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CHARACTERISTIC VECTORS OF BORDERED MATRICES WITH INFINITE DIMENSIONS

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Introduction

The statistical properties of the characteristic values of a matrix the elements of which show a normal (Gaussian) distribution are well known (cf. [6] Chapter XI) and have been derived, rather recently, in a particularly elegant fashion.[1] The present problem arose from the consideration of the properties of the wave functions of quantum mechanical systems which are assumed to be so complicated that statistical considerations can be applied to them. Since the physical problem has been given rather recently in some detail in another journal [3], it will not be reviewed here. Actually, the model which underlies the present calculations shows only a limited similarity to the model which is believed to be correct. Nevertheless, the calculation which follows may have some independent interest; it certainly provided the encouragement for a detailed investigation of the model which may reproduce some features of the actual behavior of atomic nuclei.

All the remaining work will deal with real symmetric matrices of very high dimensionality. The first and last problems concern infinite bordered matrices; the second one a finite matrix the consideration of which served as an intermediate step toward the solution of the last one. We mean by a bordered matrix the sum of a diagonal matrix **k** and a border matrix **v**. The diagonal elements of **k** are all the integers $\cdots, -2, -1, 0, 1, 2, \cdots$. The border matrix **v** has non vanishing elements only up to a distance N from the diagonal, the absolute value of all the non vanishing elements is the same

$$(1)\qquad \begin{aligned} |v_{mn}| &= v \qquad &&\text{for } |m-n| \leqq N, \; (-\infty < m, n < \infty) \\ &= 0 &&\text{for } |m-n| > N. \end{aligned}$$

Since the matrix $H = \mathbf{k} + \mathbf{v}$ is symmetric, $v_{mn} = v_{nm}$. Subject to this condition, however, the signs of the v_{ij} are random, i.e. we consider ensembles of matrices with all possible signs of v_{mn} subject to the conditions of symmetry. In the first of the problems considered $N = 1$, in the third one both N and v are very large in such a way, however, that $v^2/N = q$ remains limited. The first problem will be solved completely, i.e. the characteristic values and vectors given explicitly.

In order to formulate the third problem, we denote the characteristic values and vectors of an H by λ and $\psi^{(\lambda)}$

$$(2)\qquad H\psi^{(\lambda)} = \lambda\psi^{(\lambda)} \quad \text{or} \quad \sum_n H_{mn}\psi_n^{(\lambda)} = \lambda\psi_m^{(\lambda)}.$$

[1] Personal communication of Professor V. Bargmann.

We shall inquire then for the expectation value of those

$$(\psi_0^{(\lambda)})^2 \tag{3}$$

the λ of which is in unit interval at x which will be considered to be a continuous variable. One can imagine this to be calculated by solving the characteristic value problem (2) for each of the permissible sign combinations of the v_{mn}, choosing those which have a characteristic value λ between x and $x + \delta$, calculating (3) for these H and λ, adding the expressions obtained in this way and dividing the result by δ times the number of all H considered. The expectation value obtained in this way will be denoted by $\sigma(x)$, it is the quantity which we inquired for. An integral equation will be obtained for σ and some of its properties, but not σ itself, will be obtained explicitly. However, the mathematical analysis which leads to the integral equation for σ will lack rigor; in particular the convergence of the procedure which defines σ will not be proved. Many important statistical properties of the characteristic functions and characteristic values of H can easily be obtained from σ. The fact that the density of the characteristic values of σ is 1 follows easily from the invariance properties of the set of permissible H.

The second problem concerns a finite symmetric matrix H_0 albeit of very large dimension. Its diagonal elements are zero, the off-diagonal elements $| v_{mn} | = v$ are real but have random signs as before, m and n assume only the values $-N, -N + 1, \cdots, N - 1, N$ so that this H is $2N + 1$ dimensional. The problem is again to calculate $\sigma(x)$ as defined above for the third problem. The averages have to be taken in this case only over $2^{N(2N+1)}$ matrices and the fact that $\sigma(x/(N)^{1/2}\, v)$, properly normalized, converges to a limiting function can be proved in this case. This limiting function will be explicitly determined. It also gives the density of the characteristic values of H_0.

Singly bordered symmetric matrix

We shall calculate first the characteristic vectors of

$$H_1 = \left\| \begin{matrix} & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdots & -2 & v_{-1-2} & 0 & 0 & 0 & \cdots \\ \cdots & v_{-1-2} & -1 & v_{-10} & 0 & 0 & \cdots \\ \cdots & 0 & v_{-10} & 0 & v_{01} & 0 & \cdots \\ \cdots & 0 & 0 & v_{01} & 1 & v_{12} & \cdots \\ \cdots & 0 & 0 & 0 & v_{12} & 2 & \cdots \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \end{matrix} \right\| \tag{4}$$

in which the absolute values of all v are the same. Neither the characteristic values nor the squares of the components of the characteristic vectors will be changed if H_1 is transformed by a diagonal matrix s all the diagonal elements of which are $s_k = \pm 1$. Such a transformation will leave the diagonal elements of (4) unchanged but will replace v_{kk+1} by $s_k s_{k+1} v_{kk+1}$. By choosing $s_0 = 1$, s_k to have the sign of $-v_{k-1,k}$ for $k > 0$ and the sign of $-v_{k,k+1}$ for $k < 0$, all non diagonal elements will become negative, their value will be denoted by $-v$.

The resulting matrix $\hat{H}_1$ can be transformed into $\hat{H}_1 + 1$ by renumbering the rows and columns. Hence, the set of characteristic values λ_k will be unchanged if one replaces each λ_k by $\lambda_k + 1$. Furthermore, $\hat{H}_1$ can be transformed into $-\hat{H}_1$ by interchanging the k^{th} and $-k^{\text{th}}$ rows and columns and transforming it by an s with $s_k = 1$ for even k and $s_k = -1$ for odd k. Hence the set of characteristic values contains, with every λ_k, also $-\lambda_k$. Since the λ_k will change continuously as v is increased and since for $v = 0$ the set of λ_k consists of all integers, this will be true for all v. Without loss of generality, one can set

(4a) $$\lambda_k = k.$$

The characteristic vector of k shall be denoted by $\psi^{(k)}$. It then follows from the remarks about transforming $\hat{H}_1$ into $\hat{H}_1 + 1$ that $\psi_l^{(k)} = \psi_{l+1}^{(k+1)}$ i.e. the l^{th} component of $\psi^{(k)}$ depends only on the difference $l - k$

(4b) $$\psi_l^{(k)} = \psi_{l-k}^{(0)} = \psi_{l-k}$$

where, for simplicity $\psi^{(0)}$ has been replaced by ψ. It finally follows from the remark about the transformation of H_1 into $-H_1$ that

$$\psi_{-l}^{(0)} = c(-)^l \psi_l^{(0)}.$$

Clearly $c = \pm 1$ and again by continuity from the $v = 0$ case, $c = 1$. This gives

