# Chapter 1

# **Braid Group Cryptography**

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In the last decade, a number of public key cryptosystems based on combinatorial group theoretic problems in braid groups have been proposed. We survey these cryptosystems and some known attacks on them.

This survey includes: Basic facts on braid groups and on the Garside normal form of its elements, some known algorithms for solving the word problem in the braid group, the major public-key cryptosystems based on the braid group, and some of the known attacks on these cryptosystems. We conclude with a discussion of future directions (which includes also a description of cryptosystems which are based on other non-commutative groups).

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#### 1.1. Introduction

References

In many situations, we need to transfer data in a secure way: credit cards information, health data, security uses, etc. The idea of public-key cryptography in general is to make it possible for two parties to agree on a shared secret key, which they can use to transfer data in a secure way (see [73]).

There are several known public-key cryptosystems which are based on the discrete logarithm problem, which is the problem of finding x in the equation  $g^x = h$  where g, h are given, and on the factorization problem, which is the problem of factoring a number to its prime factors: Diffie-Hellman [38] and RSA [106]. These schemes are used in most of the present-day applications using public-key cryptography

There are several problems with this situation:

- Subexponential attacks on the current cryptosystems' underlying problems: Diffie-Hellman and RSA are breakable in time that is subexponential (i.e. faster than an exponential) in the size of the secret key [2]. The current length of secure keys is at least 1000 bits. Thus, the length of the key should be increased every few years. This makes the encryption and decryption algorithms very heavy.
- Quantum computers: If quantum computers will be implemented in a satisfactory way, RSA will not be secure anymore, since there are polynomial (in  $\log(n)$ ) run-time algorithms of Peter Shor [110] which solve the factorization problem and the discrete logarithm problem. Hence, it solves the problems which RSA and Diffie-Hellman are based on (for more information, see for example [3]).
- Too much secure data is transferred in the same method: It is not healthy that most of the secure data in the world will be transferred in the same method, since in case this method will be

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broken, too much secure data will be revealed.

Hence, for solving these problems, one should look for a new public-key cryptosystem which on one hand will be efficient for implementation and use, and on the other hand will be based on a problem which is different from the discrete logarithm problem and the factorization problem. Moreover, the problem should have no subexponential algorithm for solving it, and it is preferable that it has no known attacks by quantum computers.

Combinatorial group theory is a fertile ground for finding hard problems which can serve as a base for a cryptosystem. The braid group defined by Artin [7] is a very interesting group from many aspects: it has many equivalent presentations in entirely different disciplines; its word problem (to determine whether two elements are equal in the group) is relatively easy to solve, but some other problems (as the conjugacy problem, decomposition problem, and more) seem to be hard to solve.

Based on braid group and its problems, two cryptosystems were suggested about a decade ago: by Anshel, Anshel and Goldfeld in 1999 [5] and by Ko, Lee, Cheon, Han, Kang and Park in 2000 [72]. These cryptosystems initiated a wide discussion about the possibilities of cryptography in the braid group especially, and in groups in general.

An interesting point which should be mentioned here is that the conjugacy problem in the braid group attracted people even before the cryptosystems on the braid groups were suggested (see, for example, [43; 51]). After the cryptosystems were suggested, some probabilistic solutions were given [48; 49; 65], but it gave a great push for the efforts to solve the conjugacy problem theoretically in polynomial time (see [14; 15; 16; 53; 54; 55; 56; 78; 79; 80] and many more).

The potential use of braid groups in cryptography led to additional proposals of cryptosystems which are based on apparently hard problems in braid groups (Decomposition problem [113], Triple Decomposition problem [75], Shifted Conjugacy Search problem [30], and more) and in other groups, like Thompson Groups [112], polycyclic groups [41] and more. For more information, see the new book of Myasnikov, Shpilrain and Ushakov [98].

In these notes, we try to survey this fascinating subject. Section 1.2 deals with some different presentations of the braid group. In Section 1.3, we describe two normal forms for elements in the braid groups. In Section 1.4, we give several solutions for the word problem in the braid group. Section 1.5 introduces the notion of public-key cryptography. In Section 1.6,

the first cryptosystems which are based on the braid group are presented. Section 1.7 is devoted to the theoretical solution to the conjugacy search problem, using the different variants of Summit Sets. In Section 1.8, we describe some more attacks on the conjugacy search problem. In Section 1.9, we discuss some more suggestions for cryptosystems based on the braid group and their cryptanalysis. Section 1.10 deals with the option of changing the distribution for choosing a key. In Section 1.11, we deal with some suggestions for cryptosystems which are based on other non-commutative groups.

