER ANALYSIS OF THE WIGNER CLASSIFICATION

The equivalence Theorems I and II are so alike in form that one might suppose them to be two statements of the same triple alternative. We shall show that in fact the precise opposite is true. The two triple alternatives are entirely independent. Within the same irreducible co-representation of G, any one of the three types of algebra D may occur in combination with any one of the three types of algebra A.

To study the relation between the two theorems, we fix a particular irreducible co-representation $\Lambda(g)$ of G and investigate the possible structure of the six algebras A, B, D, X, Y, Z in combination. Since $\Lambda(g)$ is irreducible, the possible structures for D, Z, B, Y are described by Eqs. (47) to (52). The representation $\Lambda(u)$ of G_1 is however not necessarily irreducible. Theorem I, and the three alternatives given by Eqs. (40) to (42), apply directly only to the irreducible components of $\Lambda(u)$.

When D is of type R, then $\Lambda(u)$ is irreducible and Eqs. (40) to (42) apply unchanged. When D is of type Q, then, according to Eq. (52), $\Lambda(u)$ splits into two identical irreducible components, or symbolically $\Lambda(u) = 2\Lambda'(u)$. In this case Eqs. (40) to (42) apply to $\Lambda'(u)$. When D is of type C, then Eq. (51) holds, and so $\Lambda(u)$ splits into two irreducible components inequivalent over C,

$$\Lambda(u) = \Lambda_1(u) + \Lambda_2(u). \quad (53)$$

The real representation $M(u)$ of G_1 splits correspondingly into two components

$$M(u) = M_1(u) + M_2(u). \quad (54)$$

Equations (40) to (42) apply to Λ_1 and Λ_2 separately. However, we shall prove that the algebra A is necessarily of the same type (R, C, or Q) for the representations Λ_1 and Λ_2. Thus one of Eqs. (40) to (42) applies to both components of $\Lambda(u)$.

Let a be any one of the antiunitary operators in G. The transformation

$$u \rightarrow V(u) = a^{-1}ua \quad (55)$$

is an automorphism V of the unitary group G_1. The representations

$$\Lambda_V(u) = \Lambda(V(u)), \qquad M_V(u) = M(V(u)) \quad (56)$$

differ from $\Lambda(u)$ and $M(u)$ only by a relabeling of the elements of G_1. Thus $M_V(u)$ and $M(u)$ generate isomorphic group algebras. Moreover, Eq. (54) implies

$$M_V(u) = [M(a)]^{-1}M(u)M(a) \equiv M(u), \quad (57)$$

where the equivalence is over R and not over C. Suppose now that D is of type C and Eq. (54) holds. Then Eq. (57) means either

$$M_{1V}(u) = [M(a)]^{-1}M_1(u)M(a),$$
$$M_{2V}(u) = [M(a)]^{-1}M_2(u)M(a), \quad (58)$$

or

$$M_{1V}(u) = [M(a)]^{-1}M_2(u)M(a),$$
$$M_{2V}(u) = [M(a)]^{-1}M_1(u)M(a). \quad (59)$$

Because the algebra D generated by $M(u)$, $M(a)$, and i is irreducible, Eq. (58) cannot hold. Therefore Eq. (59) must hold and

$$M_{1V}(u) \equiv M_2(u), \qquad M_{2V}(u) \equiv M_1(u). \quad (60)$$

The algebra A generated by $M_1(u)$ is therefore necessarily of the same type as that generated by $M_2(u)$.

We may thus classify irreducible co-representations of G into nine possible cases, which we denote by $RR, RC, RQ, CR, \cdots, QQ$. Case CR, for example, means that algebra D is of type C while algebra A is of type R, i.e., we have a co-representation of Wigner type III whose unitary part splits into two irreducible inequivalent representations each of which is potentially real.

Using Eqs. (40) to (42) we can write down the possible forms of the algebras A and X in each of the nine cases:

case RR,	$A = 2R_n$,	$X = nR_2$,	(61)
case RC,	$A = C_n$,	$X = nC$,	(62)
case RQ,	$A = Q_m$,	$X = m\bar{Q}$,	(63)
case CR,	$A = 2R_m + 2R_m$,	$X = mR_2 + mR_2$,	(64)
case $CC1$,	$A = C_m + C_m$,	$X = mC + mC$,	(65)
case $CC2$,	$A = 2C_m$,	$X = mC_2$,	(66)
case CQ,	$A = Q_p + Q_p$,	$X = p\bar{Q} + p\bar{Q}$,	(67)
case QR,	$A = 4R_m$,	$X = mR_4$,	(68)
case QC,	$A = 2C_m$,	$X = mC_2$,	(69)
case QQ,	$A = 2Q_p$,	$X = p\bar{Q}_2$,	(70)

For convenience we wrote here $m = \frac{1}{2}n$, $p = \frac{1}{4}n$. The forms of A and X are uniquely fixed in all cases except CC. Case CC divides into two alternatives $CC1$ and $CC2$. Case $CC1$ holds when the representations $M_1(u)$ and $M_2(u)$ are inequivalent over R; case $CC2$ holds when M_1 and M_2 are equivalent over R.

The results (61) to (70) follow immediately from Eqs. (40) to (42) when D is of type R or Q. However, when D is of type C some further argument is needed. Suppose then that D is of type C, so that Eqs. (48) and (51) hold, and the representation $M(u)$ splits according to Eq. (54). When M_1 and M_2 are inequivalent over R, every matrix commuting with the $M(u)$ must commute separately with $M_1(u)$ and $M_2(u)$. The algebra X is then the direct sum of the commutator algebras of M_1 and M_2. Therefore for M_1 and M_2 inequivalent, Eq. (64), (65), or (67) holds according as A is of type R, C, or Q.

It remains to consider the case in which D is of type C while M_1 and M_2 are eauivalent over R. There is then a real matrix L which commutes with all the $M(u)$ but does not commute with $M_1(u)$, $M_2(u)$ separately. This L satisfies

$$M_1(u) = L^{-1}M_2(u)L, \quad M_2(u) = L^{-1}M_1(u)L. \quad (71)$$

Since $\Lambda_1(u)$ and $\Lambda_2(u)$ are inequivalent over C, L must anticommute with i. Now suppose if possible that A were of type R or Q. Then there would exist also a matrix L' in X, anticommuting with i and commuting with each of $M_1(u)$, $M_2(u)$ separately. The product $U = LL'$ would be a matrix commuting with i and also satisfying Eq. (71). This is impossible since Λ_1 and Λ_2 are inequivalent over C. We have thus proved that, if D is of type C and M_1 and M_2 are equivalent, A is also necessarily of type C. There exists then only the case $CC2$ with A and X given by Eq. (66).

