FOLDED CONTINUED FRACTIONS

A. J. van der Poorten\textsuperscript{1} \text{ AND } J. Shallit\textsuperscript{2}

Macquarie University and Dartmouth College

We exhibit uncountably many binary decimals together with their explicit continued fraction expansions. These expansions require only the partial quotients 1 or 2. The pattern of valleys and ridges in a sheet of paper repeatedly folded in half plays a critical rôle in our construction.

1. Introduction

It is notorious that it is generally damnably difficult to explicitly compute the continued fraction of a quantity presented in some other form. However, we shall exhibit a class of series, and thence of numbers presented in effect as binary decimals for which we can display the continued fraction expansions. In particular, the uncountably many numbers

\[ 2 \sum_{h=0}^{\infty} \pm 2^{-2^h} \]

all have continued fraction expansions with partial quotients 1 or 2 only.

Our result that the series

\[ X \sum_{h=0}^{\infty} \pm X^{-2^h} \]

all have folded continued fractions is new, whilst our specialisations generalise and complete remarks on special cases in [3] and [8].

By a result of Loxton and van der Poorten [5], generalising \textit{inter alia} a result of Kempner [4], these numbers all are transcendental. Thus our result gives no information on the conjecture that all algebraic numbers of degree at least 3 have unbounded partial quotients.

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2. CONTINUED FRACTIONS

A continued fraction is an object of the form
\[ c_0 + \frac{1}{c_1 + \frac{1}{c_2 + \frac{1}{c_3 + \cdots}}}. \]

denoted, to save vertical space, in the flat notation \([c_0, c_1, c_2, c_3, \ldots]\). Essentially all one needs to understand the behaviour of these objects is contained in the fundamental correspondence whereby:

**Proposition 1.** For \( h = 0, 1, 2, \ldots \)
\[ \frac{p_h}{q_h} = [c_0, c_1, \ldots, c_h] \]

if and only if
\[ \begin{pmatrix} c_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} c_h & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_h & p_{h-1} \\ q_h & q_{h-1} \end{pmatrix}. \]

**Proof.** The correspondence defines the convergents \( p_h/q_h \) by matrix products and is readily verified by induction on the number \( h + 1 \) of partial quotients \( c_n \), respectively the number of matrices.

The fundamental correspondence is just the observation that the well known recursion formulae
\[ p_{n+1} = a_{n+1}p_n + p_{n-1} \]
\[ q_{n+1} = a_{n+1}q_n + q_{n-1} \]

together with
\[ \begin{pmatrix} p_{-1} & p_{-2} \\ q_{-1} & q_{-2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \]

define the sequence of convergents. \( \blacksquare \)

By taking the transpose of the matrix product we have, for example,
\[ \begin{pmatrix} c_n & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_{n-1} & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} c_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_n & q_n \\ p_{n-1} & q_{n-1} \end{pmatrix} \]
\[ \longleftrightarrow [c_n, c_{n-1}, \ldots, c_1, c_0], \]

where \( \longleftrightarrow \) denotes the correspondence between matrix products and continued fractions, so
\[ [c_n, c_{n-1}, \ldots, c_1] = \frac{q_n}{q_{n-1}}. \]

In this spirit, we also recall that:
Proposition 2.
\[
\frac{p_n}{q_n} + \frac{(-1)^n}{xq_n^2} = [c_0, \overrightarrow{w}, x - \frac{q_{n-1}}{q_n}] = [c_0, \overrightarrow{w}, x, -\overrightarrow{w}].
\]

Here \(\overrightarrow{w}\) is a convenient abbreviation for the word \(c_1, c_2, \ldots, c_n\) and, accordingly, \(-\overrightarrow{w}\) denotes the word \(-c_n, -c_{n-1}, \ldots, -c_1\).

Proof. Indeed,
\[
[c_0, \overrightarrow{w}, x - \frac{q_{n-1}}{q_n}] \longleftrightarrow \left( \begin{array}{cc} p_n & p_{n-1} \\ q_n & q_{n-1} \end{array} \right) \left( \begin{array}{cc} x - \frac{q_{n-1}}{q_n} & 1 \\ 1 & 0 \end{array} \right) = \left( x p_n - \frac{p_nq_{n-1} - p_{n-1}q_n}{q_n} \right) \frac{p_n}{q_n} \longleftrightarrow \frac{p_n}{q_n} + \frac{(-1)^n}{xq_n^2}
\]
since \((p_nq_{n-1} - p_{n-1}q_n) = (-1)^{n-1}\); and, of course, \(x - \frac{q_{n-1}}{q_n} = [x, -\overrightarrow{w}]\).