### 1.2. The braid group

### 1.2.1. Basic definitions

The braid groups were introduced by Artin [7]. There are several definitions for these groups (see [13; 107]), and we need two of them for our purposes.

### 1.2.1.1. Algebraic presentation

**Definition 1.1.** For  $n \geq 2$ , the braid group  $B_n$  is defined by the presentation:

$$\left\langle \sigma_1, \dots, \sigma_{n-1} \middle| \begin{array}{c} \sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } |i-j| \ge 2\\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ for } |i-j| = 1 \end{array} \right\rangle. \tag{1.2.1}$$

This presentation is called the *Artin presentation* and the generators are called *Artin's generators*.

An element of  $B_n$  will be called an n-braid. For each n, the identity mapping on  $\{\sigma_1, \ldots, \sigma_{n-1}\}$  induces an embedding of  $B_n$  into  $B_{n+1}$ , so that we can consider an n-braid as a particular (n+1)-braid. Using this, one can define the limit group  $B_{\infty}$ .

Note that  $B_2$  is an infinite cyclic group, and hence it is isomorphic to the group  $\mathbb{Z}$  of integers. For  $n \geq 3$ , the group  $B_n$  is not commutative and its center is an infinite cyclic subgroup.

When a group is specified using a presentation, each element of the group is an equivalence class of words with respect to the congruence generated by the relations of the presentation. Hence, every n-braid is an equivalence class of n-braid words under the congruence generated by the relations in Presentation (1.2.1).

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## 1.2.1.2. Geometric interpretation

The elements of  $B_n$  can be interpreted as geometric braids with n strands. One can associate with every braid the planar diagram obtained by concatenating the elementary diagrams of Figure 1.1 corresponding to the successive letters.

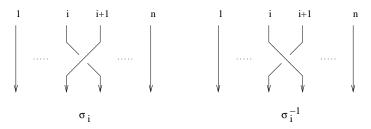


Fig. 1.1. The geometric Artin generators

A braid diagram can be seen as induced by a three-dimensional figure consisting on n disjoint curves connecting the points  $(1,0,0),\ldots,(n,0,0)$  to the points  $(1,0,1),\ldots,(n,0,1)$  in  $\mathbb{R}^3$  (see Figure 1.2).

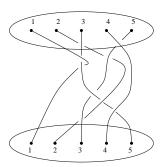


Fig. 1.2. An example of a braid in  $B_5$ 

Then the relations in Presentation (1.2.1) correspond to ambient isotopy, that is: to continuously move the curves without moving their ends and without allowing them to intersect (see Figures 1.3 and 1.4); the converse implication, i.e., the fact that the projections of isotopic 3-dimensional figures can always be encoded in words connected by presentation (1.2.1) was proved by Artin in [7]. Hence, the word problem in the braid group for the Presentation (1.2.1) is also the *braid isotopy problem*, and thus it is closely related to the much more difficult knot isotopy problem.

## Braid Group Cryptography

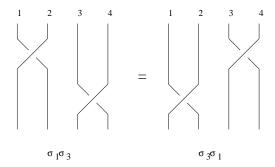


Fig. 1.3. The commutative relation for geometric Artin generators

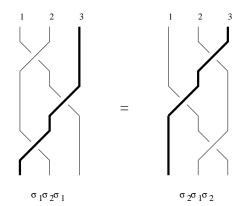


Fig. 1.4. The triple relation for geometric Artin generators

## 1.2.2. Birman-Ko-Lee presentation

Like Artin's generators, the generators of Birman-Ko-Lee [17] are braids in which exactly one pair of strands crosses. The difference is that Birman-Ko-Lee's generators includes arbitrary transpositions of strands (i, j) instead of adjacent transpositions (i, i + 1) in the Artin's generators. For each t, s with  $1 \le s < t \le n$ , define the following element of  $B_n$ :

$$a_{ts} = (\sigma_{t-1}\sigma_{t-2}\cdots\sigma_{s+1})\sigma_s(\sigma_{s+1}^{-1}\cdots\sigma_{t-2}^{-1}\sigma_{t-1}^{-1})$$

See Figure 1.5 for an example (note that the braid  $a_{ts}$  is an elementary interchange of the tth and sth strands, with all other strands held fixed, and with the convention that the strands being interchanged pass in front of all intervening strands). Such an element is called a *band generator*.

