FOLDED CONTINUED FRACTIONS

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We exhibit uncountably many binary decimals together with their explicit continued fraction expansions. These expansions require only the partial quotients 1 or 2. The pattern of valleys and ridges in a sheet of paper repeatedly folded in half plays a critical rôle in our construction.

1. Introduction

It is notorious that it is generally damnably difficult to explicitly compute the continued fraction of a quantity presented in some other form. However, we shall exhibit a class of series, and thence of numbers presented in effect as binary decimals for which we can display the continued fraction expansions. In particular, the uncountably many numbers

$$2\sum_{h=0}^{\infty} \pm 2^{-2^h}$$

all have continued fraction expansions with partial quotients 1 or 2 only.

Our result that the series

$$X\sum_{h=0}^{\infty} \pm X^{-2^h}$$

all have folded continued fractions is new, whilst our specialisations generalise and complete remarks on special cases in [3] and [8].

By a result of Loxton and van der Poorten [5], generalising *inter alia* a result of Kempner [4], these numbers all are transcendental. Thus our result gives no information on the the conjecture that all algebraic numbers of degree at least 3 have unbounded partial quotients.

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2. Continued fractions

A continued fraction is an object of the form

$$c_0 + \cfrac{1}{c_1 + \cfrac{1}{c_2 + \cfrac{1}{c_3 + \ddots}}}$$

denoted, to save vertical space, in the flat notation $[c_0\,,\,c_1\,,\,c_2\,,\,c_3\,,\,\ldots]$. Essentially all one needs to understand the behaviour of these objects is contained in the *fundamental correspondence* whereby:

Proposition 1. *For* $h = 0,\ 1,\ 2,\ \ldots$

$$\frac{p_h}{q_h} = [c_0\,,\,c_1\,,\,\ldots\,,\,c_h]$$

if and only if

$$\begin{pmatrix} c_0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c_1 & 1 \\ 1 & 0 \end{pmatrix}\cdots\cdots\begin{pmatrix} c_h & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_h & p_{h-1} \\ q_h & q_{h-1} \end{pmatrix}.$$

Proof. The correspondence defines the *convergents* p_h/q_h by matrix products and is readily verified by induction on the number $h+1$ of *partial quotients* c_n, respectively the number of matrices.

The fundamental correspondence is just the observation that the well known recursion formulæ

$$p_{n+1} = a_{n+1}p_n + p_{n-1}$$
$$q_{n+1} = a_{n+1}q_n + q_{n-1}$$

together with

$$\begin{pmatrix} p_{-1} & p_{-2} \\ q_{-1} & q_{-2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

define the sequence of convergents. ∎

By taking the transpose of the matrix product we have, for example,

$$\begin{pmatrix} c_n & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c_{n-1} & 1 \\ 1 & 0 \end{pmatrix}\cdots\cdots\begin{pmatrix} c_1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c_0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_n & q_n \\ p_{n-1} & q_{n-1} \end{pmatrix}$$
$$\longleftrightarrow [c_n\,,\,c_{n-1}\,,\,\ldots\ldots\,,\,c_1\,,\,c_0]\,,$$

where $\longleftrightarrow$ denotes the correspondence between matrix products and continued fractions, so

$$[c_n\,,\,c_{n-1}\,,\,\ldots\ldots\,,\,c_1] = \frac{q_n}{q_{n-1}}\,.$$

In this spirit, we also recall that:

Proposition 2.

$$\frac{p_n}{q_n}+\frac{(-1)^n}{xq_n^2}=[c_0\,,\,\overrightarrow{w}\,,\,x-\frac{q_{n-1}}{q_n}]=[c_0\,,\,\overrightarrow{w}\,,\,x\,,\,-\overleftarrow{w}\,].$$

Here $\overrightarrow{w}$ *is a convenient abbreviation for the word* $c_1, c_2, \ldots, c_n$ *and, accordingly,* $-\overleftarrow{w}$ *denotes the word* $-c_n, -c_{n-1}, \ldots, -c_1$.

