Virus Dynamics on Star Graphs

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Introduction

Motivation and Questions

- Epidemiology, networking, and other fields have questions concerning the spread of viruses.
- Using a model for infection and cure rates, look for a steady state or critical threshold relating two rates.
- If there is a steady state, what are the characteristics?
- What other information can we get from this steady state, provided it exists?
- Generalizations?

The Model ($a = 1 - \delta$, $b = \beta$)

A discrete-time **SIS** (**S**usceptible **I**nfected **S**usceptible) **model**. Each node is either **S**usceptible (**S**) or Infected (**I**).

Study special graphs: star graphs:

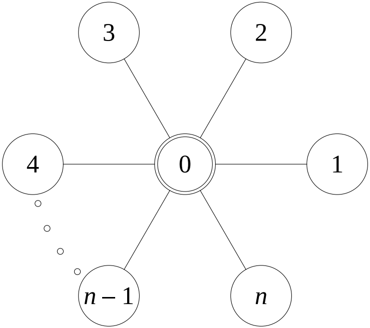
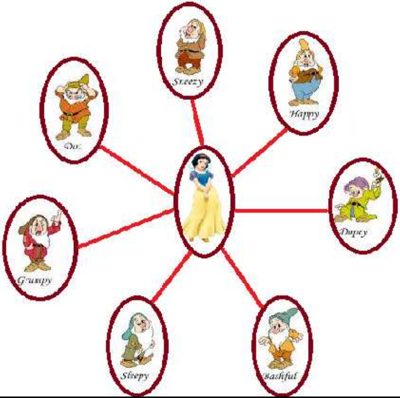


Figure: Star graph with 1 central hub and n spokes.

The Model ($a = 1 - \delta$, $b = \beta$)

A discrete-time **SIS** (**S**usceptible **I**nfected **S**usceptible) **model**. Each node is either **S**usceptible (**S**) or Infected (**I**).

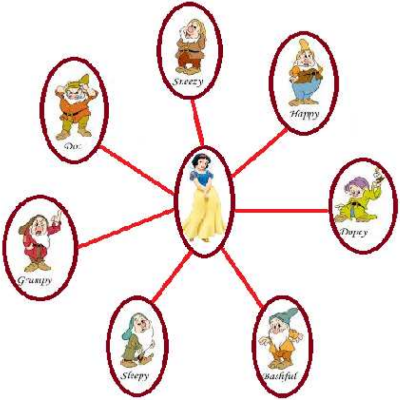
Or, a more entertaining view....



The Model ($a = 1 - \delta$, $b = \beta$)

A discrete-time **SIS** (**S**usceptible **I**nfected **S**usceptible) **model**. Each node is either **S**usceptible (**S**) or Infected (**I**).

With infection and cure rates....



The Model ($a = 1 - \delta$, $b = \beta$)

A discrete-time **SIS** (**S**usceptible **I**nfected **S**usceptible) **model**. Each node is either **S**usceptible (**S**) or Infected (**I**).

Parameters

- β: probability at any time step that an infected node infects its neighbors.
- δ: probability at any time step that an infected node is cured.
- $1 - p_{i,t} = (1 - p_{i,t-1})\,\zeta_{i,t} + \delta p_{i,t}\zeta_{i,t}$, where $\zeta_{i,t}$ is the probability node i not infected by neighbors at time t.
- $\zeta_{i,t} = \prod_{j\sim i} p_{j,t-1}\,(1-\beta) + (1 - p_{j,t-1}) = \prod_{j\sim i} 1 - \beta p_{j,t-1}$, where $j \sim i$ means i and j are neighbors (share an edge).
- $1 - p_{i,t} = (1 - p_{i,t-1})\,\zeta_{i,t} + \delta p_{i,t}\zeta_{i,t}$, where $\zeta_{i,t}$ is the probability that node i is not infected by its neighbors at time t.