Unless one adopts conventions restricting partial quotients in some appropriate manner, a continued fraction expansion is not unique. For example, the computation
\[
x - \frac{q_{n-1}}{q_n} = x - 1 + \frac{q_n - q_{n-1}}{q_n} \\
q_n/(q_n - q_{n-1}) = 1 + q_{n-1}/(q_n - q_{n-1}) \\
(q_n - q_{n-1})/q_n = -1 + q_n/q_{n-1} \\
q_{n-1}/q_n = 0 + q_n/q_{n-1} \\
q_n/q_{n-1} = \overrightarrow{w}
\]
allows one to rewrite the principal remark above as
\[
\frac{p_n}{q_n} + \frac{(-1)^n}{xq_n^2} = [c_0, \overrightarrow{w}, x - 1, 1, -1, 0, \overrightarrow{w}].
\]
This formulation seems convenient in numerical examples and is the one employed in the survey [3]. Whilst such reformulations may momentarily seem mysterious, the present one is no more than the pair of remarks

Proposition 3.
\[
[\ldots, a, 0, b, \ldots] = [\ldots, a + b, \ldots]
\]
and
\[
-[a, b, c, \ldots] = [0, -1, 1, -1, 0, a, b, c, \ldots] = [0, -1, 1, a - 1, b, c, \ldots].
\]

Proof. The first remark is just
\[
\begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a + b & 1 \\ 1 & 0 \end{pmatrix}.
\]
The second is most easily seen by noting that

\[-y = 0 + \frac{1}{-1} + \frac{1}{1} + \frac{1}{y-1}.
\]

This result is rather better known in the form

\[-[a, b, c, \ldots] = [-a, 0, -1, 1, -1, 0, b, c, \ldots] = [-a - 1, 1, b - 1, c, \ldots];
\]

for example

\[-\pi = -[3, 7, 15, 1, 292, 1, \ldots] = [-4, 1, 6, 15, 1, 292, 1, \ldots].
\]

Our preceding remarks are formal. Turning to analysis, we note that taking determinants in the fundamental correspondence yields

\[p_n q_{n-1} - p_{n-1} q_n = (-1)^{n+1},
\]

which we have already used above.

It follows that

\[\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = (-1)^{n-1} \frac{1}{q_{n-1} q_n},
\]

whence, by induction on \(n\),

\[\frac{p_n}{q_n} = c_0 + \frac{1}{q_0 q_1} - \frac{1}{q_1 q_2} + \cdots + (-1)^{n-1} \frac{1}{q_{n-1} q_n}.
\]

Hence, if, in some appropriate metric, the sequence \((q_h)\) is strictly increasing, the sequence of convergents \((p_h/q_h)\) converges to an element of the relevant completion.

We will be interested in two cases: The first is the familiar one of regular continued fractions of real numbers in which the partial quotients \(c_h\) are \textit{admissible} if they are positive integers for \(h \geq 1\) (\(c_0\) may be any element of \(\mathbb{Z}\)). The second case, that of continued fractions of formal Laurent series in \(X^{-1}\), requires the partial quotients to be polynomials in \(X\) of degree at least 1 (except \(c_0\), which may be constant). In this case the sequence of convergents converges to a Laurent series

\[\sum_{h=d}^{\infty} a_h X^{-h} \quad d \in \mathbb{Z}
\]

in the variable \(X^{-1}\). Incidentally, this latter case can be made to include continued fractions converging to \(p\)-adic rationals just by writing \(X^{-1} = p\).
3. Paperfolding

A sheet of paper can be folded in half lengthways in two ways: right half over left and right half under left. We refer to the first as a positive fold and to the second as a negative fold. After having been folded a number of times, say \( n \) times, a sheet of paper may be unfolded to display, reading from left to right, a sequence of \( 2^n - 1 \) creases. It will be convenient to denote valleys ∨ by 1 and ridges ∧ by \(-1\).