Proof. Indeed,

$$\begin{aligned}[c_0\,,\,\overrightarrow{w}\,,\,x-\frac{q_{n-1}}{q_n}] &\longleftrightarrow \begin{pmatrix} p_n & p_{n-1}\\ q_n & q_{n-1}\end{pmatrix}\begin{pmatrix} x-q_{n-1}/q_n & 1\\ 1 & 0\end{pmatrix}\\ &= \begin{pmatrix} xp_n-\big(p_nq_{n-1}-p_{n-1}q_n\big)/q_n & p_n\\ xq_n & q_n\end{pmatrix}\longleftrightarrow \frac{p_n}{q_n}+\frac{(-1)^n}{xq_n^2}\end{aligned}$$

since $\big(p_nq_{n-1}-p_{n-1}q_n\big)=(-1)^{n-1}$; and, of course, $x-q_{n-1}/q_n=[x\,,\,-\overleftarrow{w}\,]$. ∎

Unless one adopts conventions restricting partial quotients in some appropriate manner, a continued fraction expansion is not unique. For example, the computation

$$\begin{aligned} x-q_{n-1}/q_n &= x-1+(q_n-q_{n-1})/q_n\\ q_n/(q_n-q_{n-1}) &= 1+q_{n-1}/(q_n-q_{n-1})\\ (q_n-q_{n-1})/q_{n-1} &= -1+q_n/q_{n-1}\\ q_{n-1}/q_n &= 0+q_{n-1}/q_n\\ q_n/q_{n-1} &= \overleftarrow{w}\end{aligned}$$

allows one to rewrite the principal remark above as

$$\frac{p_n}{q_n}+\frac{(-1)^n}{xq_n^2}=[c_0\,,\,\overrightarrow{w}\,,\,x-1\,,\,1\,,\,-1\,,\,0\,,\,\overleftarrow{w}\,].$$

This formulation seems convenient in numerical examples and is the one employed in the survey [3]. Whilst such reformulations may momentarily seem mysterious, the present one is no more than the pair of remarks

Proposition 3.

$$[\ldots\,,\,a\,,\,0\,,\,b\,,\,\ldots]=[\ldots\,,\,a+b\,,\,\ldots]$$

and

$$-[a\,,\,b\,,\,c\,,\,\ldots]=[0\,,\,-1\,,\,1\,,\,-1\,,\,0\,,\,a\,,\,b\,,\,c\,,\,\ldots]=[0\,,\,-1\,,\,1\,,\,a-1\,,\,b\,,\,c\,,\,\ldots].$$

Proof. The first remark is just

$$\begin{pmatrix} a & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} b & 1\\ 1 & 0\end{pmatrix}=\begin{pmatrix} a+b & 1\\ 1 & 0\end{pmatrix}.$$

The second is most easily seen by noting that

$$-y = 0 + \cfrac{1}{-1 + \cfrac{1}{1 + \cfrac{1}{y-1}}} .$$

This result is rather better known in the form

$$-[a\,,b\,,c\,,\ldots] = [-a\,,0\,,-1\,,1\,,-1\,,0\,,b\,,c\,,\ldots] = [-a-1\,,1\,,b-1\,,c\,,\ldots]\,;$$

for example

$$-\pi = -[3\,,7\,,15\,,1\,,292\,,1\,,\ldots] = [-4\,,1\,,6\,,15\,,1\,,292\,,1\,,\ldots]\,. \qquad \blacksquare$$

Our preceding remarks are formal. Turning to analysis, we note that taking determinants in the fundamental correspondence yields

$$p_nq_{n-1} - p_{n-1}q_n = (-1)^{n+1}\,,$$

which we have already used above.

It follows that

$$\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = (-1)^{n-1}\frac{1}{q_{n-1}q_n}\,,$$

whence, by induction on n,

$$\frac{p_n}{q_n} = c_0 + \frac{1}{q_0q_1} - \frac{1}{q_1q_2} + \cdots + (-1)^{n-1}\frac{1}{q_{n-1}q_n}\,.$$

Hence, if, in some appropriate metric, the sequence (q_h) is strictly increasing, the sequence of convergents (p_h/q_h) converges to an element of the relevant completion.

We will be interested in two cases: The first is the familiar one of regular continued fractions of real numbers in which the partial quotients c_h are *admissible* if they are positive integers for $h \ge 1$ (c_0 may be any element of $\mathbb{Z}$). The second case, that of continued fractions of formal Laurent series in X^{-1}, requires the partial quotients to be polynomials in X of degree at least 1 (except c_0, which may be constant). In this case the sequence of convergents converges to a Laurent series

$$\sum_{h=d}^{\infty} a_hX^{-h} \qquad\qquad d \in \mathbb{Z}$$

in the variable X^{-1}. Incidentally, this latter case can be made to include continued fractions converging to p-adic rationals just by writing $X^{-1} = p$.