(4c) $$\psi_{-l} = (-)^l \psi_l .$$

The equation $\hat{H}_1 \psi = 0$ reads explicitly

(5) $$-v\psi_{l-1} + l\psi_l - v\psi_{l+1} = 0.$$

This reminds one of the recursive formula for Bessel functions, ([5], 17.21, p. 359)

(5a) $$-J_{l-1}(z) + (2l/z)J_l(z) - J_{l+1}(z) = 0.$$

Hence it follows that one can set

(5b) $$\psi_l = J_l(2v) = \psi_{l+k}^{(k)} .$$

This satisfies all conditions since for large l

(5c) $$\psi_l \approx v^l/l! \approx (ev)^l l^{-l},$$

ψ_k goes to zero fast enough for $\sum \psi_l^2 < \infty$. We note that the orthogonality of $\psi^{(0)}$ and $\psi^{(k)}$ gives

(6) $$\sum_l J_l(2v) J_{l-k}(2v) = 0 \qquad k \neq 0.$$

The summation runs over all integers from $-\infty$ to ∞. The essential result of the calculation is (5c) which shows how ψ_l decreases as $l \to \infty$. Because of (4c), the decrease is essentially the same as $l \to -\infty$.

It should be noted that one can obtain a formal solution of the equation $H_1\varphi = \lambda\varphi$ for any λ. It is necessary for this purpose only to set

$$\varphi = Z_{l-\lambda}(2v)$$

where the $Z_l = \alpha J_l + \beta N_l$ are all the same linear combinations of regular and irregular Bessel functions. However, the φ obtained in this way cannot be normalized.

Preliminary remarks on the strength function

We shall consider, below, very large sets of real symmetric matrices. Each of these will have certain characteristic values λ; the corresponding real normalized characteristic vector will be denoted by $\psi^{(\lambda)}$, its components by $\psi_k^{(\lambda)}$. The orthogonality relations of the $\psi_k^{(\lambda)}$ are

$$\sum_\lambda \psi_k^{(\lambda)}\psi_l^{(\lambda)} = \delta_{kl}\,; \qquad \sum_k \psi_k^{(\lambda)}\psi_k^{(\mu)} = \delta_{\lambda\mu}\,. \tag{7}$$

We next define a function $S(x)$ as follows. Let us consider all the characteristic values λ of all the matrices of the set which are below x. Their number shall be denoted by $\mathfrak{N}\Lambda(x)$ where $\mathfrak{N}$ is the number of matrices in the set. Let us consider then the sum of the squares of a particular component $k = 0$ of all the characteristic vectors $\psi^{(\lambda)}$ for which $\lambda < x$. Then

$$\mathfrak{N}S(x) = \sum_{\text{set}} \sum_{\lambda<x} (\psi_0^{(\lambda)})^2, \tag{8}$$

the summation is extended over all matrices of the set and all the characteristic vectors of these which satisfy the condition $\lambda < x$. The right side is clearly independent of the sign and other arbitrarinesses in the definition of the characteristic vectors should multiple characteristic values occur. Because of the normalization relations (7), if $x \to \infty$, the sum over λ gives 1 for each matrix separately and one finds

$$S(\infty) = 1. \tag{8a}$$

We shall calculate now the moments

$$M_\nu = \int_{-\infty}^{\infty} x^\nu \, dS(x) = \mathfrak{N}^{-1} \sum_{\text{set}} \sum_\lambda \lambda^\nu (\psi_0^{(\lambda)})^2 \tag{9}$$

the summation is to be extended over all characteristic vectors of all matrices of the set. If a typical matrix of the set is denoted by $H = (H_{mn})$ one can write

$$\sum_n H_{mn}\psi_n^{(\lambda)} = \lambda\psi_m^{(\lambda)}$$

and repeated application of H to both sides gives

$$\sum_n (H^\nu)_{mn}\psi_n^{(\lambda)} = \lambda^\nu\psi_m^{(\lambda)}.$$

Multiplication with $\psi_k^{(\lambda)}$ and summation over λ yields by means of (7) the well known equation

(9a) $$(H^\nu)_{mk} = \sum_\lambda \lambda^\nu \psi_m^{(\lambda)} \psi_k^{(\lambda)}.$$

Setting $m = k = 0$ herein and summing over all matrices of the set gives

(9b) $$M_\nu = \mathfrak{N}^{-1} \sum_{\text{set}} (H^\nu)_{00} = \text{Av}(H^\nu)_{00}\ .$$

Av will denote the average of the succeeding expression over all matrices of the set.

The M_ν will be calculated in the following section for a certain set of matrices in the limiting case that the dimension $2N + 1$ of these matrices becomes infinite. It will be shown, then, that $S(x)$, which is a step function for every finite N, becomes a differentiable function and its derivative $S'(x) = \sigma(x)$ will be called the strength function. In the last section, infinite sets of infinite matrices will be considered. However, all powers of these matrices will be defined and $(H^\nu)_{00}$ involves, for every ν, only a finite part of the matrix. It will be seen that the definition of the average of this quantity for the infinite set of H does not involve any difficulty. However, a similar transition to a limiting case $N \to \infty$ will be carried out with this set as with the aforementioned set and this transition will not be carried through in a rigorous manner in either case.

The expression "strength function" originates from the fact that the absorption of an energy level depends, under certain conditions, only on the square of a definite component of the corresponding characteristic vector. This component was taken, in (8), to be the 0 component. Hence $S(x_1) - S(x_2)$ is the average strength of absorption by all energy levels in the $(x_1\ , x_2)$ interval.

Random sign symmetric matrix

The matrices to be considered are $2N + 1$ dimensional real symmetric matrices; N is a very large number. The diagonal elements of these matrices are zero, the non diagonal elements $v_{ik} = v_{ki} = \pm v$ have all the same absolute value but random signs. There are $\mathfrak{N} = 2^{N(2N+1)}$ such matrices. We shall calculate, after an introductory remark, the averages of $(H^\nu)_{00}$ and hence the strength function $S'(x) = \sigma(x)$. This has, in the present case, a second interpretation: it also gives the density of the characteristic values of these matrices. This will be shown first.

Let us consider one of the above matrices and choose a characteristic value λ with characteristic vector $\psi^{(\lambda)}$. Clearly, λ will be a characteristic value also of all those matrices which are obtained from the chosen one by renumbering rows and columns. However, the components $\psi_k^{(\lambda)}$ of the corresponding characteristic vectors will be all possible permutations of the components of the original matrix' characteristic vector. It follows that if we average $(\psi_k^{(\lambda)})^2$ over the aforementioned matrices, the result will be independent of k. Because of the normalization condition (7), it will be equal to $1/(2N + 1)$.