Note that the usual Artin generator  $\sigma_t$  is the band generator  $a_{t+1,t}$ .

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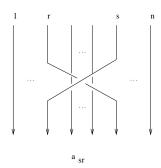


Fig. 1.5. The band generator

This set of generators satisfies the following relations (see [17, Proposition 2.1] for a proof):

- $\bullet \ a_{ts}a_{rq}=a_{rq}a_{ts} \text{ if } [s,t]\cap [q,r]=\emptyset.$
- $a_{ts}a_{sr} = a_{tr}a_{ts} = a_{sr}a_{tr}$  for  $1 \le r < s < t \le n$ .

For a geometric interpretation of the second relation, see Figure 1.6.

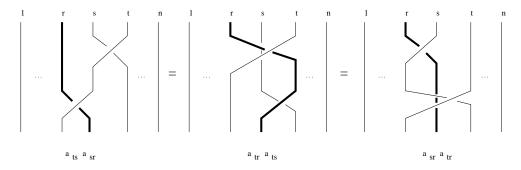


Fig. 1.6. The second relation of the Birman-Ko-Lee presentation

# 1.2.2.1. A geometric viewpoint on the difference between presentations

A different viewpoint on the relation between the two presentations is as follows: one can think on the braid group as the isotopy classes of boundary-fixing homeomorphisms on the closed disk  $D_n \subset \mathbb{C}^2$  centered at 0 with n punctures [7].

In this viewpoint, for presenting the Artin generators, we locate the

punctures on the real line, and the generator  $\sigma_i$  is the homeomorphism which exchanges the points i and i+1 along the real line (see Figure 1.7).

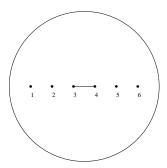


Fig. 1.7. The Artin generator  $\sigma_3$ 

On the other hand, for illustrating the generators  $a_{ts}$  of the Birman-Ko-Lee presentation, let us take the punctures organized as the vertices of a n-gon contained in the disk  $D_n$ . Now, the generator  $a_{ts}$  is the homeomorphism which exchanges the points t and s along the chord connecting them (see Figure 1.8).

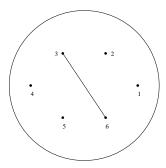


Fig. 1.8. The Birman-Ko-Lee generator  $a_{63}$ 

For more information, see [9, 19].

# 1.3. Normal forms of elements in the braid group

A *normal form* of an element in a group is a unique presentation to each element in the group.

Having a normal form for elements in the group is very useful, since it

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lets us compare two elements, so it gives a solution for the word problem:

**Problem 1.1.** Given a braid w, does  $w \equiv \varepsilon$  hold, i.e., does w represent the unit braid  $\varepsilon$  (see Figure 1.9)?

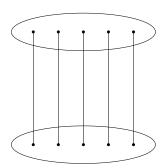


Fig. 1.9. The unit braid  $\varepsilon \in B_5$ 

Since  $B_n$  is a group, the above problem is equivalent to the following problem:

**Problem 1.2.** Given two braids w, w', does  $w \equiv w'$  hold, i.e., do w and w' represent the same braid?

Indeed,  $w \equiv w'$  is equivalent to  $w^{-1}w' \equiv \varepsilon$ , where  $w^{-1}$  is the word obtained from w by reversing the order of the letters and exchanging  $\sigma_i$  and  $\sigma_i^{-1}$  everywhere.

Also, the normal form gives a canonical representative of each equivalence class.

We present here two known normal forms of elements in the braid group. For more normal forms, see [20; 31; 40].

## 1.3.1. Garside normal form

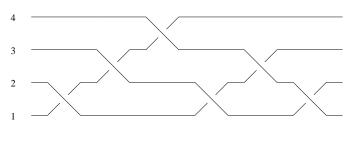
The Garside normal form is initiated in the work of Garside [51], and several variants have been described in several partly independent papers [1; 37; 43; 44; 121].

We start by defining a positive braid which is a braid which can be written as a product of positive powers of Artin generators. We denote the set of positive braids by  $B_n^+$ . This set has a structure of a monoid under the operation of braid concatenation.

