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ON THE DISTRIBUTION OF THE ROOTS OF CERTAIN SYMMETRIC MATRICES

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The present article is concerned with the distribution of the latent roots (characteristic values) of certain sets of real symmetric matrices of very high dimensionality. Its purpose is to point out that the distribution law obtained before[1] for a very special set of matrices is valid for much more general sets. The dimension of the matrices will be denoted by N, the matrix elements by v_{ij}. These are real. The condition of symmetry is

$$v_{ij} = v_{ji} \tag{1}$$

The matrix elements v_{ij} of the set of matrices are distributed according to the following laws:

(a) The distribution $p_{ij}(v_{ij})$ of the v_{ij} for $i \leqq j$ are independent. In other words, there are no statistical correlations between the matrix elements, except for the condition of symmetry.

(b) The distribution law for each v_{ij} is symmetric.

(c) The distribution laws for all v_{ij} are such that all moments of v_{ij} exist and have an upper bound which is independent of i and j. Because of (b) the odd moments all vanish.

(d) The second moment of all v_{ij} is the same and will be denoted by m^2.

Actually, the last condition can be relaxed considerably so that it holds only for the large majority of the matrix elements. However, this point will not be pursued further. The preceding postulates can be summarized by the postulate that the fraction of the matrices of the set for which the i, j matrix element is within unit interval at v_{ij} is

$$P(v_{11}, v_{12}, v_{13}, \cdots, v_{NN}) = \textstyle\prod_{i\leqq j} p_{ij}(v_{ij}) \tag{2}$$

where

$$\int p_{ij}(v)dv = 1 \tag{2a}$$

and

[1] E. P. WIGNER, Ann. of Math., 62 (1955), 548. The title of the relevant section is *Random Sign Symmetric Matrix*, pp. 552–557. The distribution of the roots of "singly bordered" matrices in which only the diagonal elements are subject to random fluctuations was given by F. J. DYSON, Phys. Rev., 92 (1953), 1331. Cf. also H. SCHMIDT, ibid. 105 (1957), 425.

(2b) $$p_{ij}(v) = p_{ij}(-v)$$

(2c) $$\int p_{ij}(v)v^n dv \leqq B_n$$

(2d) $$\int p_{ij}(v)v^2 dv = m^2 .$$

We consider that the distribution functions p_{ij} are defined for all i, j and that the bound B_n is independent of i and j. All integrals are to be extended from $-\infty$ to ∞.

Under the conditions enumerated, crudely speaking, the fraction of roots within unit interval at x becomes

(3) $$\sigma(x) = \frac{(4Nm^2 - x^2)^{1/2}}{2\pi Nm^2} \qquad \text{for } x^2 < 4Nm^2$$

and

(3a) $$\sigma(x) = 0 \qquad \text{for } x^2 > 4Nm^2$$

as N grows beyond all limits. The distribution law (3) was stated before only for the case in which all p_{ij} for $i < j$ are equal and gave the probability 1/2 to the values m and $-m$ of v_{ij} for $i \neq j$ and the probability 1 to the value 0 of v_{ii}. Note that condition (d) is not fulfilled in this case except in the sense of the remark after the statement of that condition.

The theorem can be stated more accurately as follows. Denote by $S_{\alpha,\beta}(v, N)$ (where v, N is an abbreviation for all v_{ij} with $i \leqq j \leqq N$) the number of roots of the N dimensional symmetric matrix $\| v_{ij} \|$ which lie between $\alpha\sqrt{N}$ and $\beta\sqrt{N}$. Then, if the distribution P of the v_{ij} satisfies the conditions given, the fraction of the roots between $\alpha\sqrt{N}$ and $\beta\sqrt{N}$

(4) $$N^{-1}\int \cdots \int P(v, N)S_{\alpha,\beta}(v, N)dv_{11} \cdots dv_{NN} \rightarrow \int_\alpha^\beta \frac{(4m^2 - \xi^2)^{1/2}d\xi}{2\pi m^2}$$

as $N \rightarrow \infty$ if $-2m < \alpha < \beta < 2m$. If $\alpha < \beta < -2m$ or if $2m < \alpha < \beta$, the left side of (4) tends to zero as $N \rightarrow \infty$. Note that the theorem gives the distribution of the roots of sequences of sets of matrices, the matrices of successive sets of the sequence being obtained, from the matrices of the preceding set, by augmenting the matrices with further rows and columns. The distribution of the elements in these added columns is subject, apart from the two conditions of symmetry (1) and (2b), only to the conditions (2c) and (2d). This shows that the distribution of roots depends, under the conditions stated, only on the second moment of the matrix elements.

The heuristic proof given for the special case considered before[1] applies equally under the more general conditions here specified.

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