Density Functions for Families of Dirichlet Characters

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Abstract

We calculate the 1-Level Density for the following families of primitive Dirichlet characters:

1. all primitive characters of conductor m, m a fixed prime;
2. all primitive characters of conductor m, m an odd square-free number with r factors (r fixed);
3. all primitive characters whose conductor is a square-free odd integer $m \in [N, 2N]$.

As M. Rubinstein [Ru] has already considered all primitive quadratic characters with prime conductor $q \in [N, 2N]$, we do not include our notes of this case. For these families we show the 1-Level Densities agree with the Unitary Group for even Schwartz functions $\hat{\phi}$ with supp$(\hat{\phi}) \subset (-2, 2)$. We conclude with an appendix on "reasonable" assumptions which would allow us to extend the support of the test functions, possibly up to $(-4, 4)$.

These notes were written between 1999 and 2000 as a precursor to my dissertation; they are meant to be a rapid introduction to the calculations, with little motivation. For more on such calculations, see the paper by C. Hughes and Z. Rudnick [HR], where the first family was independently considered (and by additional arguments they obtained results for supp$(\hat{\phi}) \subset [-2, 2]$). For motivation as to why one considers such quantities, see [ILS, Mil, Ru].

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1 Review of Dirichlet Characters

Below we'll examine density functions for families of primitive Dirichlet characters, as well as sums involving Dirichlet characters.

1.1 L-Function and Functional Equation

Let χ be a primitive character mod m. Let

$$c(m,\chi) = \sum_{k=0}^{m-1} \chi(k) e^{2\pi i k/m}. \tag{1.1}$$

$c(m,\chi)$ is a Gauss sum of modulus $m^{\frac{1}{2}}$. The associated L-function and its analytic continuation are given by

$$\begin{aligned} L(s,\chi) &= \prod_p (1-\chi(p)p^{-s})^{-1} \\ \Lambda(s,\chi) &= \pi^{-\frac{1}{2}(s+\epsilon)}\Gamma(\frac{s+\epsilon}{2})m^{\frac{1}{2}(s+\epsilon)}L(s,\chi), \end{aligned} \tag{1.2}$$

where

$$\begin{aligned} \epsilon &= \begin{cases} 0 & \text{if } \chi(-1) = \ \ 1 \\ 1 & \text{if } \chi(-1) = -1 \end{cases} \\ \Lambda(s,\chi) &= (-i)^{\epsilon}\frac{c(m,\chi)}{m^{\frac{1}{2}}}\Lambda(1-s,\bar{\chi}). \end{aligned} \tag{1.3}$$

1.2 Explicit Formula and Density Conjecture

Let ϕ be an even Schwartz function with compact support, say contained in the interval $(-\sigma,\sigma)$, and let χ be a non-trivial primitive Dirichlet character of conductor m.

$$\begin{aligned} \sum \phi\left(\gamma\frac{\log(\frac{m}{\pi})}{2\pi}\right) &= \int_{-\infty}^{\infty}\phi(y)dy \\ &\quad -\sum_p \frac{\log p}{\log(m/\pi)}\hat{\phi}\Big(\frac{\log p}{\log(m/\pi)}\Big)[\chi(p)+\overline{\chi}(p)]p^{-\frac{1}{2}} \\ &\quad -\sum_p \frac{\log p}{\log(m/\pi)}\hat{\phi}\Big(2\frac{\log p}{\log(m/\pi)}\Big)[\chi^2(p)+\overline{\chi}^2(p)]p^{-1} \\ &\quad +O\left(\frac{1}{\log m}\right). \end{aligned} \tag{1.4}$$

We then sum over all curves in a family.

Definition 1.1 (First and Second Sums)**.** *We call the two sums above the First Sum and the Second Sum (respectively).*

The Density Conjecture states that the family average should converge to the Unitary Density:

$$\int_{-\infty}^{\infty}\phi(y)dy. \tag{1.5}$$

We will prove this for ϕ with suitable support.

2 Dirichlet Characters from a Prime Conductor

If m is prime, then $(\mathbb{Z}/m\mathbb{Z})^*$ is cyclic of order $m-1$ with generator g (so any element is of the form g^a for some a). Let $\zeta_{m-1} = e^{2\pi i/(m-1)}$. The principal character χ_0 is given by

$$\chi_0(k) \;=\; \begin{cases} 1 & \text{if } (k,m) = 1 \\ 0 & \text{if } (k,m) > 1. \end{cases} \tag{2.6}$$

Each of the $m-2$ primitive characters are determined (because they are multiplicative) once their action on a generator g is specified. As each $\chi : (\mathbb{Z}/m\mathbb{Z})^* \to \mathbb{C}^*$, for each χ there exists an l such that $\chi(g) = \zeta_{m-1}^l$. Hence for each $l, 1 \le l \le m-2$ we have

$$\chi_l(k) \;=\; \begin{cases} \zeta_{m-1}^{la} & \text{if } k \equiv g^a \bmod m \\ 0 & \text{if } (k,m) > 0. \end{cases} \tag{2.7}$$

So $\{\chi_0\} \cup \{\chi_l\}_{1 \le l \le m-2}$ are all the characters mod m, and as each χ_l is primitive, we may use the Explicit Formula. Consider the family of primitive characters mod a prime m. There are $m-2$ elements in this family. Then we must study

$$\begin{aligned} \int_{-\infty}^{\infty} \phi(y)dy \;&-\; \frac{1}{m-2} \sum_{\chi \ne \chi_0} \sum_p \frac{\log p}{\log(m/\pi)} \hat{\phi}\Big(\frac{\log p}{\log(m/\pi)}\Big)[\chi(p) + \overline{\chi}(p)]p^{-\frac{1}{2}} \\ &-\; \frac{1}{m-2} \sum_{\chi \ne \chi_0} \sum_p \frac{\log p}{\log(m/\pi)} \hat{\phi}\Big(2\frac{\log p}{\log(m/\pi)}\Big)[\chi^2(p) + \overline{\chi}^2(p)]p^{-1} \\ &+\; O\left(\frac{1}{\log m}\right). \end{aligned} \tag{2.8}$$