System of Interest

- In limit all spokes behave the same.
- Label hub behavior at time t by x_t, spokes by y_t. Evolve by

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = F\begin{pmatrix} x_t \\ y_t \end{pmatrix},$$

where

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix} = \begin{pmatrix} 1-(1-x)(1-\beta y)^n - \delta x(1-\beta y)^n \\ 1-(1-y)(1-\beta x) - \delta y(1-\beta x) \end{pmatrix}$$

$$= \begin{pmatrix} 1-(1-ax)(1-by)^n \\ 1-(1-ay)(1-bx) \end{pmatrix}.$$

Simulation and New Results

Simulation

Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

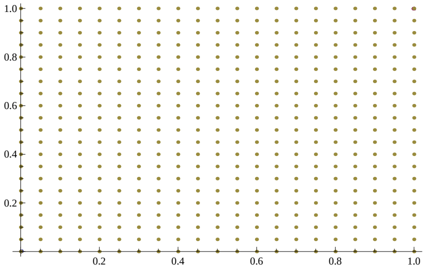


Figure: $t = 0$ (point in upper right needed for display purposes)

Simulation

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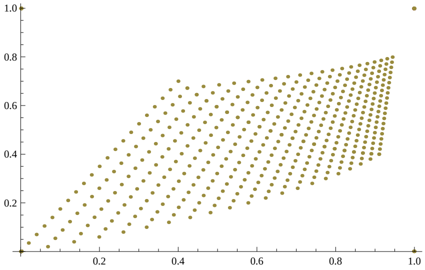

Figure: $t = 1$ (point in upper right needed for display purposes)

Simulation

Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

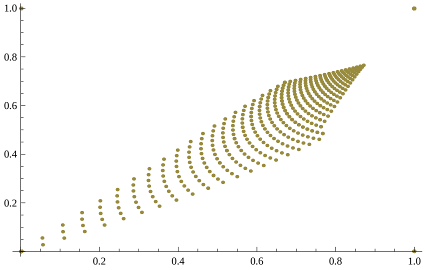

Figure: $t = 2$ (point in upper right needed for display purposes)

Simulation

Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

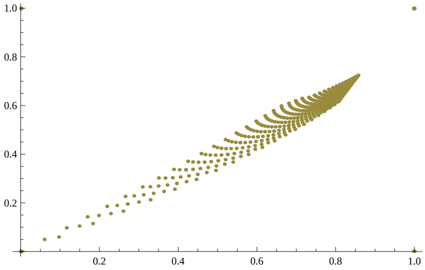

Figure: $t = 3$ (point in upper right needed for display purposes)

Simulation

Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

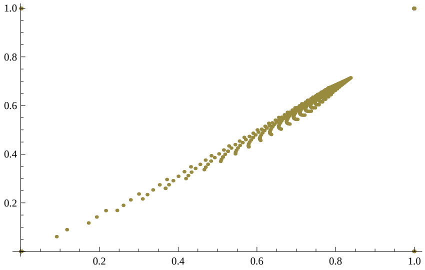

Figure: $t = 4$ (point in upper right needed for display purposes)

Simulation

Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

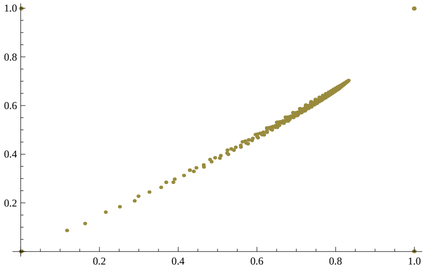

Figure: $t = 5$ (point in upper right needed for display purposes)

Simulation

Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

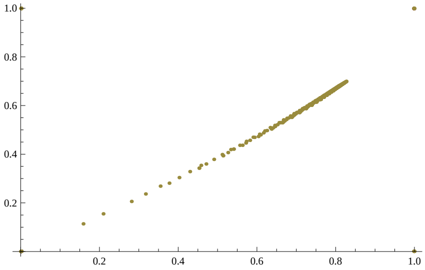

Figure: $t = 6$ (point in upper right needed for display purposes)