Let us denote now the average number of characteristic values of the matrices of our set which are smaller than x by $\Lambda(x)$. More accurately, this is the number of all characteristic values of all matrices of our set which are smaller than x, divided by the number of matrices in our set. Clearly $\Lambda(-\infty) = 0$, $\Lambda(\infty) = 2N + 1$. Since every characteristic value λ which is common to n_λ of the matrices contributes $n_\lambda/\mathfrak{N}$ to $\Lambda(x)$ if $\lambda < x$ and since it contributes $\mathfrak{N}^{-1}n_\lambda/(2N + 1)$ to $S(x)$, we have

$$\Lambda(x) = (2N + 1)S(x). \tag{10}$$

This is true for any value of N, as $N \to \infty$, the step functions $\Lambda(x)$ and $S(x)$ become differentiable and the density of characteristic values per unit interval becomes

$$\Lambda'(x) = (2N + 1)\sigma(x). \tag{10a}$$

Calculation of the moments

We now proceed with the calculation of the average of $(H^\nu)_{kk}$ or $(H^\nu)_{00}$ since the former quantity is independent of k. We first note that the diagonal elements of H^2 are

$$\sum_k v_{0k}v_{k0} = \sum v_{0k}^2 = 2Nv^2. \tag{11}$$

Hence the average of this quantity is also

$$M_2 = 2Nv^2. \tag{11a}$$

Let us consider now the expression for the ν^{th} moment

$$M_\nu = \mathfrak{N}^{-1} \sum_{\text{set}} \sum_{i_1\cdots i_{\nu-1}} v_{0i_1}v_{i_1i_2} \cdots v_{i_{\nu-1}0} . \tag{12}$$

Since the diagonal elements of our matrix vanish, the summation in (12) can be restricted to those sequences $0, i_1, i_2, \cdots, i_{\nu-1}, 0$ in which no two succeeding members are equal. A pair of unequal numbers jl, each of which characterizes a row or column of our matrices (i.e. for which $-N \leqq j, l \leqq N$) will be called a "step"; the step lj will be called the reverse of jl. The sequence $0, i_1, i_2, \cdots, i_{\nu-1}, 0$ contains ν steps: $0i_1, i_1i_2, i_2i_3, \cdots, i_{\nu-1}0$. Each matrix of our set attributes a sign to every step but the signs of a step and of its reverse are always the same.

Set summation and the summation over the i can be interchanged in (12). The set summation will give 0 for each sequence $0, i_1, i_2, \cdots, i_{\nu-1}, 0$ unless it contains every step jl and its reverse an even number of times. If it does and if its first and last members are the same, it will be called a valid sequence. Thus, $0, i, j, i, l, m, n, m, n, m, l, i, 0$ is a valid sequence for $\nu = 12$ if the i, j, l, m, n, are all different. For a valid sequence, the set summation of (12) gives $\mathfrak{N}v^\nu$. Hence M_ν is equal to v^ν times the number of valid sequences of length $\nu + 1$.

Clearly, there is no valid sequence if ν is odd. Hence, $M_{2\nu+1} = 0$ and we can restrict ourselves to the calculation of the $M_{2\nu}$, i.e. the number of valid sequences of length $2\nu + 1$. A typical valid sequence of length $2\nu + 1$ is $0, i_1, i_2, \cdots,$

$i_{\nu-1}, i_\nu, i_{\nu-1}, \cdots, i_2, i_1$, 0, no matter what the i are as long as no two successive members of the sequence are equal. There are $(2N)^\nu$ valid sequences of this type and it will be shown that the leading term in the expression for the total number of all valid sequences of length $2\nu + 1$ is a numerical multiple of this, which will be denoted by $t_\nu(2N)^\nu$. The accurate expression for the number of valid sequences contains terms with lower powers of N but we shall assume that N is so large that we can restrict ourselves to the leading term. Hence

$$M_{2\nu} = t_\nu(2N)^\nu v^{2\nu}. \tag{13}$$

Because of (11a), $t_1 = 1$ and we proceed to calculate the general t_ν.

Excluding the first member, there cannot be more than ν different members in a valid sequence of length $2\nu + 1$. Let us call the steps "free" which end in a member which did not occur in the sequence before and let us call the steps "repetitive" which end in a member which did occur before. The sequence can be valid only if the number of repetitive steps is at least equal to the number of free steps. Since the number of free steps is equal to the number of different members of the sequence, the latter number cannot be more than one half of the number 2ν of all steps. This proves the assertion.

The number of valid sequences which contain less than ν different members is proportional to $N^{\nu-1}$ and their number is, therefore, negligible. The total number of sequences with only $\nu - 1$ different members is less than $(2\nu)!\,(2N + 1)^{\nu-1}$ and the number of valid sequences is smaller than this.

It follows from the preceding observation that it will be sufficient to count those valid sequences which have ν different members, i.e. contain ν free steps. It is permissible, therefore, to call only those sequences valid which contain ν different members. Since the number of repetitive steps in these sequences is also ν and since every free step must be repeated (in the same or the opposite direction) if the sequence is to be valid, every repetitive step will in fact be equal to a free step or to the reverse thereof. This justifies the name "repetitive" for the not free steps of a valid sequence.

A valid sequence in the above restricted sense defines a type sequence. The type sequence has 2ν members, its μ^{th} member is the number of the free steps minus the number of the repetitive steps among the first μ steps of the sequence to which it is coordinated. Thus, for instance, the sequence 0, 5, 3, 5, 0 and 0, 3, 0, -2, 0 are both valid sequences for $2\nu = 4$ (if $N \geqq 5$). The type sequence of the former is 1, 2, 1, 0, that of the latter is 1, 0, 1, 0. *All type sequences start with* 1 *and end with* 0, *successive members of it differ by* ± 1.

There are exactly $2N(2N - 1) \cdots (2N - \nu + 1) = (2N)!/(2N - \nu)! \sim (2N)^\nu$ *valid sequences for every type sequence.* If the μ^{th} member of the type sequence is larger than the preceding one, the corresponding step in the valid sequence is free and may lead to any number $j(-N \leqq j \leqq N)$ which has not been used before. If, on the other hand, the μ^{th} member of the type sequence is smaller than the preceding member, the μ^{th} step of the valid sequence is a repetitive one and must be the reverse of the step which originally led to the element from which this step starts. Hence, in the latter case, the μ^{th} step of the valid sequence is completely determined. (It follows that in a valid sequence in the restricted sense the steps are not actually repeated but balanced by the reverse step. This fact is not material for the remainder of the calculation.) *It follows that t_ν in* (13) *is the number of type sequences of length* 2ν.

