Combinatorial and Additive Number Theory Problem Sessions

Participants of CANT 2010

June 1, 2011

Abstract

These notes are a summary of the problem session discussions during the last three days of CANT 2011 (May 25th to May 27th, 2011 at the CUNY Graduate Center). A list of participants is given at the end; whenever possible, comments are attributed to the speaker.

For more information, visit the conference homepage at

`http://www.theoryofnumbers.com/`

or email either the typist at sjm1@williams.edu or the organizer at melvyn.nathanson@lehman.cuny.edu.

Warning: These notes were LaTeX-ed in real-time by Steven J. Miller; all errors should be attributed solely to the typist. Many thanks to Steven Senger for typing up the final day's session.

1 Seva Lev

Problem: Let $A \subset \mathbb{F}_2^n$, $p \in \mathbb{F}_2[x_j]_{j=1}^n$, and for all $a, b \in A$ if $a \neq b$ then $p(a+b) = 0$. Does this imply that $p(0) = 0$?

For example, if $A = \{a, b\}$ then $p(a+b) = 0$ does not imply $p(0) = 0$.

If A is large and the degree of p is small, what is true? For a given p, how large must $|A|$ be for this to be true? We have the following:

$\deg P$	need
0	$\|A\| \geq 2$
1	$\|A\| \geq 3$
2	$\|A\| \geq n+3$
3	$\|A\| \geq 2n$
$\leq (frac12o(1))\, n$	$???.$

2 Giorgis Petridis

P-R: $D_2 \geq 1$ implies that there exists v_0 vertex disjointed paths of length 2 in G.

Problem: What can be said when $D_2 \geq k \in \mathbb{Z}$?

Guess: there exist v_0 vertex disjoint trees in G each having at least k_i vertices in V_i. Note: there is an example which shows that one cannot hope to prove this guess using max flow - min out. Guess confirmed in $k = |V_0| = 2$ by Petridis.

3 Mel Nathanson

Believe the following is an unsolved problem by Hamidoune (he proposed it and no one has solved it):

Problem: Let G be a torsion free group, $G \neq \{e\}$. Let S be a finite subset of G, $e \in S$,

$$\kappa_k(S) \;=\; \min\left\{|XS| - |X| : \text{finitesets } X \subset G, |X| \geq k\right\}.$$

Hamidoune conjectured that there is an $A \subset G$ with $|AS| - |A| = \kappa_k(S)$ and $|A| = k$.

True for $k = 1$, unknown for $k \geq 2$. It is true for ordered groups. As every free abelian group of finite rank can be ordered, true here. In general for $k = 2$ still unknown.

4 Matthew DeVos

Problem: Let G be a multiplicative group, $S \subset G$ a finite set, and set

$$\Pi(S) \ = \ \{s_1 \cdots s_k : s_i \in S, s_i = s_j \iff i = j\} \cup \{1\}.$$

Not allowed to use an element multiple times. Conjecture: there is a $c > 0$ such that for every group G and set $S \subset G$ there exists $H \subset G$ with $|\Pi(S)| \geq |H| + c|H| \cdot |S \setminus H|^2$.

True with $c = 1/64$ when G is abelian.

5 David Newman

Problem: How many partitions are there where no frequency is used more than once?

For example, the partitions of 4 are $\{4\}$, $\{3, 1\}$, $\{2, 2\}$, $\{2, 1, 1\}$ and $\{1, 1, 1, 1\}$. The ones that are okay are all but $\{3, 1\}$. The problem here is that the two decompositions each occur just once: we have one 3 and one 1.

6 Steven J. Miller, Sean Pegado, Luc Robinson

Problem: For each positive integer k, consider all A such that $|kA + kA| > |kA - kA|$ and $1 \in A$ (for normalization purposes). Let C_k be the smallest of the largest elements of such A's. What can you say about the growth of C_k?

$C_1 = 15$, $C_2 = 31$, $\ldots$.

7

Problem: Assume that you have A, B in a general group and $|AB| < \alpha|A|$ and $|AbB| \leq \alpha|A|$ for all $b \in B$. Does there exist an absolute c such that $X \subset A$ then $|XB^h| \leq \alpha^{ch}|X|$?

Rusza showed that if you have $|A+B_j| \leq \alpha_j|A|$ for $j=1,2$, then there is an X such that $|X+B_1+B_2| \leq \alpha_1\alpha_2|X|$.

Problem: Is there a prescription for X given that Rusza's theorem shows the existence of X.

8

Problem: Let $\mathcal{B}$ be a partition of n. Consider the partition where $c_1+\cdots+c_k=n$, and $1^{d_1}2^{d_2}\cdots n^{d_n}$. The d_i's are the number of the c_j's and $d_1+\cdots+d_n=m$. Consider

$$\sum_{\mathcal{B}\in\mathcal{P}(n)} \binom{n}{c_1,\ldots,c_k}\binom{n+m+1}{n+1,d_1,\ldots,d_n}\left[\frac{1}{m+n+1}\right].$$

What can be said?

Try putting in an r^n and summing over n. Maybe this is a holomorphic part of a non-holomorphic Maass form.