Simulation

Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

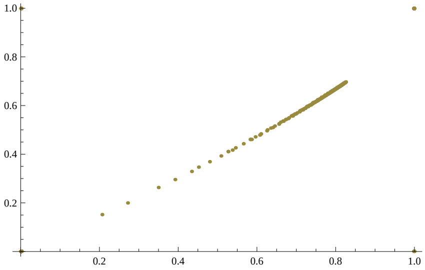

Figure: $t = 7$ (point in upper right needed for display purposes)

Simulation

Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

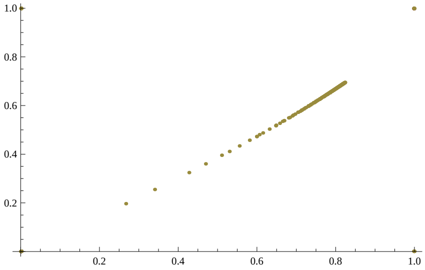

Figure: $t = 8$ (point in upper right needed for display purposes)

Simulation

Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

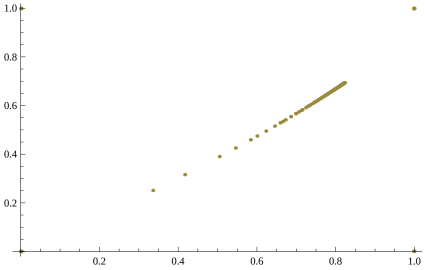

Figure: $t = 9$ (point in upper right needed for display purposes)

Simulation

Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

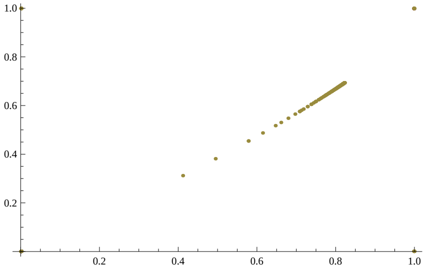

Figure: $t = 10$ (point in upper right needed for display purposes)

Simulation

Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

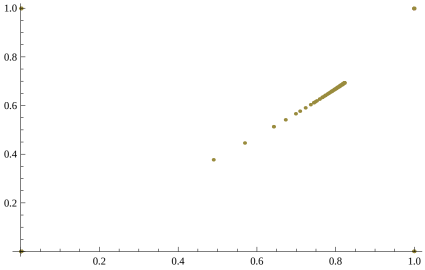

Figure: $t = 11$ (point in upper right needed for display purposes)

Simulation

Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

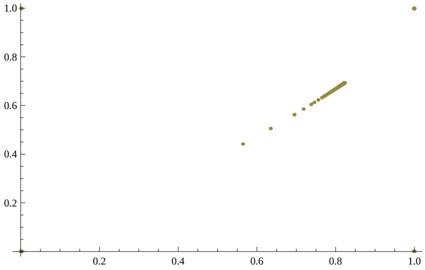

Figure: $t = 12$ (point in upper right needed for display purposes)

Simulation

Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

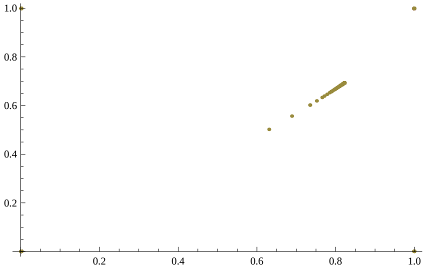

Figure: $t = 13$ (point in upper right needed for display purposes)

Simulation

Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

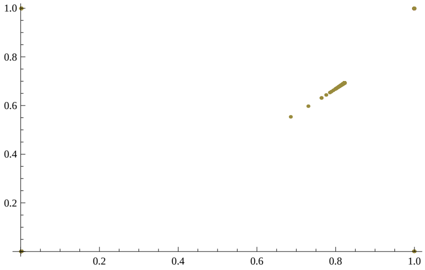

Figure: $t = 14$ (point in upper right needed for display purposes)