3. Paperfolding

A sheet of paper can be folded in half lengthways in two ways: right half over left and right half under left. We refer to the first as a *positive* fold and to the second as a *negative* fold. After having been folded a number of times, say n times, a sheet of paper may be unfolded to display, reading from left to right, a sequence of $2^n - 1$ creases. It will be convenient to denote valleys $\vee$ by 1 and ridges $\wedge$ by -1.

Accordingly, a word $i_1 i_2 \ldots i_n$ with $i_h \in \{-1, 1\}$ may be deemed to be instructions to fold so that the h th fold is positive or negative corresponding to the sign of i_h, and these folding instructions then induce a word $f_1 f_2 \ldots f_{2^n - 1}$, again with the $f_h \in \{-1, 1\}$, denoting the sequence of creases in the sheet of paper. It is in the nature of paper, because the f_{2^h} are the original creases, that for $h = 0,\ 1,\ \ldots,\ n-1$ one has $f_{2^h} = i_{n-h}$. It is only slightly less obvious that, for k in the range so that the cited crease actually occurs,

$$f_{2^h + k2^{h+1}} = (-1)^k i_{n-h}\,.$$

This gives the paperfolding word explicitly in terms of its folding instructions.

We shall make use of an alternative description of the sequence of creases: Namely, if the sheet is folded n times according to the instructions $i_1 i_2 \ldots i_n$, then the left half of the sheet of paper is folded $n-1$ times according to the instructions $i_2 i_3 \ldots i_n$, and this induces creases on this half-sheet given by the paperfolding word $\overrightarrow{w}$, say. Then comes the central fold i_1, and now the right half of the sheet, which lies over or under the left half (according to the sign of i_1) necessarily displays creases given by the paperfolding word $-\overleftarrow{w}$ induced by the instructions $-i_2, i_3 \ldots i_n$.

Given this description of the sequence of creases, let $\mathcal{F}_i$ to be the folding map

$$\mathcal{F}_i : \overrightarrow{w} \mapsto \overrightarrow{w}\, i\, -\overleftarrow{w}\,.$$

Then the folding word $f_1 f_2 \ldots f_{2^n - 1}$ is induced by the folding instructions $i_1 i_2 \ldots i_n$ if and only if

$$\mathcal{F}_{i_1} \cdots \mathcal{F}_{i_{n-1}} \mathcal{F}_{i_n}(\quad) = f_1 f_2 \ldots f_{2^n - 1}\,.$$

As is apparent, $(\quad)$ denotes the empty word; or, if one prefers, a *tabula rasa* — a fresh sheet of paper. These and the matters immediately below are detailed in the survey 'Folds!' [3].

If we now place the usual topology on the set of paperfolding words — whereby two sequences are 'close together' if they commence with the same word — then the cluster points, the *paperfolding sequences*

$$f_1 f_2 f_3 \ldots\ldots\,,$$

are all sequences on symbols from $\{-1, 1\}$ so that $f_2 f_4 f_6 \ldots\ldots$ is again a paperfolding sequence and for $h \geq 0$

$$f_{2h+1} = (-1)^h f_1\,.$$

So paperfolding sequences are determined by their sequence of *unfolding instructions* $f_1 f_2 f_4 \ldots f_{2^h} \ldots\ldots = j_0 j_1 j_2 \ldots j_h \ldots\ldots$, say. Thus, by setting $(-1)^{a_h} = f_{2^h} = j_h$, different binary decimals $0.a_1 a_2 a_3 \ldots$ may be interpreted as distinct sequences of unfolding instructions and they induce distinct paperfolding sequences.

Whilst paperfolding sequences are not very complicated they are never trivial; no paperfolding sequence is periodic.