An important example of a positive braid, which has a central role in the Garside normal form, is the fundamental braid  $\Delta_n \in B_n$ :

$$\Delta_n = (\sigma_1 \cdots \sigma_{n-1})(\sigma_1 \cdots \sigma_{n-2}) \cdots \sigma_1$$

Geometrically,  $\Delta_n$  is the braid on n strands, where any two strands cross positively exactly once (see Figure 1.10).



 $\Delta = \sigma_1 \sigma_2 \sigma_3 \sigma_1 \sigma_2 \sigma_1$ 

Fig. 1.10. The fundamental braid  $\Delta_4$ 

The fundamental braid has several important properties:

- (1) For any generator  $\sigma_i$ , we can write  $\Delta_n = \sigma_i A = B\sigma_i$  where A, B are positive braids.
- (2) For any generator  $\sigma_i$ , the following holds:

$$\tau(\sigma_i) = \Delta_n^{-1} \sigma_i \Delta_n = \sigma_{n-i}$$

(the inner automorphism  $\tau$  on  $B_n$  is called the *shift map*).

(3)  $\Delta_n^2$  is the generator of the center of  $B_n$ .

Now, we introduce *permutation braids*. One can define a partial order on the elements of  $B_n$ : for  $A, B \in B_n$ , we say that A is a *prefix* of B and write  $A \leq B$  if B = AC for some C in  $B_n^+$ . Its simple properties are:

- $(1) \ B \in B_n^+ \Leftrightarrow \varepsilon \preceq B$
- (2)  $A \leq B \Leftrightarrow B^{-1} \leq A^{-1}$ .

 $P \in B_n$  is a permutation braid (or a simple braid) if it satisfies:  $\varepsilon \leq P \leq \Delta_n$ . Its name comes from the fact that there is a bijection between the set of permutation braids in  $B_n$  and the symmetric group  $S_n$  (there is a natural surjective map from  $B_n$  to  $S_n$  defined by sending i to the ending place of the strand which starts at position i, and if we restrict ourselves to permutation braids, this map is a bijection). Hence, we have n! permutation braids.

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Geometrically, a permutation braid is a braid on n strands, where any two strands cross positively  $at\ most$  once.

Given a permutation braid P, one can define a starting set S(P) and a finishing set F(P) as follows:

$$S(P) = \{i | P = \sigma_i P' \text{ for some } P' \in B_n^+\}$$

$$F(P) = \{i | P = P'\sigma_i \text{ for some } P' \in B_n^+\}$$

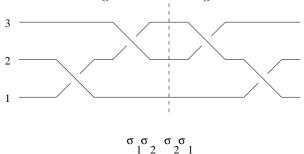
The starting set is the indices of the generators which can start a presentation of P. The finishing set is defined similarly. For example,  $S(\Delta_n) = F(\Delta_n) = \{1, \dots, n-1\}.$ 

A left-weighted decomposition of a positive braid  $A \in B_n^+$  into a sequence of permutation braids is:

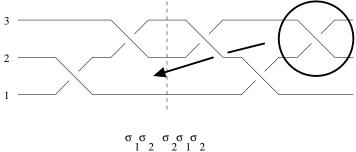
$$A = P_1 P_2 \cdots P_k$$

where  $P_i$  are permutation braids, and  $S(P_{i+1}) \subset F(P_i)$ , i.e. any addition of a generator from  $P_{i+1}$  to  $P_i$ , will convert  $P_i$  into a braid which is not a permutation braid.

**Example 1.1.** The following braid is left-weighted:



The following braid is not left-weighted, due to the circled crossing which can be moved to the first permutation braid:



Now, we show it algebraically:

$$\sigma_1\sigma_2 \cdot \sigma_2\sigma_1\sigma_2 = \sigma_1\sigma_2 \cdot \sigma_1\sigma_2\sigma_1 = \sigma_1\sigma_2\sigma_1 \cdot \sigma_2\sigma_1$$

The following theorem introduces the *Garside normal form* (or *left normal form* or *greedy normal form*) and states its uniqueness:

**Theorem 1.2.** For every braid  $w \in B_n$ , there is a unique presentation given by:

$$w = \Delta_n^r P_1 P_2 \cdots P_k$$

where  $r \in \mathbb{Z}$  is maximal,  $P_i$  are permutation braids,  $P_k \neq \varepsilon$  and  $P_1 P_2 \cdots P_k$  is a left-weighted decomposition.