Accordingly, a word \( i_1i_2\ldots i_n \) with \( i_h \in \{-1,1\} \) may be deemed to be instructions to fold so that the \( h \)th fold is positive or negative corresponding to the sign of \( i_h \), and these folding instructions then induce a word \( f_1f_2\ldots f_{2^n-1} \), again with the \( f_h \in \{-1,1\} \), denoting the sequence of creases in the sheet of paper. It is in the nature of paper, because the \( f_{2^h} \) are the original creases, that for \( h = 0, 1, \ldots, n - 1 \) one has \( f_{2^h} = i_{n-h} \). It is only slightly less obvious that, for \( k \) in the range so that the cited crease actually occurs,

\[
f_{2^h + 2^h + 1} = (-1)^{k}i_{n-h}.
\]

This gives the paperfolding word explicitly in terms of its folding instructions.

We shall make use of an alternative description of the sequence of creases: Namely, if the sheet is folded \( n \) times according to the instructions \( i_1i_2\ldots i_n \), then the left half of the sheet of paper is folded \( n - 1 \) times according to the instructions \( i_2i_3\ldots i_n \), and this induces creases on this half-sheet given by the paperfolding word \( \overrightarrow{w} \), say. Then comes the central fold \( i_1 \), and now the right half of the sheet, which lies over or under the left half (according to the sign of \( i_1 \)) necessarily displays creases given by the paperfolding word \( \overleftarrow{w} \) induced by the instructions \(-i_2, i_3\ldots i_n \).

Given this description of the sequence of creases, let \( F_i \) to be the folding map

\[
F_i : \overrightarrow{w} \mapsto \overrightarrow{w} \overleftarrow{i} - \overleftarrow{w}.
\]

Then the folding word \( f_1f_2\ldots f_{2^n-1} \) is induced by the folding instructions \( i_1i_2\ldots i_n \) if and only if

\[
F_{i_1}\ldots F_{i_n} = f_1f_2\ldots f_{2^n-1}.
\]

As is apparent, \( (\quad) \) denotes the empty word; or, if one prefers, a tabula rasa — a fresh sheet of paper. These and the matters immediately below are detailed in the survey ‘Folds!’ [3].

If we now place the usual topology on the set of paperfolding words — whereby two sequences are ‘close together’ if they commence with the same word — then the cluster points, the paperfolding sequences

\[
f_1f_2f_3\ldots,
\]

are all sequences on symbols from \( \{-1,1\} \) so that \( f_2f_4f_6\ldots \) is again a paperfolding sequence and for \( h \geq 0 \)

\[
f_{2^{h+1}} = (-1)^{h}f_1.
\]

So paperfolding sequences are determined by their sequence of unfolding instructions \( f_1f_2f_4\ldots f_{2^n}\ldots = j_0j_1j_2\ldots j_h\ldots \), say. Thus, by setting \( (-1)^{a_h} = f_{2^h} = j_h \), different binary decimals \( 0.a_1a_2a_3\ldots \) may be interpreted as distinct sequences of unfolding instructions and they induce distinct paperfolding sequences.

Whilst paperfolding sequences are not very complicated they are never trivial; no paperfolding sequence is periodic.
4. A folded continued fraction

**Theorem 1.** Let \( a \) be a binary decimal \( a = 0.a_1a_2a_3 \ldots \), set \( a_0 = 0 \) and denote by \( F_a \) the formal series

\[
F_a(X) = X \sum_{h=0}^{\infty} (-1)^{a_h} X^{-2^h}.
\]

Given a word \( \vec{p} \), let \( \mathcal{F}_{\vec{p}} \) be the folding map which acts on words \( \vec{w} \) of partial quotients

\[
\mathcal{F}_{\vec{p}} : \vec{w} \mapsto \vec{w}, \vec{p}, -\vec{w}
\]

producing the word \( \vec{w}, \vec{p}, -\vec{w} \) of partial quotients. Then, with \( j_h \) denoting \( (-1)^{a_h} \), the continued fraction expansion of \( F_a \) is given by

\[
F_a(X) = [1, \prod_{h=2}^{\infty} \mathcal{F}_{-X}((-1)^{a_h} ((-1)^{a_1} X))] = [1, \cdots \mathcal{F}_{-j_4} X \mathcal{F}_{-j_3} X \mathcal{F}_{-j_2} X (j_1 X)].
\]