4. A folded continued fraction

Theorem 1. *Let a be a binary decimal $a = 0.a_1a_2a_3\ldots$, set $a_0 = 0$ and denote by F_a the formal series*

$$F_a(X) = X\sum_{h=0}^{\infty}(-1)^{a_h}X^{-2^h}.$$

Given a word $\overrightarrow{p}$, let $\mathcal{F}_{\overrightarrow{p}}$ be the folding map *which acts on words $\overrightarrow{w}$ of partial quotients*

$$\mathcal{F}_{\overrightarrow{p}} : \overrightarrow{w} \mapsto \overrightarrow{w}, \overrightarrow{p}, -\overleftarrow{w}$$

producing the word $\overrightarrow{w}, \overrightarrow{p}, -\overleftarrow{w}$ of partial quotients. Then, with j_h denoting $(-1)^{a_h}$, the continued fraction expansion of F_a is given by

$$F_a(X) = [1\,,\ \prod_{h=2}^{\infty}\mathcal{F}_{-X(-1)^{a_h}}((-1)^{a_1}X)] = [1\,,\ \cdots\mathcal{F}_{-j_4X}\mathcal{F}_{-j_3X}\mathcal{F}_{-j_2X}(j_1X)].$$

Proof. For tidiness we detail the case with all the $a_h = 0$ since the general case raises no new principle. Evidently

$$1 + X^{-1} = [1\,,\ X] = \frac{p_1}{q_1},$$

with $q_1 = X$. If

$$1 + X^{-1} + X^{-3} + \cdots + X^{-2^h+1} = [1\,,\ \overrightarrow{w}] = [1\,,\ \mathcal{F}_{-X}^{h-1}(X)] = \frac{p_n}{q_n}$$

we have $|w| = n$ odd, $q_n = X^{2^h-1}$ and, applying Proposition 2,

$$\begin{aligned}1 + X^{-1} + X^{-3} + \cdots + X^{-2^h+1} + X^{-2^{h+1}+1} &= \frac{p_n}{q_n} + \frac{(-1)^n}{-Xq_n^2}\\ &= [1\,,\ \overrightarrow{w}\,,\ -X\,,\ -\overleftarrow{w}] = [1\,,\ \mathcal{F}_{-X}^{h}(X)].\end{aligned}$$

It follows by induction on h that

$$F_0 = [1\,,\ \mathcal{F}_{-X}^{\infty}(X)].$$

The general case follows by the same argument. ∎

5. Main result

The formal convergence that makes sense of the foregoing only presumes an absolute value $|\ \ |$ with $|X| > 1$. Hence, we should be able to set $X = 2$ and obtain a meaningful numerical result. With j_h written for $(-1)^{a_h}$, we get

$$F_a(2) = 2\sum_{h=0}^{\infty}(-1)^{a_h}2^{-2^h} = [1\,,\ \cdots\mathcal{F}_{-2j_4}\mathcal{F}_{-2j_3}\mathcal{F}_{-2j_2}(2j_1)].$$

The series $F_a(2)$ converges but the partial quotients appearing in the continued fraction expansion include -2 and are not all admissible. Nevertheless, one may readily transform the present remark to obtain:

Theorem 2. *Let a be a binary decimal $a = 0.a_1a_2a_3\ldots$, set $a_0 = 0$ and denote by $F_a(2)$ the number*

$$F_a(2) = 2\sum_{h=0}^{\infty}(-1)^{a_h}2^{-2^h}\,.$$

Then the continued fraction expansion of $F_a(2)$ consists of just the partial quotients 1 and 2.

Proof. Given Theorem 1, we have to show that a continued fraction

$$[1\,,\,2f_1\,,\,2f_2\,,\,2f_3\,,\,\ldots\ldots]\,,$$

with (f_h) a folded sequence, simplifies to a continued fraction with admissible partial quotients 1 or 2 only. We shall show more, using only the property that the sequence (f_h) can be unfolded at least once: that is, that the sequence (f_{2h+1}) is alternating in sign.

We need the auxiliary result:

$$[a\,,\,-b\,,\,c] = [a-1\,,\,1\,,\,b-2\,,\,1\,,\,c-1]\,.$$

This is readily checked by the fundamental correspondence or by invoking Proposition 3 whereby

$$\begin{aligned}&[a\,,\,-b\,,\,c\,,\,d\,,\,\ldots] = [a\,,\,0\,,\,-1\,,\,1\,,\,-1\,,\,0\,,\,b\,,\,-c\,,\,-d\,,\,\ldots]\\ &= [a-1\,,\,1\,,\,b-1\,,\,0\,,\,-1\,,\,1\,,\,-1\,,\,0\,,\,c\,,\,d\,,\,\ldots] = [a-1\,,\,1\,,\,b-2\,,\,1\,,\,c-1\,,\,d\,,\,\ldots]\,.\end{aligned}$$