For converting a given braid w into its Garside normal form we have to perform the following steps:

- (1) For any negative power of a generator, replace  $\sigma_i^{-1}$  by  $\Delta_n^{-1}B_i$  where  $B_i$  is a permutation braid.
- (2) Move any appearance of  $\Delta_n$  to the left using the relation:  $\Delta_n^{-1}\sigma_i\Delta_n=\tau(\sigma_i)=\sigma_{n-i}$ . So we get:  $w=\Delta_n^{r'}A$  where A is a positive braid.
- (3) Write A as a left-weighted decomposition of permutation braids. The way to do this is as follows: Take A, and break it into permutation braids (i.e. we take the longest possible sequences of generators which are still permutation braids). Then we get:  $A = Q_1Q_2\cdots Q_j$  where each  $Q_i$  is a permutation braid. For each i, we compute the finishing set  $F(Q_i)$  and the starting set  $S(Q_{i+1})$ . In case the starting set is not contained in the finishing set, we take a generator  $\sigma \in S(Q_{i+1}) \setminus F(Q_i)$ , and using the relations of the braid group we move it from  $Q_{i+1}$  to  $Q_i$ . Then, we get the decomposition  $A = Q_1Q_2\cdots Q_i'Q_{i+1}'\cdots Q_j$ . We continue this process till we have  $S(Q_{i+1}) \subseteq F(Q_i)$  for every i, and then we have a left-weighted decomposition as needed. For more details, see [43] and [56, Proposition 4.2] (in the latter reference, it is done based on their new idea of local slidings, see Section 1.7.5 below).

**Example 1.2.** Let us present the braid  $w = \sigma_1 \sigma_3^{-1} \sigma_2 \in B_4$  in Garside normal form. First, we should replace  $\sigma_3^{-1}$  by:  $\Delta_4^{-1} \sigma_3 \sigma_2 \sigma_1 \sigma_3 \sigma_2$ , so we get:

$$w = \sigma_1 \cdot \Delta_4^{-1} \sigma_3 \sigma_2 \sigma_1 \sigma_3 \sigma_2 \cdot \sigma_2$$

Now, moving  $\Delta_4$  to the left yields:

$$w = \Delta_4^{-1} \cdot \sigma_3 \sigma_3 \sigma_2 \sigma_1 \sigma_3 \sigma_2 \sigma_2$$

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Decomposing the positive part into a left-weighted decomposition, we get:

$$w = \Delta_4^{-1} \cdot \sigma_2 \sigma_1 \sigma_3 \sigma_2 \sigma_1 \cdot \sigma_1 \sigma_2$$

The complexity of transforming a word into a canonical form with respect to the Artin presentation is  $O(|W|^2 n \log n)$  where |W| is the length of the word in  $B_n$  [44, Section 9.5].

In a similar way, one can define a right normal form. A right-weighted decomposition of a positive braid  $A \in B_n^+$  into a sequence of permutation braids is:

$$A = P_k \cdots P_2 P_1$$

where  $P_i$  are permutation braids, and  $F(P_{i+1}) \subset S(P_i)$ , i.e. any addition of a generator from  $P_{i+1}$  to  $P_i$ , will convert  $P_i$  into a braid which is not a permutation braid.

Now, one has the following theorem about the right normal form and its uniqueness:

**Theorem 1.3.** For every braid  $w \in B_n$ , there is a unique presentation given by:

$$w = P_k \cdots P_2 P_1 \Delta_n^r$$

where  $r \in \mathbb{Z}$ ,  $P_i$  are permutation braids, and  $P_k \cdots P_2 P_1$  is a right-weighted decomposition.

For converting a given braid w into its right normal form we have to follow three steps, similar to those of the Garside normal form: We first replace  $\sigma_i^{-1}$  by  $B_i \Delta_n^{-1}$ . Then, we move any appearance of  $\Delta_n$  to the right side. Then, we get:  $w = A \Delta_n^{r'}$  where A is a positive braid. The last step is to write A as a right-weighted decomposition of permutation braids.

Now we define the *infimum* and the *supremum* of a braid w: For  $w \in B_n$ , set  $\inf(w) = \max\{r : \Delta_n^r \leq w\}$  and  $\sup(w) = \min\{s : w \leq \Delta_n^s\}$ .

One can easily see that if  $w = \Delta_n^m P_1 P_2 \cdots P_k$  is the Garside normal form of w, then:  $\inf(w) = m$ ,  $\sup(w) = m + k$ .