**Proof.** For tidiness we detail the case with all the \( a_h = 0 \) since the general case raises no new principle. Evidently

\[
1 + X^{-1} = [1, X] = \frac{p_1}{q_1},
\]

with \( q_1 = X \). If

\[
1 + X^{-1} + X^{-3} + \cdots + X^{-2^h+1} = [1, \vec{w}] = [1, \mathcal{F}_{-X}^{h-1}(X)] = \frac{p_n}{q_n}
\]

we have \( |\vec{w}| = n \) odd, \( q_n = X^{2^h-1} \) and, applying Proposition 2,

\[
1 + X^{-1} + X^{-3} + \cdots + X^{-2^h+1} + X^{-2^h+2} + X^{-2^h+3} + \cdots + X^{-2^{h+1}} = \frac{p_n}{q_n} + \frac{(-1)^n}{X q_n^2} = [1, \vec{w}, -X, -\vec{w}] = [1, \mathcal{F}_{-X}^{h}(X)].
\]

It follows by induction on \( h \) that

\[
F_0 = [1, \mathcal{F}_{-X}^{\infty}(X)].
\]

The general case follows by the same argument. \( \square \)

5. Main result

The formal convergence that makes sense of the foregoing only presumes an absolute value \(| \quad | \) with \( |X| > 1 \). Hence, we should be able to set \( X = 2 \) and obtain a meaningful numerical result. With \( j_h \) written for \( (-1)^{a_h} \), we get

\[
F_a(2) = 2 \sum_{h=0}^{\infty} (-1)^{a_h} 2^{-2^h} = [1, \cdots \mathcal{F}_{-2j_4} \mathcal{F}_{-2j_3} \mathcal{F}_{-2j_2} (2j_1)].
\]

The series \( F_a(2) \) converges but the partial quotients appearing in the continued fraction expansion include \(-2\) and are not all admissible. Nevertheless, one may readily transform the present remark to obtain:
Theorem 2. Let \( a \) be a binary decimal \( a = 0.a_1a_2a_3 \ldots \), set \( a_0 = 0 \) and denote by \( F_a(2) \) the number

\[
F_a(2) = 2 \sum_{h=0}^{\infty} (-1)^{a_h} 2^{-2h}.
\]

Then the continued fraction expansion of \( F_a(2) \) consists of just the partial quotients 1 and 2.

Proof. Given Theorem 1, we have to show that a continued fraction

\[
[1, 2f_1, 2f_2, 2f_3, \ldots],
\]

with \((f_h)\) a folded sequence, simplifies to a continued fraction with admissible partial quotients 1 or 2 only. We shall show more, using only the property that the sequence \((f_h)\) can be unfolded at least once: that is, that the sequence \((f_{2h+1})\) is alternating in sign.

We need the auxiliary result:

\[
[a, -b, c] = [a-1, 1, b-2, 1, c-1].
\]

This is readily checked by the fundamental correspondence or by invoking Proposition 3 whereby

\[
[a, -b, c, d, \ldots] = [a, 0, -1, 1, -1, 0, b, -c, -d, \ldots] = [a-1, 1, b-1, 0, -1, 1, -1, 0, c, d, \ldots] = [a-1, 1, b-2, 1, c-1, d, \ldots].
\]

Applying this result to remove the inadmissible \(-2\)s and again recalling that

\[
[\ldots, a, 0, b, \ldots] = [\ldots, a+b, \ldots],
\]

we obtain

\[
[1, 2, a, -2, b, 2, c, -2, d, 2, e, -2, f, \ldots] = [1, 2, a-1, 2, b-1, 2, c-1, 2, d-1, 2, e-1, 2, f-1, \ldots],
\]

whilst

\[
[1, -2, a, 2, b, -2, \ldots] = [0, 2, a-1, 2, b-1, 2, \ldots].
\]