2.1 The First Sum

We must analyze (for m prime)

$$S_1 = \frac{1}{m-2} \sum_{\chi \ne \chi_0} \sum_p \frac{\log p}{\log(m/\pi)} \hat{\phi}\Big(\frac{\log p}{\log(m/\pi)}\Big)[\chi(p) + \overline{\chi}(p)]p^{-\frac{1}{2}}. \tag{2.9}$$

Since

$$\sum_{\chi} \chi(k) \;=\; \begin{cases} m-1 & \text{if } k \equiv 1 \bmod m \\ 0 & \text{otherwise.} \end{cases} \tag{2.10}$$

we have for any prime $p \ne m$

$$\sum_{\chi \ne \chi_0} \chi(p) \;=\; \begin{cases} m-2 & \text{if } p \equiv 1 \bmod m \\ -1 & \text{otherwise.} \end{cases} \tag{2.11}$$

Let

$$\delta_m(p,1) \;=\; \begin{cases} 1 & \text{if } p \equiv 1 \bmod m \\ 0 & \text{otherwise.} \end{cases} \tag{2.12}$$

The contribution to the sum from $p = m$ is zero; if instead we substitute -1 for $\sum_{\chi \neq \chi_0} \chi(m)$, our error is $O(\frac{1}{\log m})$ and hence negligible.

We now calculate S_1, suppressing the errors of $O(\frac{1}{\log m})$. $\hat{\phi}$ will be an even Schwartz function with support in $(-\sigma, \sigma)$.

$$\begin{aligned}
S_1 &= \frac{1}{m-2} \sum_{\chi \neq \chi_0} \sum_p \frac{\log p}{\log(m/\pi)} \hat{\phi}\Big(\frac{\log p}{\log(m/\pi)}\Big)[\chi(p) + \overline{\chi}(p)] p^{-\frac{1}{2}} \\
&= \frac{1}{m-2} \sum_p \frac{\log p}{\log(m/\pi)} \hat{\phi}\Big(\frac{\log p}{\log(m/\pi)}\Big) \sum_{\chi \neq \chi_0} [\chi(p) + \overline{\chi}(p)] p^{-\frac{1}{2}} \\
&= \frac{2}{m-2} \sum_p \frac{\log p}{\log(m/\pi)} \hat{\phi}\Big(\frac{\log p}{\log(m/\pi)}\Big) p^{-\frac{1}{2}} (-1 + (m-1)\delta_m(p,1)) \\
&= \frac{-2}{m-2} \sum_p^{m^\sigma} \frac{\log p}{\log(m/\pi)} \hat{\phi}\Big(\frac{\log p}{\log(m/\pi)}\Big) p^{-\frac{1}{2}} \\
&\quad + 2\frac{m-1}{m-2} \sum_{p \equiv 1(m)}^{m^\sigma} \frac{\log p}{\log(m/\pi)} \hat{\phi}\Big(\frac{\log p}{\log(m/\pi)}\Big) p^{-\frac{1}{2}} \\
&\ll \frac{1}{m} \sum_p^{m^\sigma} p^{-\frac{1}{2}} + \sum_{p \equiv 1(m)}^{m^\sigma} p^{-\frac{1}{2}} \\
&\ll \frac{1}{m} \sum_k^{m^\sigma} k^{-\frac{1}{2}} + \sum_{\substack{k \equiv 1(m) \\ k \geq m+1}}^{m^\sigma} k^{-\frac{1}{2}} \\
&\ll \frac{1}{m} \sum_k^{m^\sigma} k^{-\frac{1}{2}} + \frac{1}{m} \sum_k^{m^\sigma} k^{-\frac{1}{2}} \\
&\ll \frac{1}{m} m^{\sigma/2}.
\end{aligned} \tag{2.13}$$

Note: in the above, one must be careful with the estimates of the second sum. Each residue class of k mod m has approximately the same sum, with the difference between two classes bounded by the first term of whichever class has the smallest element. Since we are dropping the first term $(k = 1)$, the class of $k \equiv 1(m)$ has the smallest sum of the m classes. Hence if we add all the classes and divide by m, we increase the sum, so the above arguments are valid.

Hence $S_1 = \frac{1}{m} m^{\sigma/2} + O\left(\frac{1}{\log m}\right)$, implying that there is no contribution from the first sum if $\sigma < 2$.

2.2 The Second Sum

We must analyze (for m prime)

$$S_2 = \frac{1}{m-2} \sum_{\chi \neq \chi_0} \sum_p \frac{\log p}{\log(m/\pi)} \hat{\phi}\Big(2 \frac{\log p}{\log(m/\pi)}\Big) [\chi^2(p) + \overline{\chi}^2(p)] p^{-1}. \tag{2.14}$$

If $p \equiv \pm 1(m)$ then $\sum_{\chi \neq \chi_0} [\chi^2(p) + \overline{\chi}^2(p)] = 2(m-2)$. Otherwise, fix a generator g and write $p \equiv g^a(m)$. As $p \not\equiv \pm 1$, $a \not\equiv 0, \frac{m-1}{2}$ mod $(m-1)$, as $(\mathbb{Z}/m\mathbb{Z})^*$ is cyclic of order $m-1$. Hence $e^{4\pi i a/(m-1)} \neq 1$. Recall $\zeta_{m-1} = e^{2\pi i/(m-1)}$. Let $x = e^{4\pi i a/(m-1)} \neq 1$.