The canonical length of w (or complexity of w), denoted by  $\ell(w)$ , is given by  $\operatorname{len}(w) = \sup(w) - \inf(w)$ . Hence, if w is given in its normal form, the canonical length is the number of permutation braids in the form.

## 1.3.2. Birman-Ko-Lee canonical form

Based on the presentation of Birman, Ko and Lee [17], they give a new canonical form for elements in the braid group.

They define a new fundamental word:

$$\delta_n = a_{n,n-1}a_{n-1,n-2}\cdots a_{2,1} = \sigma_{n-1}\sigma_{n-2}\cdots \sigma_1$$

See Figure 1.11 for an example for n = 4.

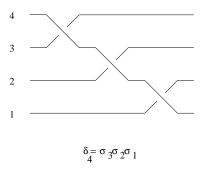


Fig. 1.11. The fundamental braid  $\delta_4$ 

One can easily see the connection between the new fundamental word and Garside's fundamental word  $\Delta_n$ :

$$\Delta_n^2 = \delta_n^n$$

The new fundamental word  $\delta_n$  has important properties, similar to  $\Delta_n$ :

- (1) For any generator  $a_{sr}$ , we can write  $\delta_n = a_{sr}A = Ba_{sr}$  where A, B are positive braids (with respect to the Birman-Ko-Lee generators)
- (2) For any generator  $a_{sr}$ , the following holds:  $a_{sr}\delta_n = \delta_n a_{s+1,r+1}$ .

Similar to Garside's normal form of braids, each element of  $B_n$  has the following unique form in terms of the band generators:

$$w = \delta_n^j A_1 A_2 \cdots A_k$$

where  $A = A_1 A_2 \cdots A_k$  is positive, j is maximal and k is minimal for all such representations, also the  $A_i$ 's are positive braids which are determined uniquely by their associated permutations (see [17, Lemma 3.1]). Note that not every permutation corresponds to a canonical factor. We will refer to Garside's braids  $P_i$  as permutation braids, and to the Birman-Ko-Lee braids  $A_i$  as canonical factors.

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Note that there are  $C_n = \frac{(2n)!}{n!(n+1)!}$  (the *n*th Catalan number) different canonical factors for the band-generators presentation [17, Corollary 3.5], whence there are n! different permutation braids for the Artin presentation. Since  $C_n$  is much smaller than n!, it is sometimes computationally easier to work with the band-generators presentation than the Artin presentation (see also Section 1.8.3.2 below).

As in Garside's normal form, there is an algorithmic way to convert any braid to this canonical form: we first convert any negative power of a generator to  $\delta_n^{-1}A$  where A is positive. Then, we move all the  $\delta_n$  to the left, and finally we organize the positive word in a left-weighted decomposition of canonical factors.

The complexity of transforming a word into a canonical form with respect to the Birman-Ko-Lee presentation is  $O(|W|^2n)$ , where |W| is the length of the word in  $B_n$  [17].

As in Garside's normal form, one can define infimum, supremum and canonical length for the canonical form of the Birman-Ko-Lee presentation.

### 1.4. Algorithms for solving the word problem in braid group

Using  $\varepsilon$  for the unit word (see Figure 1.9), the word problem is the following algorithmic problem:

**Problem 1.3.** Given one braid word w, does  $w \equiv \varepsilon$  hold, i.e., does w represent the unit braid  $\varepsilon$ ?

In this section, we will concentrate on some solutions for the word problem in the braid group.

### 1.4.1. Dehornoy's handles reduction

The process of handle reduction has been introduced by Dehornoy [28], and one can see it as an extension of the free reduction process for free groups. Free reduction consists of iteratively deleting all patterns of the form  $xx^{-1}$  or  $x^{-1}x$ : starting with an arbitrary word w of length m, and no matter on how the reductions are performed, one finishes in at most m/2 steps with a unique reduced word, i.e., a word that contains no  $xx^{-1}$  or  $x^{-1}x$ .

Free reduction is possible for any group presentation, and in particular for  $B_n$ , but it does not solve the word problem: there exist words that represent  $\varepsilon \in B_n$ , but do not freely reduce to the unit word. For example,

the word  $\sigma_1 \sigma_2 \sigma_1 \sigma_2^{-1} \sigma_1^{-1} \sigma_2^{-1}$  represents the unit word, but free reductions can not reduce it any more.