Before obtaining a recursive formula for t_ν, we note that the only two type sequences for $2\nu = 4$ are the ones given above: 1, 2, 1, 0 and 1, 0, 1, 0 the corresponding valid sequences have the form 0, i, j, i, 0 and 0, i, 0, j, 0 with $i \neq j$. Hence $t_2 = 2$. For $2\nu = 6$ the type sequences are 1, 2, 3, 2, 1, 0; 1, 2, 1, 2, 1, 0; 1, 2, 1, 0, 1, 0; 1, 0, 1, 2, 1, 0; 1, 0, 1, 0, 1, 0. Hence, $t_3 = 5$. The valid sequences of the second type are 0, i, j, i, l, i, 0 with i, j, l all different.

The number of type sequences which contain no 0 before the last member will be denoted by t_ν'. From these sequences, one can obtain a type sequence of length $2\nu - 2$ by omitting the first and last member and subtracting 1 from each remaining member. Hence

$$t_\nu' = t_{\nu-1} . \qquad (t_1' = t_0 = 1). \tag{14}$$

If the first 0 member of the type sequence is at the position 2κ (it must be at an even position) the first 2κ members of it form a 0 free type of length 2κ, the remainder an arbitrary type sequence of length $2\nu - 2\kappa$. Hence

$$t_\nu = \sum_{\kappa=1}^{\nu} t_\kappa' t_{\nu-\kappa} = \sum_{\kappa=1}^{\nu} t_{\kappa-1} t_{\nu-\kappa} . \qquad (\nu = 1, 2, 3, \cdots). \tag{14a}$$

These recursive equations permit the successive calculation of the t_ν. One can obtain a closed formula for them by writing

$$t(x) = \sum_{\nu=o}^{\infty} t_\nu x^\nu. \tag{15}$$

The recursive formula (14a) then gives

$$t(x) = 1 + xt(x)^2. \tag{15a}$$

The 1 on the right side is necessary because (14a) is not valid for $\nu = 0$. It follows that

$$t(x) = (2x)^{-1}(1 \pm (1 - 4x)^{\frac{1}{2}}). \tag{15b}$$

Actually, the lower sign has to be taken. It gives

$$t_\nu = \frac{1}{2}\binom{\frac{1}{2}}{\nu + 1}(-4)^{\nu+1} = \frac{(2\nu)!}{\nu!(\nu + 1)!}. \tag{15c}$$

This can also be proved by induction by means of (14a). Finally, by (13)

$$M_{2\nu} = \frac{(2\nu)!}{\nu!(\nu + 1)!}(2N)^\nu v^{2\nu}. \tag{16}$$

Calculation of the strength function

It follows from (9) and (15) that

$$\int_{-\infty}^{\infty} x^{2\nu}\, dS(x) = \int_{-\infty}^{\infty} x^{2\nu}\sigma(x)\, d\, x = \frac{2\nu!}{\nu!(\nu + 1)!}(2N)^\nu v^{2\nu}. \tag{17}$$

The fact that $S(x)$ is differentiable and that, therefore, the first integral can be replaced by the second one ($\sigma(x) = S'(x)$) will be a consequence of (17). From the vanishing of the odd moments $M_{2\nu+1}$ it follows that $\sigma(x)$ is an even function of x. The form of the even moments suggests the introduction of new variables

$$\rho(\xi) = N\sigma(N\xi) \tag{18}$$

in terms of which (17) reads

$$\int \xi^{2\nu}\rho(\xi)\, d\xi = \frac{(2\nu)!\, 2^{\nu}}{\nu!(\nu+1)!}\frac{v^{2\nu}}{N^{\nu}} = \frac{(2\nu)!(2q)^{\nu}}{\nu!(\nu+1)!} \tag{18a}$$

where $q = v^2/N$.

Professor W. Feller kindly pointed out to the writer that the analysis of the original manuscript leading to (20) can be simplified by calculating directly (cf. also [2], Chapters 14 and 16),

$$\begin{aligned}\int e^{ik\xi}\rho(\xi)\, d\xi &= \textstyle\sum_{\nu=0}^{\infty} \int \frac{(ik\xi)^{2\nu}}{(2\nu)!}\rho(\xi)\, d\xi \\ &= \textstyle\sum_{\nu} \frac{(-)^{\nu}(2qk^2)^{\nu}}{\nu!(\nu+1)!} = \frac{2J_1(q'k)}{q'k}\end{aligned} \tag{19}$$

where $q' = (8q)^{\frac{1}{2}}$. The second member of (19) follows from the vanishing of the odd moments, the third member is the well known series (see [5], 17.1, p. 355) for the Bessel function of order 1. In order to obtain ρ itself, the Bessel function must be represented as a Fourier integral. Such a representation is provided by the expression (see [5], Example 1, p. 366),

$$\begin{aligned}J_1(z) &= \frac{\frac{1}{2}z}{\Gamma(3/2)\Gamma(1/2)}\int_0^{\pi} e^{iz\cos\varphi}\sin^2\varphi\, d\varphi \\ &= \frac{z}{\pi}\int_{-1}^{1} e^{izw}(1-w^2)^{\frac{1}{2}}\, dw.\end{aligned} \tag{19a}$$

The last part was obtained by substituting w for $\cos\varphi$. Substitution of $q'k$ for z and ξ for $q'w$ then gives (remembering that $q'^2 = 8q$)

$$\frac{2J_1(q'k)}{q'k} = \frac{1}{4\pi q}\int_{-q'}^{q'} e^{ik\xi}(8q-\xi^2)^{\frac{1}{2}}\, d\xi. \tag{19b}$$

Comparison of (19b) with (19) yields

$$\begin{aligned}\rho(\xi) &= (4\pi q)^{-1}(8q-\xi^2)^{\frac{1}{2}} && \text{for } \xi^2 < 8q \\ &= 0 && \text{elsewhere.}\end{aligned} \tag{20}$$

The original analysis did not make use of the properties of Bessel functions. It showed, on the basis of (18a), that the moments of $(8q/\xi)\, ds/d\xi$ and of $\xi\, ds/d\xi - 3s$ are equal where $ds/d\xi = \xi\rho$. This led to the differential equation

$$(8q/\xi)\, ds/d\xi = \xi\, ds/d\xi - 3s \tag{19c}$$

and hence to (20). The integration constant was determined from the normalization of ρ, i.e. from (18a) with $\nu = 0$. It was necessary then to refer to Corollary 1.1 in Shohat and Tamarkin's book [4], p. 11, or to Carleman's general theorem [1], p. 115, or [4], p. 19, to infer from the asymptotic form of the right side of (18a) that (20) is the only solution of (18a). This also justified (17). In any case, it is easy to verify a posteriori that the ρ of (20) satisfies (17).