$$\begin{aligned}
S &= \sum_{\chi \neq \chi_0} [\chi^2(p) + \overline{\chi}^2(p)] = \sum_{l=1}^{m-2} [\chi_l^2(p) + \overline{\chi}_l^2(p)] \\
&= \sum_{l=1}^{m-2} [\chi_l^2(g^a) + \overline{\chi}_l^2(g^a)] \\
&= \sum_{l=1}^{m-2} [(\chi_l(g))^{2a} + (\overline{\chi}_l(g))^{2a}] \\
&= \sum_{l=1}^{m-2} [(\zeta_{m-1}^l)^{2a} + (\zeta_{m-1}^l)^{-2a}] \\
&= \sum_{l=1}^{m-2} [(\zeta_{m-1}^{2a})^l + (\zeta_{m-1}^{-2a})^l] \\
&= \sum_{l=1}^{m-2} [x^l + (x^{-1})^l] \\
&= \frac{x-1}{1-x} + \frac{x^{-1}-1}{1-x^{-1}} = -2.
\end{aligned} \tag{2.15}$$

The contribution to the sum from $p = m$ is zero; if instead we substitute -2 for $\sum_{\chi \neq \chi_0} \chi^2(m)$, our error is $O(\frac{1}{\log m})$ and hence negligible.

Therefore

$$\sum_{\chi \neq \chi_0} [\chi^2(p) + \overline{\chi}^2(p)] = \begin{cases} 2(m-2) & p \equiv \pm 1(m) \\ -2 & p \not\equiv \pm 1(m). \end{cases} \tag{2.16}$$

Let

$$\delta_m(p,\pm) = \begin{cases} 1 & \text{if } p \equiv \pm 1 \bmod m \\ 0 & \text{otherwise} \end{cases} \tag{2.17}$$

Up to $O\left(\frac{1}{\log m}\right)$ we find that

$$\begin{aligned}
S_2 &= \frac{1}{m-2}\sum_{\chi \neq \chi_0}\sum_p \frac{\log p}{\log(m/\pi)}\hat{\phi}\Big(2\frac{\log p}{\log(m/\pi)}\Big)[\chi^2(p)+\overline{\chi}^2(p)]p^{-1} \\
&= \frac{1}{m-2}\sum_p \frac{\log p}{\log(m/\pi)}\hat{\phi}\Big(2\frac{\log p}{\log(m/\pi)}\Big)\sum_{\chi \neq \chi_0}[\chi^2(p)+\overline{\chi}^2(p)]p^{-1} \\
&= \frac{1}{m-2}\sum_p^{m^{\sigma/2}} \frac{\log p}{\log(m/\pi)}\hat{\phi}\Big(2\frac{\log p}{\log(m/\pi)}\Big)p^{-1}[-2+(2m-2)\delta_m(p,\pm)] \\
&\ll \frac{1}{m-2}\sum_p^{m^{\sigma/2}} p^{-1} \ + \ \frac{2m-2}{m-2}\sum_{p\equiv\pm1(m)}^{m^{\sigma/2}} p^{-1} \\
&\ll \frac{1}{m-2}\sum_k^{m^{\sigma/2}} k^{-1} \ + \sum_{\substack{k\equiv 1(m)\\ k\geq m+1}}^{m^{\sigma/2}} k^{-1} \ + \sum_{\substack{k\equiv -1(m)\\ k\geq m-1}}^{m^{\sigma/2}} k^{-1} \\
&\ll \frac{1}{m-2}\log(m^{\sigma/2}) \ + \ \frac{1}{m}\sum_k^{m^{\sigma/2}} k^{-1} \ + \ \frac{1}{m}\sum_k^{m^{\sigma/2}} k^{-1} \ + \ O(\frac{1}{m}) \\
&\ll \frac{\log m}{m} + \frac{\log m}{m} + \frac{\log m}{m}.
\end{aligned} \tag{2.18}$$

Therefore $S_2 = O(\frac{\log m}{m})$, so for all σ there is no contribution.

2.3 Density Function from a Prime Conductor

Theorem 2.1 (Density Function from a Prime Conductor)**.** *Let $\hat{\phi}$ be an even Schwartz function with supp($\hat{\phi}$) $\subset (-2,2)$, m a prime, and $\mathcal{F}_m = \{\chi : \chi$ is primitive mod $m\}$. Then assuming GRH we have*

$$\frac{1}{\mathcal{F}_m}\sum_{\chi\in\mathcal{F}_m}\sum_{\gamma:L(\frac{1}{2}+i\gamma,\chi)=0}\phi\Big(\gamma\frac{\log(m/\pi)}{2\pi}\Big) = \int_{-\infty}^{\infty}\phi(y)dy + O(\frac{1}{\log m}). \tag{2.19}$$

3 Dirichlet Characters from a Square-free Number

Fix an r and let $m_1,\ldots,m_r$ be distinct odd primes. Let

$$\begin{aligned} m &= m_1 m_2 \cdots m_r \\ M_1 &= (m_1-1)(m_2-1)\cdots(m_r-1) = \phi(m) \\ M_2 &= (m_1-2)(m_2-2)\cdots(m_r-2). \end{aligned} \tag{3.20}$$

M_2 is the number of primitive characters mod m, each of conductor m. For each $l_i \in [1, m_i - 2]$ we have the primitive character discussed in the previous section, χ_{l_i}. A general primitive character mod m is given by a product of these characters:

$$\chi(u) = \chi_{l_1}(u)\chi_{l_2}(u)\cdots\chi_{l_r}(u) \tag{3.21}$$

Let $\mathcal{F} = \{\chi : \chi = \chi_{l_1}\chi_{l_2}\cdots\chi_{l_r}\}$. Then $|\mathcal{F}| = M_2$, and we are led to investigating the following sums:

$$\begin{aligned} S_1 &= \frac{1}{M_2}\sum_p \frac{\log p}{\log(m/\pi)}\hat{\phi}\Big(\frac{\log p}{\log(m/\pi)}\Big)p^{-\frac{1}{2}}\sum_{\chi\in\mathcal{F}}[\chi(p)+\overline{\chi}(p)] \\ S_2 &= \frac{1}{M_2}\sum_p \frac{\log p}{\log(m/\pi)}\hat{\phi}\Big(2\frac{\log p}{\log(m/\pi)}\Big)p^{-1}\sum_{\chi\in\mathcal{F}}[\chi^2(p)+\overline{\chi}^2(p)] \end{aligned} \tag{3.22}$$