We next discuss a special situation in which the above enumeration of possibilities simplifies considerably. We say that the group G is "factorizable" if the automorphism V given by Eq. (55) is an inner automorphism of G_1. Suppose that G is factorizable. Then there exists an element w in G_1 such that

$$V(u) = a^{-1}ua = w^{-1}uw, \quad \text{all} \quad u \quad \text{in} \quad G_1. \tag{72}$$

Then there exists an antiunitary operator

$$T = aw^{-1} \tag{73}$$

in G which commutes with all elements of G_1. Conversely, if such T exists, then $V(u)$ is an inner automorphism for any choice of the antiunitary operator a in Eq. (55). In many physical applications, when such an operator T exists it is convenient to give it the name "time-inversion operator." In any representation $M(g)$ of G, the antiunitary matrix $M(T)$ belongs to the algebra X.

We now classify the possible types of irreducible co-representation of a factorizable group G. Many cases can be immediately eliminated. First, the matrix $M(T)$ belongs to X but does not belong to Y since it anticommutes with i. Therefore $X \neq Y$ for a factorizable group. Hence, by comparing Eqs. (50) to (52) with Eqs. (62), (65), and (69), the cases RC, $CC1$, and QC are excluded. Next, suppose that D is of type C. Then Eq. (72) gives

$$M_V(u) = [M(w)]^{-1}M(u)M(w), \tag{74}$$

with $M(w)$ unitary. Since $\Lambda_1(u)$ and $\Lambda_2(u)$ are inequivalent over C, Eq. (74) implies

$$M_{1V}(u) = [M(w)]^{-1}M_1(u)M(w),$$
$$M_{2V}(u) = [M(w)]^{-1}M_2(u)M(w).$$

This together with Eq. (60) shows that $M_1(u)$ and $M_2(u)$ are equivalent. We proved earlier that cases CR, $CC1$, and CQ are then impossible.

The surviving cases for a factorizable group G are RR, RQ, QR, QQ, and $CC2$.

The operator $[M(T)]^2$ commutes with all $M(g)$ and with i, and it is also equal to $M(u)$ with $u = T^2$. Thus $[M(T)]^2$ belongs to both the algebras D and Z. By Eqs. (47) to (49), the common part of D and Z is $(2n)R$ when D is of type R or Q, and is nC when D is of type C. Since $[M(T)]^2$ is a real orthogonal matrix, it must be a scalar

$$[M(T)]^2 = \epsilon = \pm 1, \tag{75}$$

in any of the four cases RR, RQ, QR, QQ. However, Eq. (75) need not hold in case $CC2$.

We determine lastly which cases go with the plus sign and which with the minus sign in Eq. (75). When G is factorizable and Eq. (75) holds, the algebra D is a direct product of the commuting algebras A and W, where W is the algebra of order 4 generated by $[I,\ i,\ M(T),\ iM(T)]$. The structure of $W = (D/A)$ is then determined as follows:

$$\text{case } RR, \quad A = 2R_n, \quad D = R_{2n}, \quad W \sim R_2, \tag{76}$$

$$\text{case } RQ, \quad A = Q_m, \quad D = R_{2n}, \quad W \sim \bar{Q}, \tag{77}$$

$$\text{case } QR, \quad A = 4R_m, \quad D = Q_m, \quad W \sim Q, \tag{78}$$

$$\text{case } QQ, \quad A = 2Q_p, \quad D = Q_m, \quad W \sim R_2. \tag{79}$$

The sign of ϵ in Eq. (75) is plus when W is of type R_2, minus when W is of type Q. These results will now be summarized in a theorem.

Theorem III. Let $M(g)$ be an irreducible co-representation of a factorizable group G, in which $M(T)$ is anti-unitary and commutes with all the $M(g)$. Then the following three possibilities alone exist:

(i) *case RR or QQ with* $[M(T)]^2 = +1$,
(ii) *case RQ or QR with* $[M(T)]^2 = -1$,
(iii) *case $CC2$ with* $[M(T)]^2 = \cos\alpha + e \sin\alpha$,

where α may be any real angle, and e is an element of the algebra A with $e^2 = -1$.

Remark 1. It is noteworthy that the sign of $[M(T)]^2$ is determined neither by the Wigner type of the co-representation $M(g)$, nor by the reality type of the unitary subrepresentation $M(u)$, but only by these two types in combination. Thus $[M(T)]^2 = +1$ corresponds to Wigner type I and potentially real, or to Wigner type II and pseudoreal; $[M(T)]^2 = -1$ corresponds to Wigner type II and potentially real, or to Wigner type I and pseudoreal.

Remark 2. In the majority of applications of the theorem, T will be identified with the physical operation of time inversion. In these circumstances $[M(T)]^2 = +1$ for co-representations with integer spin, and $[M(T)]^2 = -1$ for co-representations with half-integer spin.[7] Therefore cases RR and QQ occur only with integer spin, cases RQ and QR only with half-integer spin. Case $CC2$ may occur with either integer or half-integer spin.

V. EXAMPLES

The classification theory of Secs. III and IV would be empty if one could not produce examples to show that each of the enumerated possibilities can actually occur. We list here one example of each of the ten possibilities (61) to (70). The first five examples are factorizable and illustrate Theorem III. The last five are nonfactorizable.

To simplify the notations we write (2×2) matrices in terms of the standard basis

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad e_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$e_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad e_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{80}$$

The quaternion units are defined by Eqs. (26) and (27). The order of the co-representation is the order of the real matrices $M(g)$; this is twice the dimension of the complex vector space H_c.

Example 1. Case RR. Order 2.

G_1 contains identity I only. $G = [I, T]$, $T^2 = I$.

$$M(I) = I_2, \qquad M(i) = e_2,$$
$$M(T) = e_3, \qquad [M(T)]^2 = I_2.$$

Example 2. Case QR. Order 4.