The handle reduction process generalizes free reduction and involves not only patterns of the form  $xx^{-1}$  or  $x^{-1}x$ , but also more general patterns of the form  $\sigma_i \cdots \sigma_i^{-1}$  or  $\sigma_i^{-1} \cdots \sigma_i$ :

**Definition 1.4.** A  $\sigma_i$ -handle is a braid word of the form

$$w = \sigma_i^e w_0 \sigma_{i+1}^d w_1 \sigma_{i+1}^d \cdots \sigma_{i+1}^d w_m \sigma_i^{-e},$$

with  $e, d = \pm 1, m \ge 0$ , and  $w_0, \dots, w_m$  containing no  $\sigma_j^{\pm 1}$  with  $j \le i + 1$ . The reduction of w is defined as follows:

$$w' = w_0 \sigma_{i+1}^{-e} \sigma_i^d \sigma_{i+1}^e w_1 \sigma_{i+1}^{-e} \sigma_i^d \sigma_{i+1}^e \cdots \sigma_{i+1}^{-e} \sigma_i^d \sigma_{i+1}^e w_m,$$

i.e., we delete the initial and final letters  $\sigma_i^{\pm 1}$ , and we replace each letter  $\sigma_{i+1}^{\pm 1}$  with  $\sigma_{i+1}^{-e}\sigma_i^{\pm 1}\sigma_{i+1}^e$  (see Figure 1.12, taken from [29]).

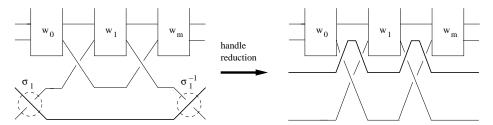


Fig. 1.12. An example for a handle reduction (for  $\sigma_1$ ). The two circled crossings in the left side are the start and the end of the handle

Note that a braid of the form  $\sigma_i \sigma_i^{-1}$  or  $\sigma_i^{-1} \sigma_i$  is a handle, and hence we see that the handle reduction process generalizes the free reduction process.

Reducing a braid yields an equivalent braid: as illustrated in Figure 1.12, the (i+1)th strand in a  $\sigma_i$ -handle forms a sort of handle, and the reduction consists of pushing that strand so that it passes above the next crossings instead of below. So, as in the case of a free reduction, if there is a reduction sequence from a braid w to  $\varepsilon$ , i.e., a sequence  $w = w_0 \rightarrow w_1 \rightarrow \cdots \rightarrow w_N = \varepsilon$  such that, for each k,  $w_{k+1}$  is obtained from  $w_k$  by replacing some handle of  $w_k$  by its reduction, then w is equivalent to  $\varepsilon$ , i.e., it represents the unit word  $\varepsilon$ .

The following result of Dehornoy [28] shows the converse implication and the termination of the process of handle reductions:

**Prop 1.1.** Assume that  $w \in B_n$  has a length m. Then every reduction

sequence starting from w leads in at most  $2^{m^4n}$  steps to an irreducible braid (with respect to Dehornoy's reductions). Moreover, the unit word  $\varepsilon$  is the only irreducible word in its equivalence class, hence w represents the unit braid if and only if any reduction sequence starting from w finishes with the unit word.

A braid may contain many handles, so building an actual algorithm requires to fix a strategy prescribing in which order the handles will be reduced. Several variants have been considered; as can be expected, the most efficient ones use a "Divide and Conquer" trick.

For our current purpose, the important fact is that, although the proved complexity upper bound of the above proposition is very high, handle reduction is extremely efficient in practice, even more than the reduction to a normal form, see [29].

**Remark 1.1.** In [33], Dehornoy gives an alternative proof for the convergence of the handle reduction algorithm of braids which is both more simple and more precise than the one in his original paper on handle reductions [28].

## 1.4.2. Action on the fundamental group

As we have pointed out at Section 1.2.2.1, the braid group can be thought of as the isotopy classes of boundary-fixing homeomorphisms on the closed disk  $D_n \subset \mathbb{C}^2$  centered at 0 with n punctures  $p_1, \ldots, p_n$  [7]. It means that two elements are the same if their actions on  $\pi_1(D_n \setminus \{p_1, \ldots, p_n\}, u)$  are equal.

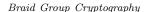
In [47], we propose the following solution for the word problem: we start with a geometric base for  $\pi_1(D_n \setminus \{p_1, \ldots, p_n\}, u)$  presented in Figure 1.13.

Now, we apply the two braids on this initial geometric base. If the resulting bases are the same up to isotopy, it means that the braids are equal, otherwise they are different.