3.1 The First Sum (m Square-free)

We must study $\sum_{\chi\in\mathcal{F}}\chi(p)$ (the sum with $\overline{\chi}$ is handled similarly). In the previous section we showed

$$\sum_{l_i=1}^{m_i-2}\chi_{l_i}(p) = \begin{cases} m_i - 1 - 1 & \text{if } p \equiv 1 \bmod m_i \\ -1 & \text{otherwise.} \end{cases} \tag{3.23}$$

Define

$$\delta_{m_i}(p,1) = \begin{cases} 1 & \text{if } p \equiv 1 \bmod m_i \\ 0 & \text{otherwise.} \end{cases} \tag{3.24}$$

Then

$$\begin{aligned} \sum_{\chi\in\mathcal{F}}\chi(p) &= \sum_{l_1=1}^{m_1-2}\cdots\sum_{l_r=1}^{m_r-2}\chi_{l_1}(p)\cdots\chi_{l_r}(p) \\ &= \prod_{i=1}^{r}\sum_{l_i=1}^{m_i-2}\chi_{l_i}(p) \\ &= \prod_{i=1}^{r}(-1+(m_i-1)\delta_{m_i}(p,1)). \end{aligned} \tag{3.25}$$

Let us denote by $k(s)$ an s-tuple $(k_1, k_2, \ldots, k_s)$ with $k_1 < k_2 < \cdots < k_s$. This is just a subset of $(1, 2, \ldots, r)$. There are 2^r possible choices for $k(s)$. We will use these to expand the above product. Define

$$\delta_{k(s)}(p,1) = \prod_{i=1}^{s} \delta_{m_{k_i}}(p,1). \tag{3.26}$$

If $s = 0$ we define $\delta_{k(0)}(p,1) = 1\ \forall p$. Then

$$\prod_{i=1}^{r}(-1+(m_i-1)\delta_{m_i}(p,1)) = \sum_{s=0}^{r}\sum_{k(s)}(-1)^{r-s}\delta_{k(s)}(p,1)\prod_{i=1}^{s}(m_{k_i}-1) \tag{3.27}$$

Let $h(p) = 2\frac{\log p}{\log(m/\pi)}\hat{\phi}\left(\frac{\log p}{\log(m/\pi)}\right) \ll ||\hat{\phi}||$. Then

$$\begin{aligned} S_1 &= \sum_{p}^{m^\sigma}\frac{1}{2}h(p)p^{-\frac{1}{2}}\frac{1}{M_2}\sum_{\chi\in\mathcal{F}}[\chi(p)+\overline{\chi}(p)] \\ &= \sum_{p}^{m^\sigma}h(p)p^{-\frac{1}{2}}\frac{1}{M_2}\sum_{s=0}^{r}\sum_{k(s)}(-1)^{r-s}\delta_{k(s)}(p,1)\prod_{i=1}^{s}(m_{k_i}-1) \\ &\ll \sum_{p}^{m^\sigma}p^{-\frac{1}{2}}\frac{1}{M_2}\Big(1+\sum_{s=1}^{r}\sum_{k(s)}\delta_{k(s)}(p,1)\prod_{i=1}^{s}(m_{k_i}-1)\Big). \end{aligned} \tag{3.28}$$

Observing that $m/M_2 \le 3^r$ we see the $s = 0$ sum contributes

$$S_{1,0} = \frac{1}{M_2}\sum_{p}^{m^\sigma}p^{-\frac{1}{2}} \ll 3^r m^{\frac{1}{2}\sigma-1}, \tag{3.29}$$

hence negligible for $\sigma < 2$. Now we study

$$S_{1,k(s)} = \frac{1}{M_2}\prod_{i=1}^{s}(m_{k_i}-1)\sum_{p}^{m^\sigma}p^{-\frac{1}{2}}\delta_{k(s)}(p,1). \tag{3.30}$$

The effect of the factor $\delta_{k(s)}(p,1)$ is to restrict the summation to primes $p \equiv 1(m_{k_i})$ for $k_i \in k(s)$. The sum will increase if instead of summing over primes satisfying the congruences we sum over all numbers n satisfying the congruences (with $n \ge 1+\prod_{i=1}^{s} m_{k_i}$). But now that the sum is over integers and not primes, we can use basic uniformity properties of integers to bound it. We are summing integers mod $\prod_{i=1}^{s} m_{k_i}$, so summing over integers satisfying these congruences is basically just $\prod_{i=1}^{s}(m_{k_i})^{-1}\sum_{n=1}^{m^\sigma} n^{-\frac{1}{2}} = \prod_{i=1}^{s}(m_{k_i})^{-1}m^{\frac{1}{2}\sigma}$. We can do this as the sum of the reciprocals from the residue classes of $\prod_{i=1}^{s} m_{k_i}$ differ by at most their first term. Throwing out the first term of the class $1+\prod_{i=1}^s m_{k_i}$ makes it have the smallest sum of the $\prod_{i=1}^s m_{k_i}$ classes, so adding all the classes and dividing by $\prod_{i=1}^s m_{k_i}$ increases the sum.