G_1 generated by $[I, T^2]$, $G = [I, T, T^2, T^3]$ with $T^4 = I$.

$$M(I) = I_4, \qquad M(i) = \tau_2,$$
$$M(T) = \tau_1, \qquad [M(T)]^2 = -I_4.$$

Example 3. Case RQ. Order 4.

G is generated by the 3-dimensional rotation group O_3, with the time-inversion operator T commuting with O_3. Representation $M(u)$ is with spin $\frac{1}{2}$.

$$M(\bar{n}, \phi) = \exp\left[\tfrac{1}{2}\phi\bar{n}\cdot\bar{\tau}\right],$$
$$M(i) = \tau_2', \qquad M(T) = \tau_1', \qquad [M(T)]^2 = -I_4.$$

Example 4. Case QQ. Order 8.

Same group as example 3. Representation $M(u)$ has two spin-$\frac{1}{2}$ components which are interchanged by the T operator.

$$M(\bar{n}, \phi) = \exp\left[\tfrac{1}{2}\phi\bar{n}\cdot\bar{\tau}\right] \times I_2, \qquad M(i) = \tau_2' \times I_2,$$
$$M(T) = \tau_1' \times e_2, \qquad [M(T)]^2 = I_8.$$

Example 5. Case $CC2$. Order 4.

G is generated by the 2-dimensional rotation group 0_2 together with an operator T commuting with 0_2. The operator T is a combination of time-inversion with space reflection. The phase angle α is a fixed parameter.

$$M(\phi) = \cos\phi[I_2 \times I_2] + \sin\phi[e_3 \times e_2],$$
$$M(i) = I_2 \times e_2.$$
$$M(T) = \cos\tfrac{1}{2}\alpha[e_1 \times e_3] + \sin\tfrac{1}{2}\alpha[e_2 \times e_1],$$
$$[M(T)]^2 = \cos\alpha[I_2 \times I_2] + \sin\alpha[e_3 \times e_2] = M(\alpha).$$

Example 6. Case RC. Order 2.

G is generated by the 2-dimensional rotation group 0_2 with an operator T not commuting with 0_2. T is now time-inversion without space reflection.

$$M(\phi) = (\cos\phi)I_2 + (\sin\phi)e_2,$$
$$M(i) = e_2, \qquad M(T) = e_3.$$

Example 7. Case QC. Order 4.

Same group as example 6.

$$M(\phi) = (\cos\phi)I_4 + (\sin\phi)\tau_2,$$
$$M(i) = \tau_2, \qquad M(T) = \tau_1.$$

Example 8. Case CR. Order 4.

G_1 is a 4-element group generated by the reflections R_x and R_y in two perpendicular planes. $G = [G_1, TG_1]$, $T^2 = I$, where T is a combination of time-inversion with a reflection in the plane $x = y$.

$$M(I) = I_2 \times I_2, \qquad M(R_x) = e_3 \times I_2,$$
$$M(R_y) = -e_3 \times I_2,$$
$$M(i) = I_2 \times e_2, \qquad M(T) = e_1 \times e_3.$$

Example 9. Case CQ. Order 8.

G_1 is a direct product $[0_3 \times 0_3]$ of two 3-dimensional rotation groups. $G = [G_1, TG_1]$, where T interchanges the two groups.

$$M(\bar{m}, \phi; \bar{n}, \psi) = \left[\tfrac{1}{2}(I_2 + e_3)\right] \times \exp\left[\tfrac{1}{2}\phi\bar{m}\cdot\tau\right] + \left[\tfrac{1}{2}(I_2 - e_3)\right] \times \exp\left[\tfrac{1}{2}\psi\bar{n}\cdot\tau\right],$$
$$M(i) = I_2 \times \tau_2', \qquad M(T) = e_1 \times \tau_1'.$$

Example 10. Case $CC1$. Order 4.

$G_1 = [0_2 \times 0_2]$, $G = [G_1, TG_1]$ where T interchanges the two 0_2 groups.

$$M(\phi, \psi) = \left[\tfrac{1}{2}(I_2 + e_3)\right] \times [\cos\phi I_2 + \sin\phi e_2] + \left[\tfrac{1}{2}(I_2 - e_3)\right] \times [\cos\psi I_2 + \sin\psi e_2],$$

$$M(i) = I_2 \times e_2, \qquad M(T) = e_1 \times e_3.$$

The most interesting of these examples are numbers 4, 5, 8. They have some features which are nontrivial and appear to be novel. We leave to the reader the exercise of verifying that in each case the commutator algebras X, Y, Z have the structure described in Eqs. (47) to (52), (61) to (70).

VI. ALGEBRAIC CHARACTERIZATION OF REPRESENTATION TYPES

In this section we conclude the study of representation types by proving a generalized version of a classical theorem of Frobenius and Schur. Let $M(g)$ be a representation of a group G, irreducible over some ground field Φ with characteristic zero. We suppose that the group G is either finite or compact, and that the matrices $M(g)$ have a finite order d. If $f(g)$ is any function of the group element g, the average of $f(g)$ over G is defined by

$$\text{av}_g\, f(g) = h^{-1} \sum_g f(g), \tag{81}$$

or by

$$\text{av}_g\, f(g) = \nu^{-1} \int f(g)\, d\mu(g), \tag{82}$$

where h is the order of G when G is finite, and where ν is the volume of G in the invariant group measure $d\mu(g)$ when G is compact. We consider the fourth-rank tensor

$$P_{ij,kl} = \text{av}_g\, [M_{ij}(g^{-1})M_{kl}(g)]. \tag{83}$$

Let K be the group algebra generated by the $M(g)$ with coefficients in Φ.

The structure of K is given by Eq. (24), since Weyl's theorem holds in any field with characteristic zero. The commutator algebra L of K has the structure

$$L = I_t \times \bar{E}, \tag{84}$$

where $\bar{E}$ is an irreducible division algebra of order e over Φ, and $d = te$.

The *type* of the representation $M(g)$ is specified by the division algebra $\bar{E}$. For example, when $\Phi = R$ is the field of real numbers, there are three types of representation corresponding to $\bar{E} = R$, C, or Q. The tensor $P_{ij,kl}$ is useful in classifying representations by virtue of the following theorem.

Theorem IV. The tensor $P_{ij,kl}$ depends only on the integer t and on the algebra $\bar{E}$, and is otherwise independent of the group G and of the representation $M(g)$.

Thus $P_{ij,kl}$ is characteristic of the type of the representation $M(g)$.