It remains only to remove those partial quotients \( a-1, b-1, \ldots \) that are inadmissible. Without losing generality we may suppose that \( a-1 \) is inadmissible, in which case we have

\[
[2, a-1, 2] = [1, 1, -a-1, 1, 1] = [1, 1, 1, 1, 1],
\]

because \( a-1 = -3 \). If also \( b-1 \) is inadmissible, then

\[
[1, 1, b-1, 2] = [1, 0, 1, -b-1, 1, 1] = [2, 1, 1, 1].
\]
Evidently, and surprisingly, this covers all cases required to demonstrate that if a sequence of $\pm 2$ s can be unfolded at least once then, considered as a sequence of partial quotients, it yields a sequence of partial quotients consisting just of 1 s or 2 s. Indeed, we could have started with yet more general sequences of partial quotients, namely those consisting of $\pm 2$ s or $\pm 3$ s folded once by $\pm 2$ s: in the sense that either $(f_{2h+1}) = (-2)^h$ for all $h > 0$ or $(f_{2h+1}) = -(2)^h$ for all $h > 0$.

In our introduction we promised explicit continued fraction expansions for explicit binary decimals. Our numbers

$$2 \sum_{h=0}^{\infty} \pm 2^{-2h}$$

become explicit binary decimals on repeated use of the remark

$$1 - 2^{-m} = 0.11\ldots 111_{\text{m digits}}$$

and Theorem 2 yields a sufficiently simple algorithm to satisfy the remainder of our undertaking. But, with only a little work, we can be less implicitly explicit. As a preliminary, we mention that maps $S_{\overrightarrow{p}} : \overrightarrow{w} \mapsto \overrightarrow{w} \overrightarrow{p} \overrightarrow{w}$ perturb symmetry in the sense that a word

$$\overrightarrow{w} \overrightarrow{p} \overrightarrow{w} \overrightarrow{p} \overrightarrow{w} \overrightarrow{p} \overrightarrow{w} \overrightarrow{p} \overrightarrow{w} \overrightarrow{p} \overrightarrow{w} \cdots$$

resulting from repeated application of such maps is symmetric if and only if the inserted perturbations $\overrightarrow{p}_h$ each are symmetric. Notice that these sequences have an arrow sequence that is folded. In fact it is just these sequences that are described as ‘folded sequences’ in [3]; see also [9].

**Theorem 3.** Let $a$ be a binary decimal $a = 0.a_1a_2a_3\ldots$, set $a_0 = 0$ and denote by $F_a(x)$ the number

$$F_a(x) = x \sum_{h=0}^{\infty} (-1)^{a_h} x^{-2^h},$$

where $x \geq 2$ is a rational integer.

Given a word $\overrightarrow{p}$, let $S_{\overrightarrow{p}}$ be the perturbed symmetry map which acts on words $\overrightarrow{w}$ of partial quotients

$$S_{\overrightarrow{p}} : \overrightarrow{w} \mapsto \overrightarrow{w}, \overrightarrow{p}, \overrightarrow{w}$$

producing the word $\overrightarrow{w}, \overrightarrow{p}, \overrightarrow{w}$ of partial quotients. Then the continued fraction expansion of $F_a(x)$ is given by:

$$F_{0.a_1a_2a_3a_4\ldots}(x) = [b_{a_1a_2} : \prod_{h=3}^{\infty} S_{p_{a_1a_2a_h}}(w_{a_1a_2a_3})]$$

$$= [1, X - 1, 1, \cdots S_{p_{a_1a_2a_3}} S_{p_{a_1a_2a_4}} S_{p_{a_1a_2a_5}}(w_{a_1a_2a_3})],$$
where

\[ b_{00} = [1, x - 1, 1], \quad w_{000} = [x - 1, x, x - 1, 1, x - 1, x - 1] \]
\[ \text{and} \quad p_{000} = [1, x - 1, x - 1, 1, x - 2, 1]; \]
\[ b_{01} = [1, x], \quad w_{010} = [x - 1, 1, x - 1, x - 1, 1, x - 2, 1, x - 1] \]
\[ \text{and} \quad p_{010} = [x, x - 1, 1, x - 1]; \]
\[ b_{10} = [0, 1, x - 1], \quad w_{100} = [x - 1, 1, x - 1, 1, x - 1, x - 1] \]
\[ \text{and} \quad p_{100} = [x - 1, 1, x - 1, x - 1]; \]
\[ b_{11} = [0, 1, x - 2, 1], \quad w_{110} = [x - 1, x - 1, 1, x - 1, x - 1, x - 1] \]
\[ \text{and} \quad p_{110} = [1, x - 2, 1, x - 1, x - 1, x - 1], \]

while

\[ \overline{w}_{a_{1}a_{2}} = \overline{w}_{a_{1}a_{2}} \text{ and } \overline{p}_{a_{1}a_{2}} = \overline{p}_{a_{1}a_{2}}. \]