Simulation

Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

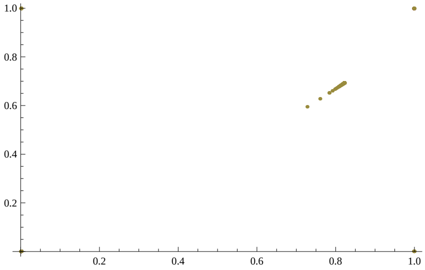

Figure: $t = 15$ (point in upper right needed for display purposes)

Simulation

Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

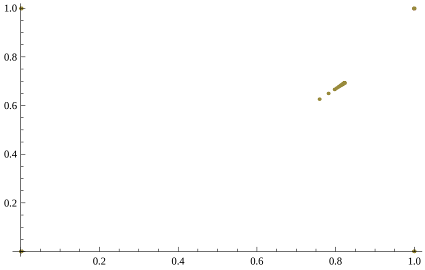

Figure: $t = 16$ (point in upper right needed for display purposes)

Simulation

Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

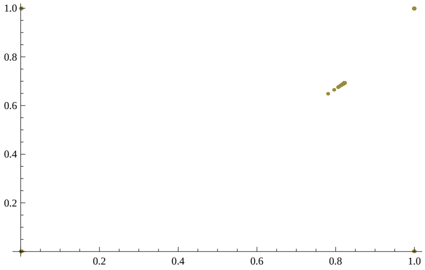

Figure: $t = 17$ (point in upper right needed for display purposes)

Simulation

Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

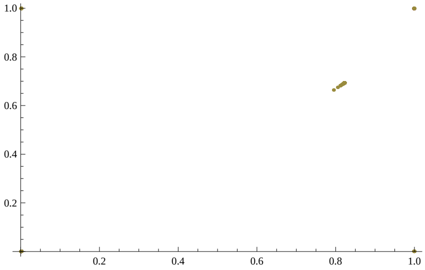

Figure: $t = 18$ (point in upper right needed for display purposes)

Simulation

Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

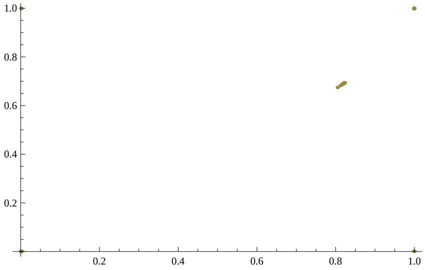

Figure: $t = 19$ (point in upper right needed for display purposes)

Simulation

Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

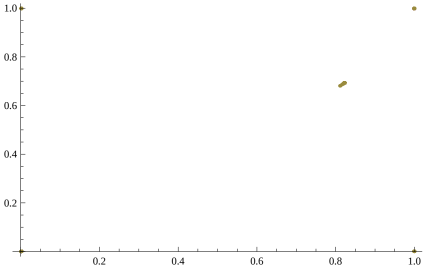

Figure: $t = 20$ (point in upper right needed for display purposes)

Simulation

Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

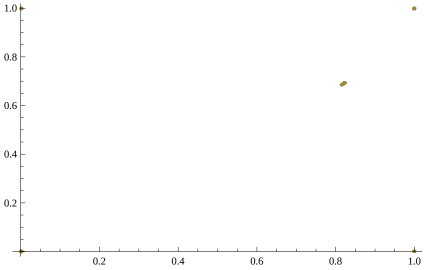

Figure: $t = 21$ (point in upper right needed for display purposes)

Simulation

Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

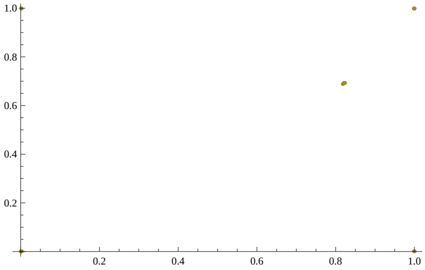

Figure: $t = 22$ (point in upper right needed for display purposes)

Simulation

Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide (x, y) space into a grid, each gridpoint a different initial configuration, iterate.