Hence (recalling $m/M_2 \leq 3^r$)

$$\begin{aligned} S_{1,k(s)} &\ll \frac{1}{M_2}\prod_{i=1}^s (m_{k_i}-1)\prod_{i=1}^s (m_{k_i})^{-1} m^{\frac{1}{2}\sigma} \\ &\ll 3^r m^{\frac{1}{2}\sigma-1}. \end{aligned} \tag{3.31}$$

Therefore, $\forall s$ the $S_{1,k(s)}$ contribute $3^r m^{\frac{1}{2}\sigma-1}$. There are 2^r choices, yielding

$$S_1 \ll 6^r m^{\frac{1}{2}\sigma-1}, \tag{3.32}$$

which is negligible as m goes to infinity for fixed r if $\sigma < 2$. We cannot let r go to infinity in the arguments above because if m is the product of the first r primes, then for r large,

$$\begin{aligned} \log m &= \sum_{k=1}^r \log p \\ &= \sum_{p\leq r} \log p \;\approx\; r \\ &\to\; 6^r \approx m^{\log 6} \;\approx\; m^{1.79}. \end{aligned} \tag{3.33}$$

3.2 The Second Sum (m Square-free)

We must study $\sum_{\chi\in\mathcal{F}} \chi^2(p)$ (the sum with $\overline{\chi}$ is handled similarly). In the previous section we showed

$$\sum_{l_i=1}^{m_i-2} \chi_{l_i}^2(p) \;=\; \begin{cases} m_i-1-1 & \text{if } p \equiv \pm 1 \bmod m_i \\ -1 & \text{otherwise.} \end{cases} \tag{3.34}$$

Then

$$\begin{aligned} \sum_{\chi\in\mathcal{F}} \chi^2(p) &= \sum_{l_1=1}^{m_1-2} \cdots \sum_{l_r=1}^{m_r-2} \chi_{l_1}^2(p)\cdots\chi_{l_r}^2(p) \\ &= \prod_{i=1}^r \sum_{l_i=1}^{m_i-2} \chi_{l_i}^2(p) \\ &= \prod_{i=1}^r (-1+(m_i-1)\delta_{m_i}(p,1)+(m_i-1)\delta_{m_i}(p,-1)). \end{aligned} \tag{3.35}$$

We now show the Second Sum is negligible for all σ. Instead of having 2^r terms we have 3^r. Let $k(s)$ be as before, and let $j(s)$ be an s-tuple of ± 1s. As s ranges from 0 to r we get each of the 3^r possibilities, as for a fixed s, there are $\binom{r}{s}$ choices for $k(s)$, each of these having 2^s choices for $j(s)$. But $\sum_{s=0}^{r} 2^s \binom{r}{k} = (1+2)^r$. Let $h(p) = 2\frac{\log p}{\log(m/\pi)} \hat{\phi}\Big(2\frac{\log p}{\log(m/\pi)}\Big) \ll ||\hat{\phi}||$. Define

$$\delta_{k(s)}(p, j(s)) = \prod_{i=1}^{s} \delta_{m_{k_i}}(p, j_i). \tag{3.36}$$

Then

$$\sum_{\chi \in \mathcal{F}} \chi^2(p) = \sum_{s=0}^{r} \sum_{k(s)} \sum_{j(s)} (-1)^{r-s} \delta_{k(s)}(p, j(s)) \prod_{i=1}^{s} (m_{k_i} - 1) \tag{3.37}$$

Therefore

$$\begin{aligned} S_2 &= \frac{1}{M_2} \sum_p \frac{\log p}{\log(m/\pi)} \hat{\phi}\Big(2\frac{\log p}{\log(m/\pi)}\Big) p^{-1} \sum_{\chi \in \mathcal{F}} [\chi^2(p) + \overline{\chi}^2(p)] \\ &= \frac{1}{M_2} \sum_p h(p) \sum_{s=0}^{r} \sum_{k(s)} \sum_{j(s)} p^{-1} (-1)^{r-s} \delta_{k(s)}(p, j(s)) \prod_{i=1}^{s} (m_{k_i} - 1) \\ &\ll \frac{1}{M_2} \sum_p \sum_{s=0}^{r} \sum_{k(s)} \sum_{j(s)} p^{-1} \delta_{k(s)}(p, j(s)) \prod_{i=1}^{s} (m_{k_i} - 1) \\ &= \sum_{s=0}^{r} \sum_{k(s)} \sum_{j(s)} S_{2,k(s),j(s)}. \end{aligned} \tag{3.38}$$

The term where $s = 0$ is handled easily (recall $m/M_2 \leq 3^r$):

$$S_{2,0,0} = \frac{1}{M_2} \sum_p^{m^\sigma} p^{-1} \ll 3^r \frac{\log m^\sigma}{m}. \tag{3.39}$$

We would like to handle the terms for $s \neq 0$ analogously as before. The congruences on p from $k(s)$ and $j(s)$ force us to sum only over certain primes mod $\prod_{i=1}^{s} m_{k_i}$, with each prime satisfying $p \geq m_{k_i} \pm 1$. We increase the sum by summing over all integers satisfying these congruences. As each congruence class mod $\prod_{i=1}^{s} m_{k_i}$ has basically the same sum, we can bound our sum over primes satisfying the congruences $k(s), j(s)$ by $\prod_{i=1}^{s} (m_{k_i})^{-1} \sum_{n=1}^{m^\sigma} n^{-1} = \prod_{i=1}^{s} (m_{k_i})^{-1} \log m^\sigma$.

There is one slight problem with this argument. Before each prime was congruent to 1 mod each prime m_{k_i}, hence the first prime occurred no earlier than at $1 + \prod_{k=1}^{s} m_{k_i}$. Now, however, some primes are congruent to $+1$ mod m_{k_i}, some to -1, and it is possible the first such prime occurs before $\prod_{k=1}^{s} m_{k_i}$.

For example, say the prime is congruent to $+1$ mod 11, and -1 mod $3, 5, 17$. We want the prime to be greater than $3 \cdot 5 \cdot 11 \cdot 17$, but $3 \cdot 5 \cdot 17 - 1$ is congruent to -1 mod $3, 5, 17$ and $+1$ mod 11. (Fortunately it equals 254, which is composite).

So, for each pair $(k(s), j(s))$ we handle all but the possibly first prime as we did in the First Sum case. We now need an estimate on the possible error for low primes. Fortunately, there is at most one for each pair, and as our sum has a $\frac{1}{p}$, we can expect cancellation if it is large.