For $\sigma(x)$, (18) gives

$$\sigma(x) = \frac{(8Nv^2 - x^2)^{\frac{1}{2}}}{4\pi Nv^2} \text{ (for } -(8N)^{\frac{1}{2}}v < x < (8N)^{\frac{1}{2}}v). \tag{20a}$$

This gives both the distribution of the characteristic values of the random sign symmetrical real matrices defined at the beginning of this section and also their strength function. Since, on the average, all the components of the characteristic vectors have the same absolute value, the two distributions are naturally identical. The reader will notice that even though some of the matrices of the set considered have characteristic values as high as $2Nv$, the characteristic values in excess of $(8N)^{\frac{1}{2}}v$ become increasingly rare as $N \to \infty$ and their total number constitutes, in the limit, a negligible fraction of all characteristic values of the matrices of the set.

Infinite symmetric matrices with wide random borders

The set of matrices $H = \mathbf{k} + \mathbf{v}$ underlying the following calculation has been described in some detail in the introduction. Of the two, $\mathbf{k}$ is unbounded but its characteristic vectors are clearly the unit vectors parallel to the coordinate axes; $\mathbf{v}$ is bounded for all N, its bounds being $\pm 2Nv$. Hence the characteristic value problem of H is defined.

Some of the remarks which apply for the singly bordered matrix also apply to the present *set* of matrices. In particular, $\mathbf{k}$ can be transformed into $-\mathbf{k}$ by renumbering rows and columns and the set of matrices $\mathbf{v}$ is also invariant under this transformation. It follows that the average number of characteristic values λ per unit interval at x is an even function of x. It also follows that if $\psi_k^{(\lambda)}$ is the characteristic vector of one matrix of the set, the set contains another matrix with characteristic vector $\psi_{-k}^{(-\lambda)}$. Hence, if $\mathfrak{N}$ is the number of matrices of the set which we consider and if this set of matrices includes either both or neither of the aforementioned matrices

$$\mathfrak{N}S(x) = \sum_{\text{set}} \sum_{\lambda < x} (\psi_0^{(\lambda)})^2 = \sum_{\text{set}} \sum_{\lambda > -x} (\psi_0^{(\lambda)})^2. \tag{21}$$

Because of (7), the sum of the expression in (21), extended over all λ, is just $\mathfrak{N}$. Hence, the last sum is also equal to

$$\mathfrak{N}S(x) = \mathfrak{N} - \sum_{\text{set}} \sum_{\lambda < -x} (\psi_0^{(\lambda)})^2 = \mathfrak{N} - \mathfrak{N}S(-x) \tag{21a}$$

so that

$$S(x) + S(-x) = 1 \tag{21b}$$

and the strength function $\sigma(x) = S'(x)$ is again even.

Second, it is again possible to transform $\mathbf{k}$ into $\mathbf{k} + 1$ by renumbering the rows and columns and the set of matrices $\mathbf{v}$ is also invariant under this transformation. Hence, the average number of characteristic values λ per unit interval at x is a periodic function of x with period 1.

It may be well to repeat here that we shall be interested, eventually, in a very wide border, i.e. in the case that N in (1) is very large. At the same time v shall be very large also, in such a way, however, that $v^2/N = q$ remains constant. The problem originates from the consideration of an "unperturbed problem" in which the spacing of the characteristic values is $1/N$, the matrix of the unperturbed problem being $N^{-1}\mathbf{k}$. The "perturbation" $N^{-1}\mathbf{v}$ has matrix elements connecting characteristic values of the unperturbed problem which differ up to 1 (this quantity being chosen as the unit of energy). The diagonal elements of the square of the perturbation $N^{-2}v^2$ are $2NN^{-2}v^2$ and this quantity is denoted by $2q$.

Calculation of the moments

Because of the even nature of the strength function, it will suffice to calculate the even moments thereof. Their calculation will be based again on (9b), i.e. on the calculation of the set average of $(H^{2\nu})_{00}$. Before carrying out the calculation, it should be noted that

$$(H^{2\nu})_{00} = \sum_n (H^\nu)_{0n}(H^\nu)_{n0} = \sum_n (H^\nu)_{0n}^2 . \tag{22}$$

It will be shown that $(H^\nu)_{0n}$ is different from zero only for $-\nu N \leqq n \leqq \nu N$ and that only those v_{mn} influence it for which $-\nu N \leqq m, n \leqq \nu N$. These statements are evident for $\nu = 1$. Since

$$(H^\nu)_{0n} = \sum_m (H^{\nu-1})_{0m}(m\delta_{mn} + v_{mn}) \tag{22a}$$

and since $v_{mn} = 0$ for $| m - n | > N$, the statements follow by induction. They show that, when calculating the set average of $(H^{2\nu})_{00}$, the average has to be taken only over a finite set of matrices, namely those in which the v_{mn} for $-2\nu N \leqq m, n \leqq 2\nu N$ have all possible signs consistent with the condition of symmetry.

The calculation of $M_{2\nu} = \text{Av}(H^{2\nu})_{00}$ will closely resemble the calculation of the preceding section. It will turn out again that the term in $M_{2\nu}$ which is proportional to the highest power of N has a factor $(v^2N)^\nu \approx N^{2\nu}$ and only the terms proportional to $N^{2\nu}$ will be calculated. The proportionality constant of $N^{2\nu}$ will depend on q and we shall write

$$M_{2\nu} = \text{Av}(H^{2\nu})_{00} = T_\nu(q)N^{2\nu}. \tag{23}$$

We use again an expression similar to (12) for

$$M_{2\nu} = \text{Av} \sum_{i_1 i_2 \cdots i_{2\nu-1}} H_{0i_1}H_{i_1 i_2}\cdots H_{i_{2\nu-1}0} \tag{24}$$

and define the steps $0i_1$, i_1i_2, i_2i_3, $\cdots$, $i_{2\nu-1}0$ as before. However, there are, in this case, three kinds of steps: (1) the free steps which lead to an i which has not occurred before in the sequence $0i_1i_2i_3 \cdots i_{2\nu-1}0$, (2) the repetitive steps which occurred before in the same or in the reverse direction, and (3) the waiting steps which correspond to diagonal elements of the matrix. It is possible again to define type sequences each member of which differs from the preceding one by ± 1 or 0. If the μ^{th} member of the type sequence is larger than the preceding one, the corresponding step is a free step: if it is smaller, it is a repetitive step. Finally, if $\mu - 1^{\text{th}}$ and μ^{th} members are equal, the corresponding step is a waiting step. It is true again that every free step must be repeated in the same or in the opposite direction. Furthermore, since the diagonal element in the 0 row vanishes, neither the first, nor the last step, can be a waiting step; the first one must be a free step, the last one a repetitive step.