To prove the theorem, let g' be any element of G. Then

$$\begin{aligned} P_{ij,kl} &= \text{av}_g\, [M_{ij}(g'^{-1}g^{-1})M_{kl}(gg')] \\ &= \sum_{mn} M_{im}(g'^{-1})P_{mj,kn}M_{nl}(g'). \end{aligned} \tag{85}$$

Thus $P_{ij,kl}$, considered as a matrix in the indices (i, l), commutes with all $M(g')$ and belongs to the commutator algebra L. Similarly, $P_{ij,kl}$ belongs to L when considered as a matrix in (k, j). Let e^μ, $\mu = 1, \cdots, e$, be a linearly independent basis for the algebra $\bar{E}$. Then Eq. (84) gives

$$tP_{ij,kl} = [(I_t)_{il}(I_t)_{kj}] \times \sum_{\mu\nu} c_{\mu\nu}(e^\mu)_{il}(e^\nu)_{kj}, \tag{86}$$

where the $c_{\mu\nu}$ are coefficients in Φ.

Now consider the sum

$$(\overset{\lambda}{\Sigma})_{jk} = \sum_{il} P_{ij,kl}[I_t \times e^\lambda]_{li}. \tag{87}$$

On the one hand, by Eq. (86),

$$(\overset{\lambda}{\Sigma})_{jk} = (I_t)_{kj} \times \sum_{\mu\nu} s_{\lambda\mu}c_{\mu\nu}(e^\nu)_{kj}, \tag{88}$$

where

$$s_{\lambda\mu} = \text{spur}\, [e^\lambda e^\mu]. \tag{89}$$

On the other hand, by Eq. (83), since e^λ commutes with all the $M(g)$,

$$\begin{aligned} (\overset{\lambda}{\Sigma})_{jk} &= \text{av}_g\, [M(g)(I_t \times e^\lambda)M(g^{-1})]_{kj} \\ &= (I_t \times e^\lambda)_{kj}. \end{aligned} \tag{90}$$

Comparison of Eqs. (88) and (90) shows that

$$\sum_\mu s_{\lambda\mu}c_{\mu\nu} = \delta_{\lambda\nu}, \tag{91}$$

so that the matrix $c_{\mu\nu}$ is the inverse of the matrix $s_{\lambda\mu}$. The coefficients $c_{\mu\nu}$ are thus uniquely determined by $\bar{E}$, and Eq. (86) establishes the truth of Theorem IV.

We shall be interested in applying Theorem IV to cases in which the matrices $M(g)$ are *orthogonal*. So we assume

$$M_{ij}(g^{-1}) = M_{ji}(g). \tag{92}$$

The algebra K then contains the transposed of every matrix in K, and L has the same property. We can therefore choose the basis elements e^λ of the algebra $\bar{E}$ to be either symmetric or antisymmetric. Suppose that the number of symmetric e^λ is q, and the number of antisymmetric e^λ is q'. The invariant

$$P = \sum_{ij} P_{ij,ij} \tag{93}$$

provides a simple criterion for the type of the representation $M(g)$.

Theorem V. When $M(g)$ is an orthogonal irreducible representation of G over a field Φ of characteristic zero,

$$P = \mathrm{av}_g \,\mathrm{spur}\, [M(g^2)] = q - q'. \tag{94}$$

The first part of Eq. (94) follows at once from the definition of P and Eq. (92). To prove the second part, we suppose the e^λ chosen so that

$$(e^\lambda)_{ki} = \eta_\lambda (e^\lambda)_{ik}, \tag{95}$$

with each η_λ equal to ± 1. Then Eqs. (86), (89), and (91) give

$$\begin{aligned} P &= \sum_{\mu\nu ij} c_{\mu\nu} (e^\mu)_{ij} \eta_\nu (e^\nu)_{ji} \\ &= \sum_{\mu\nu} c_{\mu\nu} s_{\nu\mu} \eta_\nu \\ &= \sum_\nu \eta_\nu = q - q'. \end{aligned} \tag{96}$$

Remark 1. Suppose that Φ is the field of real numbers. Then Theorem V gives the following characterization of the type of the representation $M(g)$:

$$P = +1 \text{ for } M(g) \text{ of type } R, \tag{97}$$

$$P = 0 \text{ for } M(g) \text{ of type } C, \tag{98}$$

$$P = -2 \text{ for } M(g) \text{ of type } Q. \tag{99}$$

Remark 2. We apply remark 1 to the situation discussed in Theorem I of Sec. III. Let $\Lambda(u)$ be an irreducible representation over C of a unitary group G_1. Then the corresponding real representation $M(u)$ splits into two equivalent irreducible representations $M'(u)$ when $M(u)$ is of type R, while $M(u)$ is irreducible when it is of type C or Q. The correspondence between $\Lambda(u)$ and $M(u)$ gives

$$\mathrm{spur}\, M(u) = 2 \,\mathrm{Re}\, \mathrm{spur}\, \Lambda(u), \tag{100}$$

and therefore the quantity

$$\Pi = \mathrm{av}_u \,[\mathrm{spur}\, \Lambda(u^2)] \tag{101}$$

is equal to $\frac{1}{2}P$. This P is given by Eqs. (98), (99) when $M(u)$ is of type C, Q, but is equal to $(+2)$ when $M(u)$ is of type R since Eq. (97) then refers to the irreducible component $M'(u)$. So we derive the classical criterion of Frobenius and Schur[17] for the type of an irreducible unitary representation:

$$\Pi = +1 \text{ for } \Lambda(u) \text{ potentially real}, \tag{102}$$

$$\Pi = 0 \text{ for } \Lambda(u) \text{ complex}, \tag{103}$$

$$\Pi = -1 \text{ for } \Lambda(u) \text{ pseudoreal}. \tag{104}$$

[17] G. Frobenius and I. Schur, reference 6.