Fix now a pair (remember there are at most 3^r pairs). As we never specified the order of the primes m_i, without loss of generality (basically, for notational convenience) we may assume that our prime p is congruent to $+1$ mod $m_{k_1} \cdots m_{k_a}$, and -1 mod $m_{k_{a+1}} \cdots m_{k_s}$.

The contribution to the second sum from the possible low prime in this pair is

$$\frac{1}{M_2} \frac{1}{p} \prod_{i=1}^{s} (m_{k_i} - 1). \tag{3.40}$$

How small can p be? The $+1$ congruences imply that $p \equiv 1 (m_{k_1} \cdots m_{k_a})$, so p is at least $m_{k_1} \cdots m_{k_a} + 1$. Similarly the -1 congruences imply p is at least $m_{k_{a+1}} \cdots m_{k_s} - 1$. Since the product of these two lower bounds is greater than $\prod_{i=1}^{s} (m_{k_i} - 1)$, at least one must be greater than $\Big(\prod_{i=1}^{s} (m_{k_i} - 1) \Big)^{\frac{1}{2}}$. Therefore the contribution to the second sum from the possible low prime in this pair is bounded by (remember $m/M_2 \leq 3^r$)

$$\frac{1}{M_2} \Big(\prod_{i=1}^{s} (m_{k_i} - 1) \Big)^{\frac{1}{2}} \leq \frac{m^{\frac{1}{2}}}{M_2} \leq 3^r m^{-\frac{1}{2}}. \tag{3.41}$$

Combining this with the estimate for the primes larger than $\prod_{i=1}^{s} (m_{k_i} - 1)$ yields

$$S_{2,k(s),j(s)} \ll 3^r m^{-\frac{1}{2}} + \frac{3^r}{m} \log m^{\sigma}, \tag{3.42}$$

yielding (as there are 3^r pairs)

$$S_2 = \sum_{s=0}^{r} \sum_{k(s)} \sum_{j(s)} S_{2,k(s),j(s)} \ll 9^r m^{-\frac{1}{2}}. \tag{3.43}$$

3.3 Density Function in the Square-free case

Theorem 3.1 (Density Function for Square-free m). *Let $\hat{\phi}$ be an even Schwartz function with supp($\hat{\phi}$) $\subset (-2, 2)$. Fix an $r \geq 1$. Let $\mathcal{F}_m = \{\chi : \chi$ is primitive mod $m\}$, where m is a* *square-free odd integer. Then assuming GRH we have*

$$\frac{1}{\mathcal{F}_m}\sum_{\chi\in\mathcal{F}_m}\sum_{\gamma:L(\frac{1}{2}+i\gamma,\chi)=0}\phi\Big(\gamma\frac{\log(m/\pi)}{2\pi}\Big) = \int_{-\infty}^{\infty}\phi(y)dy + O(\frac{1}{\log m}). \tag{3.44}$$

We note for future reference the following bounds on the First and Second sums:

Lemma 3.2. *Let m be a square-free odd integer with $r = r(m)$ factors. Let $m = \prod_{i=1}^r m_i$ and $M_2 = \prod_{i=1}^r (m_i - 2)$. Consider the family $\mathcal{F}_m$ of primitive characters mod m. There are M_2 such characters, and the First and Second sums satisfy the following bounds:*

$$\begin{aligned} S_1 &\ll \frac{1}{M_2}2^r m^{\frac{1}{2}\sigma} \\ S_2 &\ll \frac{1}{M_2}3^r m^{\frac{1}{2}}. \end{aligned} \tag{3.45}$$

4 Dirichlet Characters from Square-free Numbers

We now generalize the results of the previous section to consider the family $\mathcal{F}_N$ of all primitive characters whose conductor is an odd square-free integer in $[N, 2N]$. Some of the bounds below can be improved, but as the improvements do not increase the range of convergence, they will only be sketched.

First we calculate the number of primitive characters arising from odd square-free numbers $m \in [N, 2N]$. Let $n = n_1 n_2 \cdots n_r$. Then n contributes $(n_1 - 2)\cdots(n_r - 2)$ characters. On average we might expect this to be (up to a constant) N, and as a positive percent of numbers are square-free, we might expect there to be cN^2 characters.

Instead we prove there are at least $N^2/\log^2 N$ primitive characters in the family. There are at least $N/\log^2 N + 1$ primes in the interval. For each prime p (except possibly the first) we have $p - 2 \geq N$. Hence there are at least $N \cdot \frac{N}{\log^2 N} = N^2\log^{-2} N$ primitive characters. Let $M = |\mathcal{F}|$. Then

$$M \ \geq \ N^2\log^{-2} N \quad \Rightarrow \quad \frac{1}{M} \ \leq \ \frac{\log^2 N}{N^2}. \tag{4.46}$$

We recall the results from the previous section. Fix an odd square-free number $m \in [N, 2N]$, and say m has $r = r(m)$ factors. Before we divided the First and Second sums by $M_2 = (m_1 - 2)\cdots(m_r - 2)$, as this was the number of primitive characters in our family. Now we divide by M. Hence the contribution to the First and Second sum from this m is

$$\begin{aligned} S_{1,m} &\ll \frac{1}{M} 2^{r(m)} m^{\frac{1}{2}\sigma} \\ S_{2,m} &\ll \frac{1}{M} 3^{r(m)} m^{\frac{1}{2}}. \end{aligned} \tag{4.47}$$

Note that $2^{r(m)} = \tau(m)$, the number of divisors of m. While it is possible to prove

$$\sum_{n \le x} \tau^l(n) \ll x(\log x)^{2^l - 1} \tag{4.48}$$

the crude bound

$$\tau(n) \le c(\epsilon) n^{\epsilon} \tag{4.49}$$

yields the same region of convergence. Note $3^{r(m)} \le \tau^2(m)$. Therefore the contributions to the first sum is majorized by

$$\begin{aligned} S_1 &= \sum_{\substack{m=N \\ m\ squarefree}}^{2N} S_{1,m} \\ &\ll \sum_{m=N}^{2N} \frac{1}{M} 2^{r(m)} m^{\frac{1}{2}\sigma} \\ &\ll \frac{1}{M} N^{\frac{1}{2}\sigma} \sum_{m=N}^{2N} \tau(m) \\ &\ll \frac{1}{M} N^{\frac{1}{2}\sigma} c(\epsilon) N^{1+\epsilon} \\ &\ll \frac{\log^2 N}{N^2} N^{\frac{1}{2}\sigma} c(\epsilon) N^{1+\epsilon} \\ &\ll c(\epsilon) N^{\frac{1}{2}\sigma+\epsilon-1} \log^2 N. \end{aligned} \tag{4.50}$$