We shall single out again that part of the sum (24) which belongs to types which have no 0 members, excepting the last. In the corresponding terms of (24), none of the i is 0. The sum of these terms will be denoted by $T'_\nu(q)N^{2\nu}$. The first and last factors in a typical sum of this type are v_{oi_1} and v_{i_1o}, their product v^2. The rest of the terms give all the terms of

$$\sum_{i_2i_3\cdots i_{2\nu-2}} H_{i_1i_2}H_{i_2i_3}\cdots H_{i_{2\nu-2}i_1} = (H^{2\nu-2})_{i_1i_1}\,. \tag{25}$$

However, the set of matrices (H_{i_1+m,i_1+n}) is identical with the set of matrices $(H_{mn} + i_1 1)$ so that the average of (25) is equal to the average of $[(H + i_1 1)^{2\nu-2}]_{00}$. This gives for the contribution of the zero-free types to (24)

$$\begin{aligned} T'_\nu(q)N^{2\nu} &= \textstyle\sum_k v^2 \mathrm{Av}[(H + k1)^{2\nu-2}]_{00} \\ &= \textstyle\sum_k v^2 \mathrm{Av} \sum_\lambda \binom{2\nu - 2}{\lambda} k^{2\nu-2-\lambda}(H^\lambda)_{00}\,. \end{aligned} \tag{25a}$$

The summation over k can be replaced by integration from $-N$ to N. It gives 0 for odd λ, for even λ it gives $2N^{2\nu-1-\lambda}/(2\nu - 1 - \lambda)$. It simplifies the notation to write 2λ for λ and let λ run over the integers. At the same time, κ will be written for ν and the expression (23) substituted for the Av

$$\begin{aligned} T'_\kappa(q)N^{2\kappa} &= 2v^2 \textstyle\sum_{\lambda=0}^{\kappa-1} \binom{2\kappa - 2}{2\lambda} T_\lambda(q)N^{2\lambda} \dfrac{N^{2\kappa-2\lambda-1}}{2\kappa - 2\lambda - 1} \\ &= 2qN^{2\kappa} \textstyle\sum_{\lambda=0}^{\kappa-1} \binom{2\kappa - 2}{2\lambda} \dfrac{T_\lambda(q)}{2\kappa - 2\lambda - 1} \qquad (\text{for } \kappa > 0). \end{aligned} \tag{25b}$$

This is the analogue of the very much simpler equation (14) of the preceding section. The highest T_λ which occurs on the right side is $T_{\kappa-1}$.

The various terms which enter the sum (24) can be distinguished according to the position 2κ at which the first 0 occurs in their type sequence. In the terms in which 0 occurs at the position 2κ, the index $i_{2\kappa} = 0$ so that they contribute altogether to the sum (24)

$$T'_\kappa(q)N^{2\kappa}\mathrm{Av}H_{0i_{2\kappa+1}}H_{i_{2\kappa+1}i_{2\kappa+2}} \cdots H_{i_{2\nu-1}0} = T'_\kappa(q)N^{2\kappa}\mathrm{Av}(H^{2\nu-2\kappa})_{00}\,. \tag{26}$$

The whole sum (24) will contain terms which correspond to any κ from 1 to ν; it will be a sum of expressions (26) with all these κ. Hence

$$\begin{aligned} T_\nu(q)N^{2\nu} &= \sum_{\kappa=1}^{\nu} T'_\kappa(q)N^{2\kappa}\mathrm{Av}(H^{2\nu-2\kappa})_{00} \\ &= \sum_{\kappa=1}^{\nu} T'_\kappa(q)T_{\nu-\kappa}(q)N^{2\nu} \qquad (\text{for } \nu > 0) \end{aligned} \tag{26a}$$

where (23) was used to express the Av. This last expression is the analogue of the second member of (14a). Substitution of (25b) into (26a) gives a recursive formula for the T_ν alone

$$\begin{aligned} T_\nu &= 2q \sum_{\kappa=1}^{\nu} \sum_{\lambda=0}^{\kappa-1} \binom{2\kappa-2}{2\lambda} \frac{T_\lambda T_{\nu-\kappa}}{2\kappa-2\lambda-1} + \delta_{\nu o} \\ &= 2q \sum_{\kappa=0}^{\nu-1} \sum_{\lambda=0}^{\kappa} \binom{2\kappa+1}{2\lambda} \frac{T_\lambda T_{\nu-\kappa-1}}{2\kappa+1} + \delta_{\nu o}. \end{aligned} \tag{26b}$$

The last term was added to make (26b) valid also for $\nu = 0$. The highest T that occurs on the right side has index $\nu - 1$ and the equations (26b) determine successively all T_ν. One obtains, either by direct calculation or from (26b)

$$T_0 = 1 \qquad T_1 = 2q \qquad T_2 = 8q^2 + 2q/3. \tag{26c}$$

The equations of the preceding section for the M_ν are contained in (26b) as a limiting case for $q = \infty$. Hence the coefficient of q^ν in T_ν is $2^\nu(2\nu)!/(\nu!(\nu+1)!)$. The coefficient of $q^{\nu-1}$ is

$$2^{3\nu-4}\nu/3 - 2^{\nu-2}(2\nu-1)!((\nu-1)!)^{-2}/3.$$

The coefficients of q and q^2 are $2/(2\nu-1)$ and $4(2\nu-2)/(2\nu-3)$, respectively. However, no closed expression could be obtained for T_ν.

The following estimate of T_ν, although very crude, will show by Carleman's theorem [1], p. 115 or [4] p. 19, that the moments (26b) uniquely determine the distribution function. The explicit expressions (26c) show that

$$T_\nu < 2(q+1)^\nu \nu^{2\nu} \tag{27}$$

is valid for $\nu = 0, 1$. It will be proved, in general, by induction. If (27) is valid for $\lambda = 0, 1, 2, \cdots, \nu - 1$, we have, for $\kappa \leqq \nu - 1$

$$\begin{aligned} \sum_{\lambda=0}^{\kappa} \binom{2\kappa+1}{2\lambda} T_\lambda &< 2\sum_{\lambda=0}^{\kappa} \binom{2\kappa+1}{2\lambda} (q+1)^\lambda \lambda^{2\lambda} < 2\sum_0^\kappa \binom{2\kappa+1}{2\lambda} \\ &\cdot (q+1)^\kappa \kappa^{2\lambda} < (q+1)^\kappa[(1+\kappa)^{2\kappa+1} + (1-\kappa)^{2\kappa+1}]. \end{aligned} \tag{27a}$$