Remark 3. We apply remark 1 to the situation discussed in Theorem II of Sec. III. Let $\Lambda(g)$ be an irreducible co-representation over C of a group G. According to remark 6 of Sec. II, the group algebra D is generated over R not by the group G itself but by an extended group Γ. The representation of Γ which generates D consists of the matrices

$$M(u), \quad iM(u), \quad M(a), \quad iM(a), \tag{105}$$

which are all real and orthogonal. When Theorem V is applied to the group Γ, the contributions from $M(u)$, $iM(u)$ to P cancel each other, while the contributions from $M(a)$, $iM(a)$ are equal. Thus

$$P = \tfrac{1}{2} \mathrm{av}_a \,\mathrm{spur}\, [M(a^2)], \tag{106}$$

averaged over the antiunitary part only of G. If $\Lambda(u)$ is the unitary part of the co-representation, $\Lambda(u)$ is irreducible when D is of type R, while $\Lambda(u)$ has two irreducible components when D is of type C or Q. In any case we let $\Lambda'(u)$ be one of the (one or two) irreducible components of $\Lambda(u)$, and we write

$$\Pi' = \mathrm{av}_a \,[\mathrm{spur}\, \Lambda'(a^2)]. \tag{107}$$

By Eq. (100), this Π' is equal to P when D is of type R, and is equal to $\frac{1}{2}P$ when D is of type C or Q. The criterion of Eqs. (97)–(99) then becomes

$$\Pi' = +1 \text{ for } \Lambda(g) \text{ of Wigner type I}, \tag{108}$$

$$\Pi' = 0 \text{ for } \Lambda(g) \text{ of Wigner type III}, \tag{109}$$

$$\Pi' = -1 \text{ for } \Lambda(g) \text{ of Wigner type II}. \tag{110}$$

This elegant analog to the Frobenius-Schur criterion was discovered by Bargmann.[18]

VII. THEORY OF MATRIX ENSEMBLES

In this section we deal with the problem for which the theory of Sec. II was specifically introduced, namely the classification of ensembles of matrices with given symmetry properties. An ensemble is a set of objects with an assigned probability distribution. We shall define the probability distributions later; it is necessary first of all to study the classification of sets of matrices invariant under some symmetry-group G.

As in Sec. II, we suppose that the matrices S which we are studying operate in a complex vector space H_c of finite dimension n. We are given a representation of the group G in H_c, consisting of unitary operators $\Lambda(u)$ and antiunitary operators $\Lambda(a)$. The matrices S are supposed to be invariant under G, but this notion of invariance already introduces an ambiguity. There is a choice between two

[18] V. Bargmann (private communication).

definitions of invariance. We say that S is "formally invariant under G" if

$$S\Lambda(g) = \Lambda(g)S, \quad \text{all} \quad g \quad \text{in} \quad G. \tag{111}$$

Formal invariance means that S is unchanged by any of the transformations

$$S \to \Lambda(g)S[\Lambda(g)]^{-1}, \tag{112}$$

whether g be unitary or antiunitary. We say that S is "physically invariant under G" if for every pair of vectors (ϕ, ψ) in H_c

$$(\phi, S\psi) = (\Lambda(u)\phi, S\Lambda(u)\psi) = (\Lambda(a)\psi, S\Lambda(a)\phi). \tag{113}$$

Note that the initial and final state vectors are interchanged in Eq. (113) in the case of antiunitary elements of G. The effect of Eq. (113) is that we have instead of Eq. (111)

$$S\Lambda(u) = \Lambda(u)S, \qquad S\Lambda(a) = \Lambda(a)S^{+}, \tag{114}$$

where S^{+} means the Hermitian conjugate of S.

The two types of invariance are relevant in different circumstances. If S is, for example, a unitary operator describing a change in the representation of states, then formal invariance under G is a meaningful requirement, signifying that this change in representation does not disturb the symmetry relations of the states under the operations of G. If S is an operator characterizing a physical system, for example a scattering matrix, then the antiunitary operations of G are associated with a reversal of the physical roles of initial and final states; in this case physical invariance of S is the physically meaningful requirement, signifying that the system to which S belongs is invariant under the operations of G in the usual dynamical sense. The two definitions of invariance under G become equivalent only when the matrix S is Hermitian, for example when S is the Hamiltonian of a system.

It is convenient to transcribe the matrix S into a real $(2n \times 2n)$ matrix operating in the real vector space H_R according to Eq. (2). The real form of S then satisfies

$$Si = iS, \tag{115}$$

with i defined by Eq. (5). For S to be invariant under the unitary subgroup G_1 (in either sense) it is necessary and sufficient that

$$SM(u) = M(u)S, \qquad u \quad \text{in} \quad G_1, \tag{116}$$

where the matrices $M(g)$ are the representation of G defined in Sec. II. The condition for S to be formally invariant under G is

$$SM(a) = M(a)S, \qquad a \quad \text{in} \quad G, \tag{117}$$

in addition to Eq. (116). The condition for S to be physically invariant under G is Eq. (116) and

$$SM(a) = M(a)S^{T}, \qquad a \quad \text{in} \quad G, \tag{118}$$

where S^T means the transpose of S.

From Eqs. (115), (116) we see that the set Y of matrices in H_R invariant under G_1 is identical with the commutator algebra Y defined by Eqs. (9) and (12). From Eqs. (115)–(117), the set Z of matrices formally invariant under G is identical with the commutator algebra Z defined by Eqs. (10) and (13). We define the set W to consist of those matrices which are both formally and physically invariant under G. Then W is the set of all symmetric matrices in Z. Lastly, we define V to be the set of matrices physically invariant under G. Then we shall prove

Theorem VI. For S to be in V, it is necessary and sufficient that

$$S = S_1 + iS_2, \tag{119}$$

where S_1 and S_2 are matrices in W.

The sufficiency follows immediately from the relations

$$i^T = -i, \qquad iM(a) = -M(a)i. \tag{120}$$

To prove the necessity, we observe that all the matrices $M(g)$ are orthogonal, and thus

$$M(g^{-1}) = [M(g)]^T. \tag{121}$$

Hence S^T belongs to V whenever S does, and we may then write

$$S = S' + S'', \tag{122}$$

where S' is symmetric and S'' antisymmetric, and both S', S'' belong to V. The matrices $S_1 = S'$ and $S_2 = -iS''$ now satisfy both Eq. (117) and Eq. (118), and therefore belong to W.

The results of Sec. II, and in particular Weyl's theorem, provide us with a complete structural analysis of the sets V, W, Y, Z. We use Frobenius' theorem (Sec. I) in order to replace the division algebras E_i of Weyl's theorem by the standard trio R, C, and Q. The integers t_i of Weyl's theorem are now irrelevant since they contribute to the structure of the group algebra D but not to the commutator algebra Z. We thus state the main result of the theory of matrix ensembles as follows.