If \( x = 2 \), the simplification that ensues from removing the inadmissible 0 partial quotients is plain.

**Proof.** Folding yields perturbed symmetry because

\[ [b, F_{x}(c, d, \overline{w}, d, c)] = [b, c, d, \overline{w}, d, c, x, -c, -d, -\overline{w}, -d, -c] \]
\[ = [b, c, d, \overline{w}, d, c, x - 1, 1, c - 1, d, \overline{w}, d, c] \]
\[ = [b, c, d, S_{\overline{p}}(\overline{w}), d, c], \]

with \( \overline{p} = d, c, x - 1, 1, c - 1, d \). Moreover, folding once with \(-x\) yields the same result other than that the perturbation must be replaced by \( \overline{p} \).

There are four possible beginnings for the continued fractions of \( F_{0,a_{1}a_{2}}(x) \), namely

\[ f_{0.00}(x) = [1, x, -x, -x] = [1, x - 1, 1, x - 1, x] \]
\[ f_{0.01}(x) = [1, x, x, -x] = [1, x, x - 1, 1, x - 1] \]
\[ f_{0.10}(x) = [1, -x, -x, x] = [0, 1, x - 1, x - 1, 1, x - 2, 1] \]
\[ f_{0.11}(x) = [1, -x, x, x] = [0, 1, x - 2, 1, x - 1, x - 1, 1] \]

where foresight brought by hindsight suggests writing the latter two continued fractions so as to produce an even number of partial quotients in each case. Each yields two beginnings

\[ f_{0.000}(x) = [1, x - 1, 1, x - 1, x, x, -x, -x + 1, -1, -x + 1] \]
\[ = [1, x - 1, 1, x - 1, x, x - 1, 1, x - 1, x - 1, x - 1, 1, x - 1] \]
\[ = [1, x - 1, 1, \overline{w}_{000}, 1, x - 1] \]

and

\[ f_{0.001}(x) = [1, x - 1, 1, \overline{w}_{000}, 1, x - 1]; \]
\[ f_{0.010}(x) = [1, x, x - 1, 1, x - 1, x, -x + 1, -1, -x + 1, -x] \]
\[ = [1, x, x - 1, 1, x - 1, 1, x - 2, 1, x - 1, x] \]
\[ = [1, x, \overline{010}, x] \]

and

\[ f_{0.011}(x) = [1, x, \overline{011}, x]; \]

\[ f_{0.100}(x) = [0, 1, x - 1, 1, x - 1, 1, x - 2, 1, x, -1, -x + 2, -1, -x + 1, -x + 1, -1] \]
\[ = [0, 1, x - 1, x - 1, 1, x - 2, 1, x - 1, 1, x - 1, 1, x - 1, 1] \]
\[ = [0, 1, x - 1, \overline{010}, x - 1, 1] \]

and

\[ f_{0.101}(x) = [0, 1, x - 1, \overline{100}, x - 1, 1]; \]

\[ f_{0.110}(x) = [0, 1, x - 2, 1, x - 1, 1, x - 1, 1, x, -1, -x + 1, -x + 1, -1, -x + 2, -1] \]
\[ = [0, 1, x - 2, 1, x - 1, 1, x - 1, 1, x - 1, 1, x - 2, 1] \]
\[ = [0, 1, x - 1, \overline{110}, 1, x - 2, 1] \]

and

\[ f_{0.111}(x) = [0, 1, x - 2, 1, \overline{110}, 1, x - 2, 1]. \]

In each case \( w_{a_1a_20} \) has been chosen to ensure a ready interpretation of the final result in the special case \( x = 2 \).