The last term on the right side can be omitted for $\kappa \geqq 1$. For $\kappa = 0$, omission of the last term makes (27a) an equality. Hence, we have, by (26b) and (27)

$$\begin{aligned} T_\nu &\leqq 2q \sum_{\kappa=0}^{\nu-1} (q+1)^\kappa(1+\kappa)^{2\kappa+1} 2(q+1)^{\nu-\kappa-1}(\nu-\kappa-1)^{2(\nu-\kappa-1)}/(2\kappa+1) \\ &< 4(q+1)^\nu \left\{ \frac{1(\nu-1)^{2\nu-2}}{1} + \frac{2^3(\nu-2)^{2\nu-4}}{3} + \cdots + \frac{(\nu-1)^{2\nu-3}1^2}{2\nu-3} + \frac{\nu^{2\nu-1}}{2\nu-1} \right\}. \end{aligned}$$

There are ν terms in the bracket and the last one is largest among them. Hence

(27b) $$T_\nu < 4(q+1)\nu^{2\nu}/(2\nu-1) < 2(q+1)^\nu \nu^{2\nu}$$

for $\nu = 2, 3, \cdots$. This proves (27).

It follows that

(27c) $$(T_\nu)^{-\frac{1}{2}\nu} \rightarrow (q+1)^{-\frac{1}{2}}\nu^{-1}$$

so that the series with the general term (27c) diverges and the Perron-Carleman criterion for the uniqueness of the moment problem (28b) is satisfied.

Equations for the strength function

We shall write the basic equation for the strength function at once in terms of σ

(28) $$\int_{-\infty}^{\infty} x^{2\nu}\sigma(x)\,dx = M_{2\nu} = T_\nu(q)N^{2\nu}.$$

By introducing a proper scale for the variable of the strength function

(28a) $$\rho(\xi) = N\sigma(N\xi)$$

(28) transforms into

(28b) $$\int_{-\infty}^{\infty} \xi^{2\nu}\rho(\xi)\,d\xi = T_\nu(q).$$

All odd moments of σ and ρ vanish. The existence of all moments implies that ρ goes to zero at $\xi \rightarrow \pm\infty$ faster than any power of ξ. Substitution of (28b) into (26b) gives for $\nu \neq 0$

(29) $$\int d\xi\xi^{2\nu}\rho(\xi) = 2q \sum_{\kappa=0}^{\nu-1}\sum_{\lambda=0}^{\kappa}\binom{2\kappa+1}{2\lambda}\int d\xi \int d\zeta\rho(\xi)\rho(\zeta)\,\frac{\xi^{2\lambda}\zeta^{2\nu-2\kappa-2}}{2\kappa+1}.$$

All integrals in which no limit is given are to be extended from $-\infty$ to ∞. Together with $\int\rho(\xi)\,d\xi = 1$, (29) completely determines the even function ρ. The summation over λ can be carried out in (29) by the binomial theorem and gives

(29a) $$\int d\xi\xi^{2\nu}\rho(\xi) = q\sum_{\kappa=0}^{\nu-1}\int d\xi\int d\zeta\rho(\xi)\rho(\zeta)\zeta^{2\nu-2\kappa-2}\,\frac{(1+\xi)^{2\kappa+1}+(1-\xi)^{2\kappa+1}}{2\kappa+1}.$$

Integration of the identity

$$\sum_{\kappa=0}^{\nu-1} x^{2\kappa} = \frac{x^{2\nu}-1}{x^2-1}$$

gives

(29b) $$\sum_{\kappa=0}^{\nu-1}\frac{u^{2\kappa+1}}{2\kappa+1} = \frac{1}{2}\int_{-u}^{u} dx\,\frac{x^{2\nu}-1}{x^2-1}.$$

Hence (29a) can be written also as

$$\text{(29c)}\quad \int d\xi\xi^{2\nu}\rho(\xi) = \tfrac{1}{2}q\int d\xi\int d\zeta\rho(\xi)\rho(\zeta)\zeta^{2\nu-1}\left(\int_{-\alpha}^{\alpha}dx + \int_{-\beta}^{\beta}dx\right)\frac{x^{2\nu}-1}{x^2-1}$$

where $\alpha = (1+\xi)/\zeta$ and $\beta = (1-\xi)/\zeta$. Introducing $z = \zeta x$ as new variable instead of x gives

$$\text{(29d)}\quad \int d\zeta\zeta^{2\nu}\rho(\zeta) = \tfrac{1}{2}q\int d\xi\int d\zeta\rho(\xi)\rho(\zeta)\left(\int_{-1-\xi}^{1+\xi}dz + \int_{-1+\xi}^{1-\xi}dz\right)\frac{z^{2\nu}-\zeta^{2\nu}}{z^2-\zeta^2}.$$

The purpose of the following transformations is to bring the right side into a form in which ν appears only as the exponent of ζ^2. This cannot be done by simply interchanging the variables z and ζ in the term which contains $z^{2\nu}$ because the singularity which would then result at $z = \zeta$. In addition, it appears worthwhile to simplify the last equation somewhat. One can replace the two integrals with respect to z by two similar integrals extending from $\xi - 1$ to $\xi + 1$ and from $-\xi - 1$ to $-\xi + 1$. Since the remainder of the integrand is an even function of ξ, the two integrals are equal and can be replaced by twice one of them. Finally, one can interchange the integrations with respect to z and ξ to obtain, still only for $\nu \neq 0$,

$$\text{(30)}\quad \begin{aligned}\int d\zeta\zeta^{2\nu}\rho(\zeta) &= q\int dz\int d\zeta\int_{z-1}^{z+1}d\xi\rho(\xi)\rho(\zeta)\frac{z^{2\nu}-\zeta^{2\nu}}{z^2-\zeta^2}\\ &= q\int dz\int d\zeta R_1(z)\rho(\zeta)\frac{z^{2\nu}-\zeta^{2\nu}}{z^2-\zeta^2}\end{aligned}$$

in which

$$\text{(30a)}\quad R_1(z) = \int_{z-1}^{z+1}d\xi\rho(\xi)$$

is again an even function which drops to zero at $z = \pm\infty$ faster than any power of z. Since the integrand on the right side of (30) has no singularity, it is permissible to exclude two narrow strips $|z \pm \zeta| < \varepsilon$ from the integration. Then, the integration variables z and ζ can be interchanged in the term with $z^{2\nu}$ to give, as $\varepsilon \to 0$

$$\text{(30b)}\quad \int d\zeta\zeta^{2\nu}\rho(\zeta) = \iint_{|z\pm\zeta|>\varepsilon} q\,dz\,d\zeta[R_1(\zeta)\rho(z) + \rho(\zeta)R_1(z)]\frac{\zeta^{2\nu}}{\zeta^2-z^2}.$$

Even though both integrands are even functions of ζ, the application of the theorem of moments to equate the integrands is not permissible because (30b) is not valid for $\nu = 0$.