For $\sigma < 2$, choosing $\epsilon < 1 - \frac{1}{2}\sigma$ yields S_1 goes to zero as N tends to infinity. For S_2 we have

$$
\begin{aligned}
S_2 &= \sum_{\substack{m=N \\ m \ squarefree}}^{2N} S_{2,m} \\
&\ll \sum_{m=N}^{2N} \frac{1}{M} 3^{r(m)} m^{\frac{1}{2}} \\
&\ll \frac{1}{M} N^{\frac{1}{2}} \sum_{m=N}^{2N} \tau^2(m) \\
&\ll c(\epsilon) \frac{\log^2 N}{N^2} N^{\frac{1}{2}} N^{1+2\epsilon} \\
&\ll c(\epsilon) N^{2\epsilon - \frac{1}{2}} \log^2 N. \qquad (4.51)
\end{aligned}
$$

which converges to zero as N tends to infinity for all σ. Hence we have proved

Theorem 4.1 (Dirichlet Characters from Square-free Numbers)**.** *Let $\mathcal{F}_N$ denote the family of primitive Dirichlet characters arising from odd square-free numbers $m \in [N, 2N]$. Denote the conductor of χ by $c(\chi)$. Then $\forall \sigma < 2$*

$$
\frac{1}{\mathcal{F}_N} \sum_{\chi \in \mathcal{F}_N} \sum_{\gamma : L(\frac{1}{2} + i\gamma, \chi) = 0} \phi\Big(\gamma \frac{\log(c(\chi)/\pi)}{2\pi}\Big) = \int_{-\infty}^{\infty} \phi(y) dy + O(\frac{1}{\log N}). \qquad (4.52)
$$

5 Summary

In all the cases investigated, we observe the First and Second sums do not contribute for even Schwartz test functions ϕ with support $\sigma < 2$.

Thus, in the notation of Katz and Sarnak, the 1-level densities for these families is $\int_{\mathbb{R}} \phi(x) dx$, which agrees with the 1-level density for Unitary matrices.

A similar calculation should yield the 2-level densities agreeing with that of Unitary matrices, though only up to support $\sigma < 1$.

A Natural Conjectures to Extend the Support

Trivial estimation of prime sums yield the 1-level density for families of Dirichlet L-functions for supp$(\widehat{\phi}) \subset (-2, 2)$. We discuss some natural (we hope!) conjectures for the distribution of primes in residue classes, and how these would allow us to increase the support. Specifically, consider estimates of errors for the distribution of primes in residue classes. Assuming GRH (and anything else that you find reasonable!), how are the errors or excesses split among the various classes? Specifically, what is the modulus dependence on average.

A.1 Definitions and Preliminaries

Let m either be a prime or range over primes in $[N, 2N]$. Let

$$\begin{aligned} \psi(x) &= \sum_{n\le x}\Lambda(n) \\ \psi(x,q,a) &= \sum_{\substack{n\le x\\ n\equiv a \bmod q}}\Lambda(n) \\ E(x,q,a) &= \psi(x,q,a)-\frac{\psi(x)}{\phi(q)}. \end{aligned} \tag{A.53}$$

If we assume GRH, we have (we could replace ϵ with powers of $\log$ below) that

$$\begin{aligned} \psi(x) &= x+O(x^{\frac12+\epsilon}) \\ \psi(x,q,a) &= \frac{\psi(x)}{\phi(q)}+O(x^{\frac12}\cdot(xq)^{\epsilon}) \\ E(x,q,a) &= O(x^{\frac12}\cdot(xq)^{\epsilon}). \end{aligned} \tag{A.54}$$

Probabilistic arguments suggest that $E(x,q,a)$ should be much smaller. Expecting square-root cancellation, we have $\phi(q)$ residue classes. If the error of size $x^{\frac12+\epsilon}$ is spread among these $\phi(q)$ classes equally, we expect each $\psi(x,q,a)$ to be of size $\frac{\psi(x)}{\phi(q)}$ with errors of size $\sqrt{\frac{x}{\phi(q)}}\cdot(xq)^{\epsilon}$; see [Mon1]. It is by gaining some savings in q in the error that we can increase the support for families of Dirichlet L-functions.

Consider the total variance

$$V(x,q) \;=\; \sum_{\substack{a=1\\ (a,q)=1}}^{q}\left|\psi(x,q,a)-\frac{x}{\phi(q)}\right|^2; \tag{A.55}$$

dividing by $\frac{1}{\phi(q)}$ would give the average variance. Note we subtract $\frac{x}{\phi(q)}$ and not $\frac{\psi(x)}{\phi(q)}$, though assuming GRH, either gives the same results in terms of increasing the support.

Goldston and Vaughan [GV] have shown that under GRH,

$$\sum_{q\le Q}V(x,q) \;=\; Qx\log Q-cxQ+O\left(Q^2(x/Q)^{\frac14+\epsilon}+x^{\frac32}(\log 2x)^{\frac52}(\log\log 3x)^2\right). \tag{A.56}$$

If each $E(x,q,a)$ were of size $\sqrt{\frac{x}{\phi(q)}}\cdot(xq)^{\epsilon}$, we would expect

$$V(x,q) \;\approx\; x\cdot(xq)^{\epsilon} \tag{A.57}$$

and

$$\sum_{q\le Q}V(x,q) \;\approx\; Qx\cdot(xQ)^{\epsilon}. \tag{A.58}$$

In fact, Hooley has conjectured that (A.57) holds for some unspecified range of q (replacing ϵ with logarithms).