Applying this result to remove the inadmissible -2 s and again recalling that

$$[\ldots\,,\,a\,,\,0\,,\,b\,,\,\ldots] = [\ldots\,,\,a+b\,,\,\ldots]\,,$$

we obtain

$$\begin{aligned}&[1\,,\,2\,,\,a\,,\,-2\,,\,b\,,\,2\,,\,c\,,\,-2\,,\,d\,,\,2\,,\,e\,,\,-2\,,\,f\,,\,\ldots]\\ &\qquad = [1\,,\,2\,,\,a-1\,,\,2\,,\,b-1\,,\,2\,,\,c-1\,,\,2\,,\,d-1\,,\,2\,,\,e-1\,,\,2\,,\,f-1\,,\,\ldots]\,,\end{aligned}$$

whilst

$$[1\,,\,-2\,,\,a\,,\,2\,,\,b\,,\,-2\,,\,\ldots] = [0\,,\,2\,,\,a-1\,,\,2\,,\,b-1\,,\,2\,,\,\ldots]\,.$$

It remains only to remove those partial quotients $a-1$, $b-1$, $\ldots$ that are inadmissible. Without losing generality we may suppose that $a-1$ is inadmissible, in which case we have

$$[2\,,\,a-1\,,\,2] = [1\,,\,1\,,\,-a-1\,,\,1\,,\,1] = [1\,,\,1\,,\,1\,,\,1\,,\,1]\,,$$

because $a-1=-3$. If also $b-1$ is inadmissible, then

$$[1\,,\,1\,,\,b-1\,,\,2] = [1\,,\,0\,,\,1\,,\,-b-1\,,\,1\,,\,1] = [2\,,\,1\,,\,1\,,\,1]\,.$$

Evidently, and surprisingly, this covers all cases required to demonstrate that if a sequence of ± 2 s can be unfolded at least once then, considered as a sequence of partial quotients, it yields a sequence of partial quotients consisting just of 1 s or 2 s. Indeed, we could have started with yet more general sequences of partial quotients, namely those consisting of ± 2 s *or* ± 3 s folded once by ± 2 s: in the sense that either $(f_{2h+1}) = (-2)^h$ for all $h > 0$ or $(f_{2h+1}) = -(-2)^h$ for all $h > 0$. ∎

In our introduction we promised *explicit* continued fraction expansions for explicit binary decimals. Our numbers

$$2\sum_{h=0}^{\infty} \pm 2^{-2^h}$$

become explicit binary decimals on repeated use of the remark

$$1 - 2^{-m} = 0.\underbrace{11\ldots111}_{m \text{ digits}}$$

and Theorem 2 yields a sufficiently simple algorithm to satisfy the remainder of our undertaking. But, with only a little work, we can be less implicitly explicit. As a preliminary, we mention that maps $\mathcal{S}_{\overrightarrow{p}} : \overrightarrow{w} \mapsto \overrightarrow{w}\,\overrightarrow{p}\,\overleftarrow{w}$ *perturb symmetry* in the sense that a word

$$\overrightarrow{w}\,\overrightarrow{p_1}\,\overleftarrow{w}\,\overrightarrow{p_2}\,\overrightarrow{w}\,\overleftarrow{p_1}\,\overleftarrow{w}\,\overrightarrow{p_3}\,\overrightarrow{w}\,\overrightarrow{p_1}\,\overleftarrow{w}\,\overleftarrow{p_2}\,\overrightarrow{w}\,\overleftarrow{p_1}\,\overleftarrow{w}\,\overrightarrow{p_4}\,\overrightarrow{w}\,\overrightarrow{p_1}\ldots$$

resulting from repeated application of such maps is symmetric if and only if the inserted perturbations $\overrightarrow{p_h}$ each are symmetric. Notice that these sequences have an arrow sequence that is folded. In fact it is just these sequences that are described as 'folded sequences' in [3]; see also [9].

Theorem 3. *Let a be a binary decimal $a = 0.a_1a_2a_3\ldots$, set $a_0 = 0$ and denote by $F_a(x)$ the number*

$$F_a(x) = x\sum_{h=0}^{\infty}(-1)^{a_h}x^{-2^h},$$

where $x \geq 2$ is a rational integer.