The opening remark of the proof now yields the theorem. \( \square \)

**Note.** Special cases of this detailed result are first remarked on in [8]. Those observations are alluded to in [3], where a blunder, see [1], hints at the present general result. There are different ways for a sequence to display its perturbed symmetry. Thus, indeed,

\[ F_0(x) = [1, x - 1, 1, x - 1, \prod_{h=0}^{\infty} S_{1, x-1, x-1, 1, x-2, 1}(x - 1, x, x - 1, 1, x - 1, x - 1)] \]
\[ = [1, x - 1, 1, \prod_{h=0}^{\infty} S_{x-1, 1, x-1, x-1, 1, x-2, 1, x-1}(x, x - 1, 1, x - 1)] \]

confirming the correction suggested in [1].

### 6. Divide and Conquer

We shall briefly sketch a technique for obtaining, from our results, explicit continued fraction expansions of the numbers

\[ \sum_{h=0}^{\infty} \pm x^{-2^h}. \]
Indeed, we shall divide our results above by $x$. To that end we recall the fundamental correspondence whereby one defines the convergents by matrix products. For our present purpose it will be convenient to view those products a little differently. Accordingly, set

$$J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

noticing that $JR = LJ$, $JL = RJ$ and $J^2 = I$. Then, on observing that for $d \in \mathbb{Z}$,

$$R^d = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}, \quad L^d = \begin{pmatrix} 1 & 0 \\ d & 1 \end{pmatrix},$$

we may write, with a formal interpretation intended,

$$\begin{pmatrix} c_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots = R^{c_0} L^{c_1} R^{c_2} \cdots,$$

that is, with $\longleftrightarrow$ denoting the fundamental correspondence between matrix products and continued fractions,

$$[c_0, c_1, c_2, \ldots] \longleftrightarrow R^{c_0} L^{c_1} R^{c_2} \cdots.$$

Then

$$x^{-1} [c_0, c_1, c_2, \ldots] \longleftrightarrow A'R^{c_0} L^{c_1} R^{c_2} \cdots,$$

where we set

$$A = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}, \quad A' = \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}.$$

Of course the matrix product on the right in the correspondence just given does not correspond to a continued fraction, but using the transition formulæ

$$ALR^{x-1} = R^{x-1} LA' \quad A'RL^{x-1} = L^{x-1} RA$$

$$ALR^{-x-1} = -LR^{x+1} L^{-1} A' \quad A'RL^{-x-1} = -IL^{x+1} R^{-1} A$$

together with the remark that multiplication by a matrix $dI$ does not change the corresponding continued fraction, we can transit the multiplier $A'$ through the $R-L$ sequence until it disappears into the $\cdots\cdots$ on the right.

For example, recalling that

$$F_0(x) = [1, x, -x, -x, x, x, -x, -x, \ldots],$$

we can compactly show the transduction as

$$F_0(x) \quad RL^{x-1} LR^{-x-1} \quad RL^{-x-1} LR^{x-1} \quad RL^{x-1} LR^{-x-1} \quad RL^{-x-1} LR^{x-1} \quad LR^{-x-1}$$

$$A' \quad A \quad A' \quad A \quad A' \quad A \quad A' \quad A \quad \cdots$$

$$G_0(x) \quad L^{x-1} RRL^{x+1} L^{-1} L^{x+1} R^{-1} RRL^{x+1} L^{-1} L^{x-1} RRL^{x+1} L^{-1} L^{x-1}$$

and see that

$$G_0(x) = \sum_{h=0}^{\infty} x^{-2h} = [x - 1, x + 2, x, x, x - 2, x, x + 2, x, \ldots].$$

In this manner the energetic reader can readily rediscover the results of [8], [9] and more — see [6]. The technique used above is suggested by work of Raney [7]. A surprising application can be found in [2].
REFERENCES


Alfred J. van der Poorten
School of Mathematics, Physics, Computing and Electronics
Macquarie University NSW 2109
Australia
alf@mqcomp.mqcs.mq.oz.au
munnari!mqcomp.mqcs.mq.oz.au!alf@UUNET.uu.net

Jeffrey Shallit
Department of Mathematics and Computer Science
Dartmouth College
Bradley Hall
Hinman Box 6188
Hanover New Hampshire 03755
USA
shallit@dartmouth.edu