A.2 Conjectures for Distribution Among Residue Classes

Conjecture A.1. *Assume $\exists \theta \in [0,1]$ such that either of the following hold:*

1. *for all (or at least a sequence of primes tending to infinity) prime $m \ll \sqrt{u} \ll m^{2-\theta}$,*

$$E(u,m,1)^2 \ \ll \ m^\theta \cdot \frac{1}{\phi(m)} \sum_{\substack{a=1 \\ (a,m)=1}}^{m} E(u,m,a)^2. \tag{A.59}$$

2. *for prime $m \in [N, 2N]$ with $N \ll \sqrt{u} \ll N^{2-\theta}$,*

$$\sum_{\substack{m=N \\ m \ prime}}^{2N} E(u,m,1)^2 \ \ll \ N^\theta \cdot \frac{1}{N} \sum_{\substack{m=N \\ m \ prime}}^{2N} \sum_{\substack{a=1 \\ (a,m)=1}}^{m} E(u,m,a)^2. \tag{A.60}$$

Then the 1-level density for the family of Dirichlet L-functions can be extended to hold for test functions whose Fourier transforms are supported in $(-4+2\theta, 4-2\theta)$.

This conjecture is trivially true for $\theta = 1$, and is unlikely to be true for $\theta = 0$. Is it reasonable to expect either version to hold for say $\theta = \epsilon$ (for any $\epsilon > 0$)? Basically, what we need is some control over biases of primes to be congruent to $1 \bmod m$. For the residue class $a \bmod m$, $E(u,m,a)^2$ is the variance; the above conjecture can be interpreted as bounding $E(u,m,1)^2$ in terms of the average variance. Interestingly, $\theta = 1$ recovers the 1-level density result of support in $(-2,2)$.

Bounds such as these are useful as, by using the Cauchy-Schwartz inequality, the variance $E(u,m,1)^2$ surfaces in investigating the 1-level density sums. If we can express the variance $E(u,m,1)^2$ in terms of the average variance, the bounds from Goldston-Vaughan are applicable. There is also the possibility of using higher moment bounds and Holder's Inequality instead of Cauchy-Schwartz (see [Va]); unfortunately, Vaughan's results only hold for m "close" to u. Explicitly, $u^{\frac{3}{4}+\epsilon} \ll m \ll x$. To obtain better support than $(-2,2)$, we need $u \gg \sqrt{q}$.

The question is: for what θ is the above conjecture "reasonable"? Can we glean a reasonable value for θ from the arguments in say [RubSa], or from probabilistic arguments on random primes (where with probability one we know RH is true for a random sequence of primes – what is known there about error terms in congruence classes, and how that depends on the modulus)?

One could probably work with all square-free m and not just prime m in the Dirichlet L-function's densities; however, as the variances are positive, if bounds like this don't hold for m restricted to prime values, they won't hold for m square-free (because we are going for more than a logarithm savings).

References

[Da] H. Davenport, *Multiplicative Number Theory*, 2*nd edition*, Graduate Texts in Mathematics **74**, Springer-Verlag, New York, 1980, revised by H. Montgomery.

[GV] D. A. Goldston and R. C. Vaughan, *On the Montgomery-Hooley asymptotic formula*, Sieve methods, exponential sums and their applications in number theory (ed. G. R. H. Greaves, G. Harman and M. N. Huxley, Cambridge University Press, 1996), 117-142.

[HW] G. Hardy and E. Wright, *An Introduction to the Theory of Numbers*, fifth edition, Oxford Science Publications, Clarendon Press, Oxford, 1995.

[Hej] D. Hejhal, *On the triple correlation of zeros of the zeta function*, Internat. Math. Res. Notices 1994, no. 7, $294 - 302$.

[HR] C. Hughes and Z. Rudnick, *Linear Statistics of Low-Lying Zeros of L-functions*, to appear.

[ILS] H. Iwaniec, W. Luo and P. Sarnak, *Low lying zeros of families of L-functions*, Inst. Hautes Études Sci. Publ. Math. **91**, 2000, $55 - 131$.

[KS1] N. Katz and P. Sarnak, *Random Matrices, Frobenius Eigenvalues and Monodromy*, AMS Colloquium Publications **45**, AMS, Providence, 1999.

[Mes] J. Mestre, *Formules explicites et minorations de conducteurs de variétés algébriques*, Compositio Mathematica **58**, 1986, $209 - 232$.

[Mil] S. J. Miller, 1- *and* 2-*Level Densities for Families of Elliptic Curves: Evidence for the Underlying Group Symmetries*, P.H.D. Thesis, Princeton University, 2002, http://www.math.princeton.edu/∼sjmiller/thesis/thesis.pdf.

[Mon1] H. Montgomery, *Prime's in arithmetic progression*, Michigan Math. J. 17 (1970), 33-39.

[Mon2] H. Montgomery, *The pair correlation of zeros of the zeta function*, Analytic Number Theory, Proc. Sympos. Pure Math. **24**, Amer. Math. Soc., Providence, 1973, $181 - 193$.

[Ru] M. Rubinstein, *Evidence for a spectral interpretation of the zeros of L-functions*, P.H.D. Thesis, Princeton University, 1998, http://www.ma.utexas.edu/users/miker/thesis/thesis.html.

[RubSa] M. Rubinstein and P. Sarnak, *Chebyshev's bias*, Experiment. Math. 3 (1994), no. 3, 173-197.

[RS] Z. Rudnick and P. Sarnak, *Zeros of principal L-functions and random matrix theory*, Duke Journal of Math. **81**, 1996, $269-322$.

[Va] R. C. Vaughan, *On a variance associated with the distribution of primes in arithmetic progression*, Proc. London Math. Soc. (3) **82** (2001), 533-553.