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CHARACTERISTIC VECTORS OF BORDERED MATRICES WITH INFINITE DIMENSIONS II

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The matrices which form the subject of an earlier study [1] consist of a diagonal matrix **k**

$$k_{mn} = m\delta_{mn} \qquad (-\infty < m, n < \infty) \tag{1}$$

and of the "border". This is a real symmetric matrix **v**, all the elements of which vanish beyond a certain distance from the diagonal

$$v_{mn} = 0 \quad \text{for} \quad |m - n| > N \qquad (-\infty < m, n < \infty) \tag{1a}$$

while those closer to the diagonal have all the same absolute value

$$|v_{mn}| = v \quad \text{for} \quad |m - n| \leqq N. \tag{1b}$$

Since **v** is symmetric $v_{mn} = v_{nm}$. Subject to this condition, however, the signs of the v_{mn} are random; i.e., ensembles of matrices are considered with all possible signs of the v_{mn} subject to the conditions of symmetry.

The problem is connected with the real normed characteristic vectors $\psi^{(\lambda)}$ of $H = \mathbf{k} + \mathbf{v}$:

$$\textstyle\sum_n H_{mn}\psi_n^{(\lambda)} = \lambda\psi_m^{(\lambda)}. \tag{2}$$

The average value of $(\psi_0^{(\lambda)})^2$ over all matrices of the ensemble was denoted by $\sigma(\lambda)$, the characteristic value λ being considered as a continuous variable. A more nearly exact definition of $\sigma(\lambda)$ is given in the aforementioned article. A complete solution could be given for $N = 1$; the more interesting case of a very large N led, in general, to an integral equation. In order to formulate this integral equation, a more proper scale for the variable λ was introduced

$$\rho(\lambda/N) = N\sigma(\lambda). \tag{3}$$

It was further assumed that, together with N, also v tends to infinity in such a way, however, that

$$v^2/N = q \tag{3a}$$

remains finite. Then, the following integral equation (33) was obtained for ρ, partly by heuristic arguments

$$\rho(\zeta) = qR_1(\zeta)\int\frac{\rho(z) - \rho(\zeta)}{\zeta^2 - z^2}\,dz + q\rho(\zeta)\int\frac{R_1(z) - R_1(\zeta)}{\zeta^2 - z^2}\,dz. \tag{4}$$

All integrations on which no limits are given are to be extended from $-\infty$ to ∞;

$$R_1(\zeta) = \int_{\zeta-1}^{\zeta+1}\rho(z)dz. \tag{4a}$$

It is easily seen that $\rho(\zeta)$ is an even function, it tends to zero for large ζ so rapidly that all its moments exist. Its zero$^{\text{th}}$ moment (integral) is 1, its second moment $2q$.

The integral equation (4) could not be solved in closed form and no such solution will be presented here. However, it will be transformed into a much simpler equation from which the approximate expressions for ρ (for large and small values of q) which were obtained before can be obtained much more easily. For this purpose we consider the function $r(z)$ which is analytic in the upper half plane, tends to zero in that plane as $|z| \to \infty$, and the real part of which is $\rho(z)$ for real z. This function is well known to be given in the upper half plane by the integral

$$r(z) = \frac{1}{i\pi}\int \frac{\rho(\zeta)}{\zeta - z}\,d\zeta \qquad \text{(for Im } z > 0\text{)}. \tag{5}$$

It is easy to verify that the real part of $r(z)$ tends to $\rho(\zeta)$ as z approaches the real ζ from above

$$\text{Re}\, r(z) = \rho(z) \qquad \text{(for real } z\text{)}. \tag{5a}$$

The imaginary part of $r(z)$ on the real axis is obtained in the same way as the principal value integral

$$\text{Im}\, r(z) = \frac{1}{\pi} P \int \frac{\rho(\zeta)}{z - \zeta}\,d\zeta \qquad \text{(for real } z\text{)}. \tag{5b}$$

Naturally, the integral representation (5) is not valid in the lower half plane; in fact, the real part of the integral in (5) approaches $-\rho(\zeta)$ as z approaches the real ζ from below. In the lower half plane $r(z)$ must be obtained by analytic extension from the upper half plane, not by (5). As a matter of fact, it will not be necessary to make such an extension, the complex variable z can be restricted for the purposes of the present article to the upper half plane including the real axis.