9 Peter Hegarty

Problem: Consider the least residue of n modulo q, denoted $[n]_q$, which is in $\{-q/2,\ldots,q/2\}$. Want a function from $\pi:\{1,\ldots,27\}$ to itself (a permutation, so 1-1) with the property that given any a,b,c not all equal with $|[a+c-2b]_{27}| \leq 1$ then $|[\pi(a)+\pi(c)-2\pi(b)]_{27}| \geq 2$.

Motivation: replace 27 with n, ..., have a permutation avoiding a progression. Conjecture that a permutation of $\mathbb{Z}_n$ exists for every n sufficiently large.

10

Problem: Define $h:\{1,\ldots,N\}\to\mathbb{Z}/N\mathbb{Z}$; call it a partial homomorphism if it a bijection such that whenever $a,b,ab\in\{1,\ldots,N\}$ then $h(ab)=h(a)+h(b) \bmod N$. Does such a function exist for all N?

Have built by hand for all N up to 64?

11 Steven Senger

The basic idea is that an additive shift will destroy multiplicative structure. Given a large, finite set, $A \subset \mathbb{N}$, suppose that $|AA|$ = n. We know that there exists no generalized geometric progression, G, of length $c_1 n$, such that $|(AA+1) \cap G| \geq c_2 n$, where c_1 and c_2 do not depend on n. The question is, given the same conditions on A, do there exist sets $E, F \subset \mathbb{N}$, such that the following hold for c_3, c_4 independent of n, and $\delta > 0$:

- $|E|, |F| \geq n^{\delta}$
- $|EF| = c_3 n$
- $|(AA+1) \cap EF| \geq c_4 n$

Even partial results would be interesting to me. Also, considering the problem over $\mathbb{R}$ would be interesting to me.

12 Urban Larsson

2 pile Nim can be described as the set of moves on a chessboard made by a rook, moving only down and left. Players take turns moving the rook, and the person to move it to the lower-left corner is the winner. The set of legal moves is defined to be

$$\{(0,x),(x,0)\}.$$

In this case, the positions which guarantee victory following perfect play, or *p-positions* are along the diagonal. That is, the player who consistently moves the rook to the diagonal will eventually win.

In Wythoff Nim, the piece is replaced by a queen, and the diagonal move is added. The set of legal moves for Wythoff Nim is

$$\{(0,x),(x,0),(x,x)\}.$$

This game has p-positions close to the lines of slope ϕ and ϕ^{-1}, where, ϕ denotes the golden ratio. For example, the points $(\lfloor \phi x \rfloor, \lfloor \phi^2 x \rfloor)$ are p-positions in Wythoff Nim.

Now, adjoin the multiples of the last possible p-positions from Wythoff Nim which are not in Wythoff Nim, namely the multiples of the knight's move. The legal moves of the new game are

$$\{0, x), (x, 0), (x, x), (x, 2x), (2x, x)\}.$$

The p-positions for this game appear to split along lines of slopes nearly 2.25 and 1.43. Why?

13 Thomas Chartier

Let $n, k \in \mathbb{N}$, and $p = nk + 1$ be prime. Exclude 1 and 2. Fixing n does there exist a k such that

$$1^k, 2^k, 3^k, \ldots, n^k$$

are distinct mod p? The conjecture is that such a k exists for every non-trivial n.

14 Mel Nathanson

Recall the classical sum-product problem of Erdős. Given a large set of positive integers, $A \subset \mathbb{N}$, either the set of sums or the set of products should be large. The conjecture is that, for such an A, with c independent of n, for any $\epsilon > 0$,

$$\max\{|A + A|, |AA|\} \geq cn^{2-\epsilon}.$$

15 Speakers

1. Paul Baginski, Universite Claude Bernard Lyon, France
2. Mei-Chu Chang, University of California-Riverside
3. Scott Chapman, Sam Houston State University
4. Jonathan Cutler, Montclair State University
5. Matthew DeVos, Simon Fraser University
6. Aviezri Fraenkel, Weizmann Institute of Science, Israel

7. Peter Hegarty, University of Gothenburg, Sweden
8. Charles Helou, Penn State Brandywine
9. Jerry Hu, University of Houston - Victoria
10. Alex Iosevich, University of Missouri
11. Renling Jin, College of Charleston
12. Nathan Kaplan, Harvard University
13. Mizan R. Khan, Eastern Connecticut State University
14. Omar Kihel, Brock University, Canada
15. Alex Kontorovich, SUNY at Stony Brook
16. Urban Larsson, University of Gothemburg, Sweden
17. Thai Hoang Le, Institute for Advanced Study
18. Vsevolod Lev, University of Haifa, Israel
19. Zeljka Ljujic, CUNY Graduate Center
20. Neil Lyall, University of Georgia
21. Steven J. Miller, Williams College
22. Rishi Nath, York College (CUNY)
23. Mel Nathanson, Lehman College (CUNY)
24. Hoi H. Nguyen, University of Pennsylvania
25. Lan Nguyen
26. Sean Pegado, Williams College
27. Giorgis Petridis, University of Cambridge
28. Luc Robinson, WIlliams College
29. Steve Senger, University of Missouri
30. Jonathan Sondow, New York