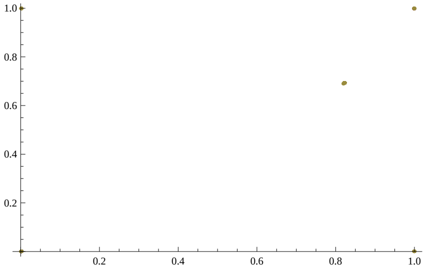

Figure: $t = 23$ (point in upper right needed for display purposes)

Main Result

Theorem (BG-TKMS '13)

Let $a, b \in (0, 1)$ and F as above.

- *For any initial configuration, as time evolves all the spokes converge to a common behavior.*
- *If $b \leq (1-a)/\sqrt{n}$ then the virus dies out.*
- *If $b > (1-a)/\sqrt{n}$ then all points except $(0, 0)$ evolve to a unique, non-trivial fixed point (x_f, y_f).*

Fixed Points and Proofs

Determining Fixed Points of F: Partial Fixed Points

Goal is to find fixed points: $F(x, y) = (x, y)$.

Easier: look for partial fixed points:

$$F(x, y) = (x, y') \quad \text{or} \quad F(x, y) = (x', y).$$

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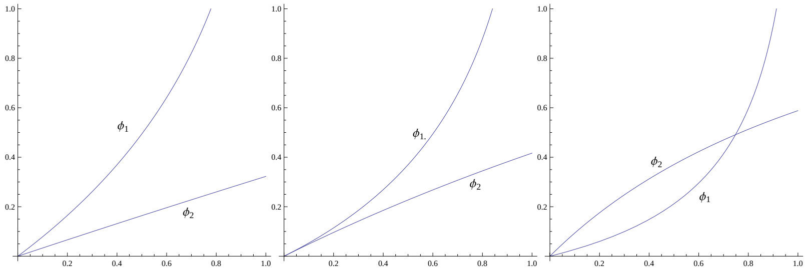
Introduce functions ϕ_1, ϕ_2 so that

- $\forall y \; \exists y'$ st $F(\phi_1(y), y) = (\phi_1(y), y')$.
- $\forall x \; \exists x'$ st $F(x, \phi_2(x)) = (x', \phi_2(x))$.

Can explicitly solve for ϕ_1, ϕ_2.

Determining Fixed Points of F: Partial Fixed Curves



Partial fixed points from ϕ_1 and ϕ_2 when (from left to right) $b < \frac{1-a}{\sqrt{n}}$, $b = \frac{1-a}{\sqrt{n}}$, $b > \frac{1-a}{\sqrt{n}}$ ($b = 3, n = 4, a = .1, .4, .7$).

$$\phi_1(y) = \frac{1-(1-by)^n}{1-a(1-by)^n} \quad \phi_2(x) = \frac{bx}{1-a+abx}.$$

Determining Fixed Points of F: Regions: $b > (1-a)/\sqrt{n}$

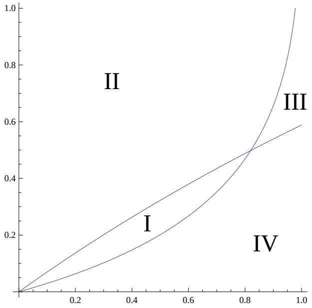


Figure: The four regions determined by the partial fixed point functions when $b > (1-a)/\sqrt{n}$.

Analysis easy if $b \leq (1-a)/\sqrt{n}$; $(0,0)$ only fixed point.

Proof unique additional fixed point when $b > (1-a)/\sqrt{n}$: concavity of the partial fixed point curves and value of derivatives at origin.