Theorem VII. The set Z of matrices in H_R formally invariant under G is a direct product of irreducible components, one component Z_i corresponding to each inequivalent irreducible co-representation of G contained in the given co-representation $\Lambda(g)$. Each com-

ponent Z_j may be written as the set of all square matrices of order s_j with elements in an algebra Φ_j. Each Φ_j is either R, the algebra of real numbers, or C, the algebra of complex numbers, or Q, the algebra of real quaternions.

Remark 1. The structure of the set Z_j depends on the Wigner type of the corresponding co-representation of G in the manner specified by equivalence Theorem II. The reality type of the unitary part of the representation, specified by equivalence Theorem I, is here entirely irrelevant, except insofar as the Wigner type and the reality type may be correlated for factorizable groups G according to Theorem III.

According to remark 7 at the end of Sec. II, the sets V, W, Y are direct products of independent components, one corresponding to each component Z_j of Z. To avoid unnecessary repetition, we describe the structure of V, W, Y corresponding to a single component of Z. Thus in the following theorems we assume that Z is irreducible, which means that all irreducible co-representations contained in $\Lambda(g)$ are equivalent. From this special case the general case is easily derived by writing Z_j for Z and taking a direct product over j.

We have seen, in Remarks 5 and 6 of Sec. III, that the algebra Y is generated by i and Z. The matrix i commutes with Z, and therefore commutes with the algebras Φ_j. Hence we may form a new algebra Φ_j^c by adding the independent unit i to Φ_j.

Theorem VIII. When Z is irreducible, the set Y of matrices in H_R invariant under G_1 may be written as the set of all square matrices of order s with elements in an algebra Φ^c, derived from Φ by allowing each element of Φ to have complex instead of real coefficients.

Remark 2. When $\Phi = R$, Φ^c is the algebra of ordinary complex numbers. When $\Phi = C$, Φ^c is the algebra of complex-complex numbers with two commuting imaginary units; in this case Φ^c is reducible and has the structure

$$\Phi^c \sim C + C. \tag{123}$$

When $\Phi = Q$, Φ^c is the algebra of complex quaternions, which is equivalent to an algebra of complex (2×2) matrices,

$$\Phi^c \sim C_2. \tag{124}$$

The algebra W consists of matrices which are symmetric when written in expanded form in H_R. When S is written, as in Theorems VII and VIII, as a smaller matrix with elements in Φ, the condition of symmetry becomes a condition of Φ duality, as follows. We define the Φ conjugate of a number in Φ to be the number obtained by reversing the signs of the coefficient of e_2 (in the case $\Phi = C$) or of the coefficients of τ_1, τ_2, τ_3 (in the case $\Phi = Q$). We define the Φ dual of a matrix to be the transposed matrix with each element Φ conjugated. Since the units e_2, τ_1, τ_2, τ_3 when written in expanded form are antisymmetric, a matrix which is symmetric in expanded form becomes Φ self-dual when written with elements in Φ.

Theorem IX. When Z is irreducible, the set W of matrices in H_R invariant under G in both physical and formal senses may be written as the set of all square self-dual matrices of order s with elements in $\Phi = R$, C, or Q.

The Φ conjugate of an element of Φ^c is obtained by changing the signs of the coefficients of the Φ units, leaving the unit i unchanged. So from Theorems VI and IX follows immediately the result:

Theorem X. When Z is irreducible, the set V of matrices in H_R invariant under G in the physical sense may be written as the set of all square self-dual matrices of order s with elements in Φ^c.

Remark 3. We now finally make contact with the theory of matrix ensembles developed earlier by the author.[4] Let V_U be the subset of unitary matrices in V. Then Theorem X states that, for the most general symmetry group G and the most general quantum-mechanical representation of G, the set V_U is a direct product of independent components, each of which is identical with one of the three ensemble-spaces T_1, T_2, and T_4 defined in reference 4. The cases T_1, T_2, T_4 correspond, respectively, to $\Phi = R$, C, Q. The spaces T_1, T_2, and T_4 were originally obtained by considering special groups G of a very simple kind. It is satisfactory to find that the same three spaces, and no others, occur in all possible circumstances.

The reason for choosing V_U as the space in which to construct an ensemble is that no natural definition of uniform probability appears to exist in V. For the same reason we study the subset Z_U of unitary matrices in Z. The following theorem follows from Theorem VII together with well-known properties of the classical groups.[2]

Theorem XI. The set Z_U of unitary matrices in H_R formally invariant under G is a direct product of irreducible components, each of which is a simple classical group. When $\Phi_j = R$, C, or Q, the corresponding component of Z_U is an orthogonal, unitary, or symplectic group of dimension s_j.

In the same way we define the unitary subset Y_U of Y. The components of Y_U are

$$Y_U = U(s), \qquad U(s) \times U(s), \qquad U(2s), \tag{125}$$

corresponding to

$$Z_U = 0(s), \qquad U(s), \qquad Sp(2s). \tag{126}$$

The unitary space V_U is not a group. But it can be represented conveniently in terms of the groups Y_U, Z_U in the following way. A matrix S belongs to V_U if and only if it can be expressed as a Φ-symmetric product

$$S = UU^D, \qquad U \text{ in } Y_U, \tag{127}$$

where D denotes Φ dual. All matrices U' of the form

$$U' = UU_1, \qquad U_1 \text{ in } Z_U, \tag{128}$$

correspond to the same S by Eq. (127), and every U' corresponding to S is of the form (128). Thus each matrix S in V_U corresponds to a unique co-set of the subgroup Z_U in the group Y_U. We have thus proved

Theorem XII. The set V_U of unitary matrices in V is abstractly equivalent to the homogeneous space (Y_U/Z_U), the quotient of the group Y_U by its subgroup Z_U.

Having defined the spaces Z_U and V_U, we are now in a position to define the corresponding invariant matrix ensembles. The ensemble E^F of unitary matrices formally invariant under G is defined as the space Z_U with probability distribution given by the invariant group measure in Z_U. Since Z_U is a direct product of simple classical groups, the group measure in Z_U is merely the product of the invariant measures in the irreducible components of Z_U. The ensemble E^P of unitary matrices physically invariant under G is defined as the space V_U with measure given according to Theorem XII by

$$d\mu(V_U) = [d\mu(Y_U)/d\mu(Z_U)]. \tag{129}$$

Here $d\mu(Y_U)$ and $d\mu(Z_U)$ are the invariant group measures in Y_U and Z_U, and the quotient measure is defined in the obvious way. Alternatively, the quotient measure may be uniquely defined as the measure in V_U which is invariant under all automorphisms

$$S \to USU^D, \qquad U \text{ in } Y_U, \tag{130}$$

of V_U into itself. The ensemble E^P is a direct product of irreducible components, each of which is identical with one of the three types E_1, E_2, E_4 which were studied in reference 4.