Given a word $\overrightarrow{p}$, let $\mathcal{S}_{\overrightarrow{p}}$ be the perturbed symmetry map *which acts on words $\overrightarrow{w}$ of partial quotients*

$$\mathcal{S}_{\overrightarrow{p}} : \overrightarrow{w} \mapsto \overrightarrow{w}\,,\overrightarrow{p}\,,\overleftarrow{w}$$

producing the word $\overrightarrow{w}\,,\overrightarrow{p}\,,\overleftarrow{w}$ of partial quotients. Then the continued fraction expansion of $F_a(x)$ is given by:

$$\begin{aligned} F_{0.a_1a_2a_3a_4\ldots}(x) &= [b_{a_1a_2}\,,\ \prod_{h=3}^{\infty}\mathcal{S}_{\overrightarrow{p_{a_1a_2a_h}}}(\overrightarrow{w_{a_1a_2a_3}})\,] \\ &= [1\,,\ X-1\,,\ 1\,,\ \cdots\mathcal{S}_{\overrightarrow{p_{a_1a_2a_5}}}\mathcal{S}_{\overrightarrow{p_{a_1a_2a_4}}}\mathcal{S}_{\overrightarrow{p_{a_1a_2a_3}}}(\overrightarrow{w_{a_1a_2a_3}})\,]\,, \end{aligned}$$

where

$$\begin{aligned}
b_{00} &= [1\,,\,x-1\,,\,1], & w_{000} &= [x-1\,,\,x\,,\,x-1\,,\,1\,,\,x-1\,,\,x-1]\\
&& \textit{and}\ \ p_{000} &= [1\,,\,x-1\,,\,x-1\,,\,1\,,\,x-2\,,\,1]\,;\\
b_{01} &= [1\,,\,x], & w_{010} &= [x-1\,,\,1\,,\,x-1\,,\,x-1\,,\,1\,,\,x-2\,,\,1\,,\,x-1]\\
&& \textit{and}\ \ p_{010} &= [x\,,\,x-1\,,\,1\,,\,x-1]\,;\\
b_{10} &= [0\,,\,1\,,\,x-1], & w_{010} &= [x-1\,,\,1\,,\,x-2\,,\,1\,,\,x-1\,,\,x-1\,,\,1\,,\,x-1]\\
&& \textit{and}\ \ p_{100} &= [x-1\,,\,1\,,\,x-1\,,\,x-1]\,;\\
b_{11} &= [0\,,\,1\,,\,x-2\,,\,1], & w_{110} &= [x-1\,,\,x-1\,,\,1\,,\,x-1\,,\,x\,,\,x-1]\\
&& \textit{and}\ \ p_{110} &= [1\,,\,x-2\,,\,1\,,\,x-1\,,\,x-1\,,\,1]\,,
\end{aligned}$$

while

$$\overrightarrow{w_{a_1a_2 1}} = \overleftarrow{w_{a_1a_2 0}} \text{ and } \overrightarrow{p_{a_1a_2 1}} = \overleftarrow{p_{a_1a_2 0}}\,.$$

If $x=2$, the simplification that ensues from removing the inadmissible 0 partial quotients is plain.

Proof. Folding yields perturbed symmetry because

$$\begin{aligned}
[b\,,\,\mathcal{F}_x(c,d,\overrightarrow{w},d,c)] &= [b\,,\,c\,,\,d\,,\,\overrightarrow{w}\,,\,d\,,\,c\,,\,x\,,\,-c\,,\,-d\,,\,-\overleftarrow{w}\,,\,-d\,,\,-c]\\
&= [b\,,\,c\,,\,d\,,\,\overrightarrow{w}\,,\,d\,,\,c\,,\,x-1\,,\,1\,,\,c-1\,,\,d\,,\,\overleftarrow{w}\,,\,d\,,\,c]\\
&= [b\,,\,c\,,\,d\,,\,\mathcal{S}_{\overrightarrow{p}}(\overrightarrow{w})\,,\,d\,,\,c]\,,
\end{aligned}$$

with $\overrightarrow{p} = d, c, x-1, 1, c-1, d$. Moreover, folding once with $-x$ yields the same result other than that the perturbation must be replaced by $\overleftarrow{p}$.