In order to render (30b) valid also for $\nu = 0$, we calculate first

$$\text{(31)}\quad \begin{aligned}\int_{|z\pm\zeta|>\varepsilon}\frac{dz}{z^2-\zeta^2} &= \frac{1}{\zeta}\ln\frac{2\zeta+\varepsilon}{\varepsilon} &&\text{for } 0<\zeta<\varepsilon\\ &= \frac{1}{\zeta}\ln\frac{2\zeta+\rho}{2\zeta-\varepsilon} &&\text{for } \varepsilon<\zeta.\end{aligned}$$

These expressions are proportional to $1/\varepsilon$ as long as ζ is of the order ε. They drop very fast as ζ increases: the integral is 1 for $\zeta \sim \varepsilon^{\frac{1}{2}}$ and of the order of $\varepsilon^{\frac{1}{2}}$ for $\zeta \sim \varepsilon^{\frac{1}{4}}$. Hence, if $f(\zeta)$ is an even function with two continuous derivatives which is small enough at $\zeta = \pm\infty$,

$$\text{(31a)} \qquad \lim_{\varepsilon=0} \iint_{|z\pm\zeta|>\varepsilon} dz\, d\zeta \frac{f(\zeta)}{z^2 - \zeta^2} = cf(0); \qquad c = \pi^2/2$$

c is the integral of (31) from $\zeta = -\infty$ to ∞ or twice the integral from 0 to ∞. The reader will recognize that the double integral of $(\zeta^2 - z^2)^{-1}$ over the whole ζz plane excluding the $|\zeta \pm z| < \varepsilon$ strips is not absolutely convergent: it gives a negative value if first integrated over z then ζ, the opposite value if the integration is carried out in the reverse order. It is clear from (31a) that the subtraction of a term $f(\zeta)$ in the square bracket of (30b) will not render this equation invalid for $\nu \geqq 1$ as long as the even function $f(\zeta)$ drops faster at $\pm\infty$ than any power of ζ because $\zeta^{2\nu}f(\zeta)$ vanishes at $\zeta = 0$. On the other hand, the value of the integral for $\nu = 0$, which is zero as the integral now stands, will become 1 if

$$\text{(32)} \qquad f(0) = 2/\pi^2 q.$$

Hence

$$\text{(32a)} \quad \int d\zeta \zeta^{2\nu}\rho(\zeta) = \lim_{\varepsilon=0} \int d\zeta \zeta^{2\nu} \int_{|z-\zeta|<\varepsilon} q\, dz \frac{R_1(\zeta)\rho(z) + \rho(\zeta)R_1(z) - f(\zeta)}{\zeta^2 - z^2}$$

is valid now for all ν. It can be written in a somewhat more transparent form

$$\text{(32b)} \quad \begin{aligned} \int d\zeta\zeta^{2\nu}\rho(\zeta) - \int d\zeta\zeta^{2\nu} q\left[R_1(\zeta)\int dz \frac{\rho(z) - \rho(\zeta)}{\zeta^2 - z^2} + \rho(\zeta)\int dz \frac{R_1(z) - R_1(\zeta)}{\zeta^2 - z^2}\right] \\ = \lim_{\varepsilon=0} \int d\zeta\zeta^{2\nu} \int_{|z\pm\zeta|>\varepsilon} q\, dz \frac{2R_1(\zeta)\rho(\zeta) - f(\zeta)}{\zeta^2 - z^2}. \end{aligned}$$

In the square bracket, the strips $|z \pm \zeta| < \varepsilon$ need not be excluded from the integration since the integrand is regular everywhere. Because of (31a), the right side can be replaced by

$$\text{(32c)} \qquad \delta_{\nu 0} c q[f(0) - 2R_1(0)\rho(0)] = \tfrac{1}{2}\pi^2 q \delta_{\nu 0}[2/\pi^2 q - 2R_1(0)\rho(0)].$$

Hence, the left side of (32b) contains the moments of an even function of and the form (32c) of the right side shows that these all vanish except the zero$^{\text{th}}$ moment. It follows that the function vanishes and so does its zero$^{\text{th}}$ moment

$$\text{(33)} \qquad \rho(\zeta) = qR_1(\zeta)\int \frac{\rho(z) - \rho(\zeta)}{\zeta^2 - z^2}\, dz + q\rho(\zeta)\int \frac{R_1(z) - R_1(\zeta)}{\zeta^2 - z^2}\, dz$$

$$\text{(33a)} \qquad R_1(0)\rho(0) = 1/\pi^2 q.$$

The last equation means, in terms of the strength function σ (cf. (28a))

$$\text{(33b)} \qquad N\sigma(0)\int_{-N}^{N} \sigma(x)\, dx = 1/\pi^2 q.$$

Discussion of the equations (33) for the strength function

As was mentioned before, the problem of the preceding section represents the limiting case $q = \infty$ of the present problem. Hence, (20) is a solution of (33), (33a) for very large q. In the opposite limiting case of small q, the total width of ρ will be much smaller than 1 and R_1 can be replaced by 1 over the significant region. Hence, in this case

$$\rho(\zeta) \sim q \int \frac{\rho(z) - \rho(\zeta)}{\zeta^2 - z^2} dz. \tag{34}$$

This equation is solved by

$$\rho(\zeta) \sim \frac{\text{const}}{q^2\pi^2 + \zeta^2} = \frac{q}{\pi^2 q^2 + \zeta^2} \tag{34a}$$

the constant being determined by (33a). However (34a) is valid only as long as $\zeta \ll 1$ since only in this region is $R_1 \sim 1$.

Neither (20) nor (34a) give the asymptotic behavior of $\rho(\zeta)$ correctly. In particular already the second moment of (34a) is divergent. The asymptotic behavior can be calculated on the basis of the observation that ρ and R_1 surely decrease to 0 at $\zeta \to \infty$ faster than any power of ζ and that, hence, the largest contribution to the integrals in (33) comes from the region at very small z. Hence, the $\zeta^2 - z^2$ in the denominators can be replaced by ζ^2. Since the integrals of ρ and R_1 are 1 and 2, respectively, we have asymptotically

$$\rho(\zeta) \approx qR_1(\zeta)/\zeta^2 + 2q\rho(\zeta)/\zeta^2. \tag{35}$$

The last term on the right side is much smaller than the left side and can be omitted. The resulting equation has the asymptotic solution

$$\rho(\zeta) \approx \text{const.}\ (2q\zeta^2 \ln \zeta/e^2)^{-\zeta}. \tag{35a}$$

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