Let us consider now the analytic function which is in the same relation to R_1 as r is to ρ. The function

$$r_1(z) = \int_{-1}^{1} r(z + \zeta)\,d\zeta \tag{5c}$$

satisfies all the conditions which this function must satisfy and is therefore this function. It is also even in z.

At very large $|z|$, because of the rapid drop of $\rho(\zeta)$ at $\zeta \to \pm\infty$, and since $\int \rho(\zeta)\,d\zeta = 1$, one obtains from (5)

$$r(z) \to i/\pi z \qquad \text{(for Im } z > 0\text{)}. \tag{6}$$

Let us now calculate the imaginary parts of $r(z)$ and $r_1(z)$ on the real axis.

Since ρ is an even function

$$\text{(7)}\qquad \begin{aligned} \operatorname{Im} r(z) &= \frac{1}{\pi} P \int \frac{\rho(\zeta)}{z - \zeta}\, d\zeta = \frac{1}{2\pi} P \int \left(\frac{\rho(\zeta)}{z - \zeta} + \frac{\rho(-\zeta)}{z + \zeta}\right) d\zeta \\ &= \frac{1}{2\pi} P \int \rho(\zeta)\, \frac{2z}{z^2 - \zeta^2}\, d\zeta. \end{aligned}$$

An easy calculation gives, furthermore,

$$\begin{aligned} P \int_{-\infty}^{\infty} \frac{2z\, d\zeta}{z^2 - \zeta^2} &= 2P \int_0^{\infty} \left(\frac{1}{\zeta + z} - \frac{1}{\zeta - z}\right) d\zeta \\ &= 2 \lim_{N\to\infty} \lim_{\epsilon\to 0} \left(\int_0^{z-\epsilon} d\zeta + \int_{z+\epsilon}^{N} d\zeta\right)\left(\frac{1}{\zeta + z} - \frac{1}{\zeta - z}\right) \\ &= 2 \lim_{N\to\infty} \lim_{\epsilon\to 0} \left\{\left[\ln \frac{\zeta + z}{z - \zeta}\right]_0^{z-\epsilon} + \left[\ln \frac{\zeta + z}{\zeta - z}\right]_{z+\epsilon}^{N}\right\} \\ &= 2 \lim_{N\to\infty} \lim_{\epsilon\to 0} \left\{\ln \frac{2z - \epsilon}{\epsilon} + \ln \frac{N + z}{N - z} - \ln \frac{2z + \epsilon}{\epsilon}\right\} = 0. \end{aligned}$$

It now follows by means of (7) that for real z

$$\text{(7a)}\qquad \operatorname{Im} r(z) = \frac{z}{\pi} P \int \frac{\rho(\zeta)}{z^2 - \zeta^2}\, d\zeta = \frac{z}{\pi} P \int \frac{\rho(\zeta) - \rho(z)}{z^2 - \zeta^2}\, d\zeta$$

since the last equation shows that the term added on the right side vanishes. The principal value integral can now be replaced by the proper integral since the integrand has no singularity any more.