There are four possible beginnings for the continued fractions of $F_{0.a_1a_2\ldots}(x)$, namely

$$\begin{aligned}
f_{0.00}(x) &= [1\,,\,x\,,\,-x\,,\,-x] = [1\,,\,x-1\,,\,1\,,\,x-1\,,\,x]\\
f_{0.01}(x) &= [1\,,\,x\,,\,x\,,\,-x] = [1\,,\,x\,,\,x-1\,,\,1\,,\,x-1]\\
f_{0.10}(x) &= [1\,,\,-x\,,\,-x\,,\,x] = [0\,,\,1\,,\,x-1\,,\,x-1\,,\,1\,,\,x-2\,,\,1]\\
f_{0.11}(x) &= [1\,,\,-x\,,\,x\,,\,x] = [0\,,\,1\,,\,x-2\,,\,1\,,\,x-1\,,\,x-1\,,\,1]
\end{aligned}$$

where foresight brought by hindsight suggests writing the latter two continued fractions so as to produce an even number of partial quotients in each case. Each yields two beginnings

$$\begin{aligned}
f_{0.000}(x) &= [1\,,\,x-1\,,\,1\,,\,x-1\,,\,x\,,\,x\,,\,-x\,,\,-x+1\,,\,-1\,,\,-x+1]\\
&= [1\,,\,x-1\,,\,1\,,\,x-1\,,\,x\,,\,x-1\,,\,1\,,\,x-1\,,\,x-1\,,\,1\,,\,x-1]\\
&= [1\,,\,x-1\,,\,1\,,\,\overrightarrow{w_{000}}\,,\,1\,,\,x-1]
\end{aligned}$$

and

$$f_{0.001}(x) = [1\,,\,x-1\,,\,1\,,\,\overleftarrow{w_{000}}\,,\,1\,,\,x-1]\,;$$

$$f_{0.010}(x) = [1\,,\, x\,,\, x-1\,,\, 1\,,\, x-1\,,\, x\,,\, -x+1\,,\, -1\,,\, -x+1\,,\, -x]$$
$$= [1\,,\, x\,,\, x-1\,,\, 1\,,\, x-1\,,\, x-1\,,\, 1\,,\, x-2\,,\, 1\,,\, x-1\,,\, x]$$
$$= [1\,,\, x\,,\, \overrightarrow{w_{010}}\,,\, x]$$

and

$$f_{0.011}(x) = [1\,,\, x\,,\, \overleftarrow{w_{010}}\,,\, x]\,;$$

$$f_{0.100}(x) = [0\,,\, 1\,,\, x-1\,,\, x-1\,,\, 1\,,\, x-2\,,\, 1\,,\, x\,,\, -1\,,\, -x+2\,,\, -1\,,\, -x+1\,,\, -x+1\,,\, -1]$$
$$= [0\,,\, 1\,,\, x-1\,,\, x-1\,,\, 1\,,\, x-2\,,\, 1\,,\, x-1\,,\, x-1\,,\, 1\,,\, x-1\,,\, x-1\,,\, 1]$$
$$= [0\,,\, 1\,,\, x-1\,,\, \overrightarrow{w_{100}}\,,\, x-1\,,\, 1]$$

and

$$f_{0.101}(x) = [0\,,\, 1\,,\, x-1\,,\, \overleftarrow{w_{100}}\,,\, x-1\,,\, 1]\,;$$

$$f_{0.110}(x) = [0\,,\, 1\,,\, x-2\,,\, 1\,,\, x-1\,,\, x-1\,,\, 1\,,\, x\,,\, -1\,,\, -x+1\,,\, -x+1\,,\, -1\,,\, -x+2\,,\, -1]$$
$$= [0\,,\, 1\,,\, x-2\,,\, 1\,,\, x-1\,,\, x-1\,,\, 1\,,\, x-1\,,\, x\,,\, x-1\,,\, 1\,,\, x-2\,,\, 1]$$
$$= [0\,,\, 1\,,\, x-2\,,\, 1\,,\, \overrightarrow{w_{110}}\,,\, 1\,,\, x-2\,,\, 1]$$

and

$$f_{0.111}(x) = [0\,,\, 1\,,\, x-2\,,\, 1\,,\, \overleftarrow{w_{110}}\,,\, 1\,,\, x-2\,,\, 1]\,.$$

In each case $w_{a_1a_20}$ has been chosen to ensure a ready interpretation of the final result in the special case $x = 2$.