Two other types of ensemble naturally suggest themselves for study, composed of Hermitian and anti-Hermitian matrices, respectively. A matrix S of V_U is of the form

$$S = \exp[iH], \qquad H \text{ in } W, \tag{131}$$

while a matrix S of Z_U is of the form

$$S = \exp[A], \qquad A \text{ in } Z_A, \tag{132}$$

where Z_A is the subset of Z containing anti-Hermitian matrices. Thus W and Z_A are the spaces of infinitesimal generators for V_U and Z_U, respectively.

We define the Hermitian Gaussian ensemble E^H as the space W of matrices H with the probability distribution

$$d\mu(H) = C \exp[-(\text{spur } H^2)/4a^2] \prod dH_{ij}^{\alpha}, \tag{133}$$

where c, a are constants and the product extends over all the independent real coefficients of the elements of H in the algebra Φ. The anti-Hermitian Gaussian ensemble E^H is defined as the space Z_A with probability distribution

$$d\mu(A) = C \exp[+(\text{spur } A^2)/4a^2] \prod dA_{ij}^{\alpha}. \tag{134}$$

These ensembles have an algebraic structure precisely analogous to that of E^P and E^F, respectively. They divide into irreducible components each of which is of one of the three types R, C, or Q. In particular, E^H is the natural ensemble to use in describing the statistical properties of the Hamiltonian H of a system known to be physically invariant under the group G.

The physical motivation for considering ensembles of matrices with probability distributions defined in these various ways has been discussed by Wigner[19] and by the author.[4] In the case of the ensembles E^F and E^P, consisting of unitary matrices, the existence of a natural uniform measure provides an intuitively plausible definition of "equal *a priori* probability." In the case of the ensembles E^H and E^A, consisting of Hermitian and anti-Hermitian matrices, the choice of a Gaussian probability distribution is mainly a matter of mathematical convenience. Rosenzweig[20] has argued that one should use in preference to Eq. (133) a "microcanonical ensemble" with the exponential replaced by a delta function

$$\delta[\text{spur }(H^2) - \sigma^2].$$

The algebraic structure of E^H and E^A would of course not be affected by such a change.

In any physical situation to which the ensembles E^P or E^H are relevant, we have a system specified by a unitary operator S or by a Hermitian H.

[19] E. P. Wigner, *Proceedings of the 4th Canadian Mathematics Congress* (University of Toronto Press, Toronto, Canada, 1959), p. 174.
[20] N. Rosenzweig, Bull. Am. Phys. Soc. 7, 91 (1962).

Since the system is invariant under G, every stationary state is associated with a particular irreducible co-representation of G. Each irreducible co-representation fixes the values of a certain set of quantum numbers (spin, parity, isotopic spin, etc.) which are attached to the energy levels belonging to that co-representation. The fact that the ensemble E^P or E^H is a direct product of irreducible components means that the energy levels belonging to different sets of quantum numbers are statistically uncorrelated. Thus the statistical properties of energy levels are entirely determined by the behavior of the individual level-series, each associated with one set of quantum numbers. A single level-series is described by an irreducible ensemble. The final result of our analysis may then be stated as follows: When we consider a single series of energy levels of a complex system, having definite values for all quantum numbers of the symmetry-group G, the statistical behavior of these levels follows one of three possible laws, corresponding to the three types of irreducible ensemble E^P or E^H.

VIII. EIGENVALUE DISTRIBUTIONS

In this section we list without proof the joint probability distributions of the eigenvalues of matrices belonging to the irreducible ensembles E^F, E^A, E^P, E^H. In each case the integer s is the dimension of the algebra Z over the field Φ which may be R, C, or Q. The constant c will not be the same each time it appears.

1. E^F. *Ensemble of Unitary Matrices Formally Invariant under G*

(α) $\Phi = R$, $Z_U = O(s)$. In this case Z_U (the orthogonal group) splits into two disconnected parts, consisting of matrices with determinant Δ equal to $+1$ and -1, respectively. There are thus four distinct eigenvalue distributions to be listed.

(i) $s = 2n$, $\Delta = 1$, eigenvalues $\exp(\pm i\theta_i)$,

$$P(\theta_1, \cdots, \theta_n) = c \prod_{i<j} [\cos\theta_i - \cos\theta_j]^2. \quad (135)$$

(ii) $s = 2n$, $\Delta = -1$, eigenvalues ± 1, $\exp(\pm i\theta_i)$,

$$P\ \theta_1, \cdots, \theta_{n-1}) = c \prod_j (1 - \cos^2\theta_j) \times \prod_{i<j} [\cos\theta_i - \cos\theta_j]^2. \quad (136)$$

(iii) $s = 2n + 1$, $\Delta = 1$, eigenvalues $+1$, $\exp(\pm i\theta_i)$,

$$P(\theta_1, \cdots, \theta_n) = c \prod_j (1 - \cos\theta_j) \times \prod_{i<j} [\cos\theta_i - \cos\theta_j]^2. \quad (137)$$

(iv) $s = 2n + 1$, $\Delta = -1$, eigenvalues -1, $\exp(\pm i\theta_i)$,

$$P(\theta_1, \cdots, \theta_n) = c \prod_j (1 + \cos\theta_j) \times \prod_{i<j} [\cos\theta_i - \cos\theta_j]^2. \quad (138)$$

(β) $\Phi = C$, $Z_U = U(s)$, eigenvalues $\exp(i\theta_j)$.

$$P(\theta_1, \cdots, \theta_s) = c \prod_{i<j} |\exp(i\theta_i) - \exp(i\theta_j)|^2. \quad (139)$$

(γ) $\Phi = Q$, $Z_U = \mathrm{Sp}(2s)$, eigenvalues $\exp(\pm i\theta_j)$.

$$P(\theta_1, \cdots, \theta_s) = c \prod_j (1 - \cos^2\theta_j) \times \prod_{i<j} [\cos\theta_i - \cos\theta_j]^2. \quad (140)$$

2. E^A. *Gaussian Ensemble of Anti-Hermitian Matrices Formally Invariant under G*

(α) $\Phi = R$, matrices real and antisymmetric.