As (4a) shows, the even character of ρ entails the even character of R_1. Hence, the same calculation gives

$$\text{(7b)}\qquad \operatorname{Im} r_1(z) = \frac{z}{\pi} \int \frac{R_1(\zeta) - R_1(z)}{z^2 - \zeta^2}\, d\zeta$$

and (4), multiplied by z/π, can be given the form

$$\text{(8)}\qquad (\operatorname{Re} zr/\pi) = q(\operatorname{Re} r_1)(\operatorname{Im} r) + q(\operatorname{Re} r)(\operatorname{Im} r_1) \qquad \text{(for real } z\text{)}.$$

It follows that the real part of the analytic function $w = zr/\pi + iqrr_1$ vanishes on the real axis. Since r and r_1 are both regular in the upper half plane, including the real axis, this holds also of w. Finally, since both r and r_1 tend to zero as $|z| \to \infty$, and because of (6), $w \to i/\pi^2$ for very large z. It follows that $w - i/\pi^2$ is regular in the upper half plane, tends to zero for $|z| \to \infty$ and its real part vanishes on the real axis. Hence $w - i/\pi^2 = 0$

$$\text{(9)}\qquad zr/\pi + iqrr_1 = i/\pi^2.$$

This is the simplified form of (4) to which we were referring. It can be written in an even slightly simpler form if one introduces $p(z) = \pi r(z)$ and $p_1(z) =$

$\pi r_1(z)$

(9a) $$zp + iqpp_1 = i$$

while the relation between p and p_1 remains

(9b) $$p_1(z) = \int_{z-1}^{z+1} p(\zeta)\, d\zeta.$$

The consequences of (4), obtained before, will now be derived from (9) or the equivalent (9a). It will be noted, first, that $\operatorname{Im} r(z)$ is an odd function of the real variable z. This follows most easily from (7a). Similarly, (7b) shows that $\operatorname{Im} r_1$ is also odd. Hence, both r and r_1 are real functions of iz

(10) $$\overline{r(iz)} = r(i\bar{z}); \qquad \overline{r_1(iz)} = r_1(i\bar{z}).$$

In particular, r and r_1 are real for $z = 0$ and equal to $\rho(0)$ and $R_1(0)$, respectively. Hence, insertion of $z = 0$ into (9) gives

(11) $$q\rho(0)R_1(0) = 1/\pi^2.$$

This is equation (33a) of Reference 1. It follows from the development given there that (11) should be a consequence of (4). The preceding argument verifies this directly.

Since $2q$ is the second moment of ρ, this will change very little in a unit interval if q is large. Hence, by (5c), $r_1 \approx 2r$ for large q. Thus (9) becomes in this case

$$2iqr^2 + (z/\pi)r - i/\pi^2 = 0$$

or

$$r(z) = +\frac{iz}{4\pi q} + \left[\frac{1}{2\pi^2 q} - \frac{z^2}{16\pi^2 q^2}\right]^{\frac{1}{2}}.$$

In this case for real z

$$\begin{aligned} \rho(z) = \operatorname{Re} r(z) &= (4\pi q)^{-1}(8q - z^2)^{\frac{1}{2}} && \text{for } z^2 < 8q \\ &= 0 && \text{for } z^2 > 8q. \end{aligned}$$

This is equation (20) of Reference 1.

If q is very small, the width of ρ will be very small and the integral of (5c) will be closely approximated by the similar integral from $-\infty$ to ∞ as long as $z < 1$. For real z, the value of this integral is 1 because this is the contribution of the real part of r and because the imaginary part of r, being odd, does not contribute at all. Hence (9) becomes in this case

$$((z/\pi) + iq)r = i/\pi^2.$$

The real part of r then is for real z

$$\rho(z) = \mathrm{Re}\, r(z) = q/(z^2 + \pi^2 q^2) \qquad (\text{for } -1 < z < 1)$$

which is (34a) of Reference 1.

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REFERENCE

1. E. P. WIGNER, *Characteristic Vectors of Bordered Matrices with Infinite Dimensions*, Ann. of Math., 62 (1955), pp. 548-564. The following errors may be disturbing: The coefficient of q^2 in T_ν given around the middle of page 560 is incorrect (the expression given is neither derived nor used subsequently). In (27c), replace $(T_\nu)^{-\frac{1}{2}\nu}$ by $(T_\nu)^{-1/2\nu}$. In (31), replace ρ by ϵ.