Proofs: $b \leq (1-a)/\sqrt{n}$

Convergence Case $b \leq \frac{(1-a)}{\sqrt{n}}$

Theorem

Assume $b < (1-a)/\sqrt{n}$. Then iterates of any point under F converge to the trivial fixed point $(0,0)$.

Proved with MVT and an eigenvalue analysis of the resulting matrix.

Lemma: Let $a, b \in (0,1)$ with $b < (1-a)/\sqrt{n}$, and let $\lambda_1 \geq \lambda_2$ denote the eigenvalues of the matrix $\begin{pmatrix} a\alpha & nb\beta \\ b\gamma & a\delta \end{pmatrix}$, where $\alpha, \beta, \gamma, \delta \in [0,1]$. Then $-1 < \lambda_1, \lambda_2 < 1$.

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- $\mathcal{F}(t) = f(c(t)) = \begin{pmatrix} 1-(1-atx)(1-bty)^n \\ 1-(1-aty)(1-btx) \end{pmatrix}$.
- $\mathcal{F}'(t) = \begin{pmatrix} a(1-bty)^n & nb(1-atx)(1-bty)^{n-1} \\ b(1-aty) & a(1-btxu) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

Proof (continued)

- Apply the one-dimensional chain rule twice, once to the x-coordinate function and once to the y-coordinate function.

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- Get t_1 and t_2 such that

$$f\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) - f\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) =$$
$$\begin{pmatrix} a(1-bt_1y)^n & nb(1-at_1x)(1-bt_1y)^{n-1} \\ b(1-at_2y) & a(1-bt_2x) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

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- We have a contraction map on a compact space, completing the proof.

Proofs: $b > (1-a)/\sqrt{n}$

Preliminary Results

Recall

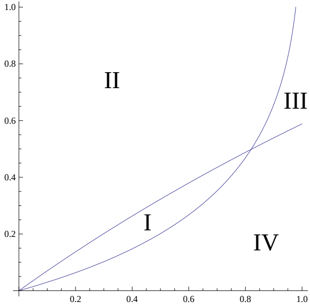

Figure: The four regions determined by the partial fixed point functions when $b > (1-a)/\sqrt{n}$.

Preliminary Results

Key lemmas (proofs by algebra):

- Points in Region I strictly increase in x and y on iteration by F, and points in Region III strictly decrease in x and y on iteration.
- Points in Region I iterate inside Region I under F, and points in Region III iterate inside Region III under F.
- All non-trivial points in Regions I and III converge to the non-trivial fixed point under F.

Armed with the above lemmas, we now complete the proof.

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- Consider any rectangle in $[0,1]^2$ whose lower left vertex is not $(0,0)$.

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- Assume the lower left and upper right vertices are in Regions I and III.
- Image of rectangle under F is strictly contained in rectangle (image of the lower left (respectively, upper right) point has both coordinates smaller (respectively, larger) than any other iterate).
- As lower left and upper right vertices iterate to the non-trivial fixed points (in Regions I and III), so too do all the other points in rectangle.

Behavior Conjectures

Corollary

The amount of time it takes for all points to converge is the maximum of the time it takes $\begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}$ *and* $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ *to converge, for* $\epsilon_1, \epsilon_2 \to 0$.

Conjecture

Points in Region II and IV exhibit one of two behaviors, dependent on a, b, n. *Either:*

1. *All points in Region II iterate outside Region II and all points in Region IV iterate outside Region IV ("flipping behavior"), or*
2. *All points in Region II iterate outside Region IV and all points in Region IV iterate outside Region II*

Conclusions and References

- Can extend to Generalized Star Graphs.

- Thealexa Becker, Alec Greaves-Tunnell, Leo Kontorovich, Steven J. Miller and Karen Shen), *Virus Dynamics on Spoke and Star Graphs*, the Journal of Nonlinear Systems and Applications **4** (2013), no. 1, 53–63. http://arxiv.org/pdf/1111.0531.