(i) $s = 2n$, eigenvalues $\pm iE_j$,

$$P(E_1, \cdots, E_n) = c\Big[\prod_{i<j} (E_i^2 - E_j^2)^2\Big] \times \exp\Big[-\sum_j E_j^2/2a^2\Big]. \quad (141)$$

(ii) $s = 2n + 1$, eigenvalues 0, $\pm iE_j$,

$$P(E_1, \cdots, E_n) = c\Big[\prod_j E_j^2\Big]\Big[\prod_{i<j} (E_i^2 - E_j^2)^2\Big] \times \exp\Big[-\sum_j E_j^2/2a^2\Big]. \quad (142)$$

(β) $\Phi = C$. Eigenvalues iE_j.

$$P(E_1, \cdots, E_s) = c\Big[\prod_{i<j} (E_i - E_j)^2\Big] \times \exp\Big[-\sum_j E_j^2/4a^2\Big]. \quad (143)$$

(γ) $\Phi = Q$. Eigenvalues $\pm iE_j$.

$$P(E_1, \cdots, E_s) = c\Big[\prod_j E_j^2\Big]\Big[\prod_{i<j} (E_i^2 - E_j^2)^2\Big] \times \exp\Big[-\sum_j E_j^2/2a^2\Big]. \quad (144)$$

3. E^P. *Ensemble of Unitary Matrices Physically Invariant under G*

Eigenvalues $\exp(i\theta_j)$, each doubly degenerate in the case $\Phi = Q$.

$$P(\theta_1, \cdots, \theta_s) = c \prod_{i<j} |\exp(i\theta_i) - \exp(i\theta_j)|^\beta, \quad (145)$$

with $\beta = 1, 2, 4$ for $\Phi = R, C, Q$, respectively.

4. E^H. Gaussian Ensemble of Hermitian Matrices Invariant (in either sense) under G

Eigenvalues E_i, each doubly degenerate in the case $\Phi = Q$.

$$P(E_1, \cdots, E_s) = c[\prod_{i<j} |E_i - E_j|^\beta] \times \exp[-\sum_j E_j^2/4a^2], \qquad (146)$$

with $\beta = 1, 2, 4$ for $\Phi = R, C, Q$.

Proofs of Eqs. (135) to (140) are to be found in Chapter 7 of Weyl's book.[2] Equations (141) to (144) can be deduced as limiting cases of Eqs. (135) to (140) when all angles θ_i are small. Similarly Eq. (146) can be deduced from Eq. (145). The proof of Eq. (145) has been given by the author.[4]

The statistical properties of the eigenvalues resulting from each of these ensembles can be studied by following the method used by the author[4] for the case of Eq. (145). The eigenvalue distribution in each ensemble has an exact mathematical analog in the form of a classical Coulomb gas.

We briefly describe the Coulomb gas analogs to E^F and E^A when $\Phi = R$ or Q. In E^F the numbers

$$x_j = \cos\theta_j \qquad (147)$$

are considered to be positions of unit charges, constrained to move on the segment $[-1 \le x \le +1]$, which may be imagined to be a straight conducting wire of length 2. Every two charges repel each other with the potential

$$W(x_i - x_j) = -\ln|x_i - x_j|. \qquad (148)$$

In addition there are fixed charges of q_+ units at $x = +1$ and of q_- units at $x = -1$. When $\Phi = R$, the angles θ_j are rotation angles of a random rotation in the orthogonal group $O(s)$. The values of q_+, q_- are

(i) $s = 2n, \Delta = 1; q_+ = q_- = -\frac{1}{4}$.

(ii) $s = 2n, \Delta = -1; q_+ = q_- = +\frac{1}{4}$.

(iii) $s = 2n + 1, \Delta = 1; q_+ = +\frac{1}{4}, q_- = -\frac{1}{4}$.

(iv) $s = 2n + 1, \Delta = -1; q_+ = -\frac{1}{4}, q_- = +\frac{1}{4}$.

When $\Phi = Q$, the angles θ_j are rotation angles of a random matrix in the symplectic group $\mathrm{Sp}(2s)$. In this case $q_+ = q_- = +\frac{1}{4}$. The temperature of the gas is the same for $\Phi = R$ or Q, namely, $T = \frac{1}{2}$.

The Gaussian antisymmetric ensembles E^A for $\Phi = R$ or Q have a Coulomb analog composed of unit charges with positions

$$x_j = E_j^2, \qquad (149)$$

constrained to move on the semi-infinite straight wire $0 \le x < \infty$. The repulsion between charges is again given by Eq. (148), and $T = \frac{1}{2}$ as before. There is a fixed charge of q units at $x = 0$, where

$$q = -\tfrac{1}{4} \quad \text{when} \quad \Phi = R, \qquad s = 2n,$$

$q = +\frac{1}{4}$ when $\Phi = R, s = 2n + 1$, or when $\Phi = Q$. In addition to the Coulomb forces, each charge x_j is subject to a constant downward force produced by a "gravitational potential"

$$V(x) = [x/4a^2]. \qquad (150)$$

When $\Phi = C$, the ensembles E^F and E^A become identical with E^P and E^H, for which the Coulomb analogs have been described previously.[4]

The whole of the previous analysis[4] of level distributions, based on the ensembles E^P, can be repeated with minor modifications for the other ensembles E^F, E^A, E^H. However, there is one basic difference between the physical ensembles E^P, E^H on the one hand and the formal ensembles E^F, E^A on the other.

Theorem XIII. Consider an irreducible ensemble of matrices over the field Φ, with order $s \to \infty$. In E^F or E^A, the local statistical behavior of eigenvalues is described by an infinite Coulomb gas with temperature $T = \frac{1}{2}$ independent of Φ. In E^P or E^H the local behavior of eigenvalues is described by an infinite Coulomb gas with temperature $T = 1, \frac{1}{2}, \frac{1}{4}$ corresponding to $\Phi = R, C, Q$.

The most striking qualitative feature of the physical ensembles E^P, E^H is that the strength of the repulsion between neighboring energy levels depends on the Wigner type of the co-representation to which these levels belong. This feature is absent in the formal ensembles E^F, E^A.

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