The opening remark of the proof now yields the theorem. ∎

Note. Special cases of this detailed result are first remarked on in [8]. Those observations are alluded to in [3], where a blunder, see [1], hints at the present general result. There are different ways for a sequence to display its perturbed symmetry. Thus, indeed,

$$F_0(x) = [1\,,\, x-1\,,\, 1\,,\, x-1\,,\, \prod_{h=0}^{\infty} \mathcal{S}_{1\,,\, x-1\,,\, x-1\,,\, 1\,,\, x-2\,,\, 1}(x-1,\, x,\, x-1,\, 1,\, x-1,\, x-1)\,]$$
$$= [1\,,\, x-1\,,\, 1\,,\, \prod_{h=0}^{\infty} \mathcal{S}_{x-1\,,\, 1\,,\, x-1\,,\, x-1\,,\, 1\,,\, x-2\,,\, 1\,,\, x-1}(x,\, x-1,\, 1,\, x-1)\,]$$

confirming the correction suggested in [1].

6. Divide and conquer

We shall briefly sketch a technique for obtaining, from our results, explicit continued fraction expansions of the numbers

$$\sum_{h=0}^{\infty} \pm x^{-2^h}\,.$$

Indeed, we shall divide our results above by x. To that end we recall the fundamental correspondence whereby one defines the convergents by matrix products. For our present purpose it will be convenient to view those products a little differently. Accordingly, set

$$J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

noticing that $JR = LJ$, $JL = RJ$ and $J^2 = I$. Then, on observing that for $d \in \mathbb{Z}$,

$$R^d = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \qquad L^d = \begin{pmatrix} 1 & 0 \\ d & 1 \end{pmatrix},$$

we may write, with a formal interpretation intended,

$$\begin{pmatrix} c_0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c_1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c_2 & 1 \\ 1 & 0 \end{pmatrix}\cdots\cdots = R^{c_0}L^{c_1}R^{c_2}\cdots\cdots,$$

that is, with $\longleftrightarrow$ denoting the fundamental correspondence between matrix products and continued fractions,

$$[c_0\,,\, c_1\,,\, c_2\,,\, \ldots] \longleftrightarrow R^{c_0}L^{c_1}R^{c_2}\cdots.$$

Then

$$x^{-1}[c_0\,,\, c_1\,,\, c_2\,,\, \ldots] \longleftrightarrow A'R^{c_0}L^{c_1}R^{c_2}\cdots,$$

where we set

$$A = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \qquad A' = \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}.$$

Of course the matrix product on the right in the correspondence just given does not correspond to a continued fraction, but using the *transition formulae*

$$\begin{aligned} ALR^{x-1} &= R^{x-1}LA' & A'RL^{x-1} &= L^{x-1}RA \\ ALR^{-x-1} &= -IR^{x+1}L^{-1}A' & A'RL^{-x-1} &= -IL^{x+1}R^{-1}A \end{aligned}$$

together with the remark that multiplication by a matrix dI does not change the corresponding continued fraction, we can transit the multiplier A' through the R–L sequence until it disappears into the $\cdots\cdots$ on the right.

For example, recalling that

$$F_0(x) = [1\,,\, x\,,\, -x\,,\, -x\,,\, -x\,,\, x\,,\, x\,,\, -x\,,\, -x\,,\, \ldots],$$

we can compactly show the *transduction* as

$$\begin{array}{lllllllll} F_0(x) & RL^{x-1} & LR^{-x-1} & RL^{-x-1} & LR^{-x-1} & RL^{x-1} & LR^{x-1} & RL^{-x-1} & LR^{-x-1} \\ & A' & A & A' & A & A' & A & A' & A \quad \cdots \\ G_0(x) & L^{x-1}R & R^{x+1}L^{-1} & L^{x+1}R^{-1} & R^{x+1}L^{-1} & L^{x-1}R & R^{x-1}L & L^{x+1}R^{-1} & R^{x+1}L^{-1} \end{array}$$

and see that

$$G_0(x) = \sum_{h=0}^{\infty} x^{-2^h} = [x-1\,,\, \overrightarrow{x+2\,,\, x}\,,\, \overrightarrow{x\,,\, x-2}\,,\, \overleftarrow{x\,,\, x+2}\,,\, x\,,\, \ldots].$$

In this manner the energetic reader can readily rediscover the results of [8], [9] and more — see [6]. The technique used above is suggested by work of Raney [7]. A surprising application can be found in [2].

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