In Figure 1.14, there is a simple example of two equal braids which result the same base.

This algorithm is very quick and efficient for short words, but its worst case is exponential. For more details on its implementation, see [47].

For more solutions for the word problem for the braid groups, see [39].



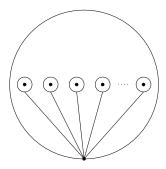


Fig. 1.13. A geometric base

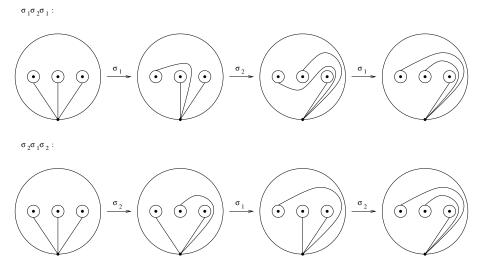


Fig. 1.14. An example of applications of two equal braids  $\sigma_1\sigma_2\sigma_1=\sigma_2\sigma_1\sigma_2$  on the initial geometric base

# 1.5. What is Public-Key Cryptography?

The idea of Public-Key Cryptography (PKC) was invented by Diffie and Hellman [38]. At the heart of this concept is the idea of using a one-way function for encryption (see the survey paper of Koblitz and Menezes [73]).

The functions used for encryption belong to a special class of *one-way* functions that remain one-way only if some information (the decryption key) is kept secret. If we use informal terminology, we can define a public-key encryption function as a map from plain text message units to cipher-

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text message units that can be feasibly computed by anyone having the public key, but whose inverse function (which deciphers the ciphertext message units) cannot be computed in a reasonable amount of time without some additional information, called the *private key*.

This means that everyone can send a message to a given person using the same enciphering key, which can simply be looked up in a public directory whose contents can be authenticated by some means. There is no need for the sender to have made any secret arrangement with the recipient; indeed, the recipient need never have had any prior contact with the sender at all.

Some of the purposes for which public-key cryptography has been applied are:

- Confidential message transmission: Two people want to exchange messages in the open airwaves, in such a way that an intruder observing the communication cannot understand the messages.
- **Key exchange** or **Key agreement**: Two people using the open airwaves want to agree upon a secret key for use in some symmetric-key cryptosystem. The agreement should be in such a way that an intruder observing the communication cannot deduce any useful information about their shared secret.
- Authentication: The prover wishes to convince the verifier that he knows the private key without enabling an intruder watching the communication to deduce anything about his private key.
- Signature: The target in this part is: The sender of the message has to send the receiver a (clear or ciphered) message together with a signature proving the origin of the message. Each signature scheme may lead to an authentication scheme: in order to authenticate the sender, the receiver can send a message to the sender, and require that the sender signs this message.

Now, we give some examples of the most famous and well-known publickey cryptosystems.

#### 1.5.1. Diffie-Hellman

In 1976, Diffie and Hellman [38] introduced a key-exchange protocol which is based on the apparent difficulty of computing logarithms over a finite field  $\mathbb{F}_q$  with q elements and on some commutative property of the exponent.

Their key-exchange protocol works as follows:

## Protocol 1.5.

Public keys: q and a primitive element  $\alpha$ . Private keys: Alice:  $X_i$ ; Bob:  $X_j$ .

Alice: Sends Bob  $Y_i = \alpha^{X_i} \pmod{q}$ . Bob: Sends Alice  $Y_i = \alpha^{X_j} \pmod{q}$ 

Shared secret key:  $K_{ij} = \alpha^{X_i X_j} \pmod{q}$ 

 $K_{ij}$  is indeed a shared key since Alice can compute  $K_{ij} = Y_j^{X_i} \pmod{q}$  and Bob can compute  $K_{ij} = Y_i^{X_j} \pmod{q}$ .

This method is secured due to the hardness of the Discrete Logarithm Problem.

### 1.5.2. RSA

Rivest, Shamir and Adleman [106] introduced one of the most famous and common cryptosystem, which is called RSA. This method is widely used in commerce.

Find two large prime numbers p and q, each about 100 decimal digits long. Let n=pq and  $\phi=\phi(n)=(p-1)(q-1)$  (the Euler number). Choose a random integer E between 3 and  $\phi$  that has no common factors with  $\phi$ . It is easy to find an integer D that is the "inverse" of E modulo  $\phi$ , that is,  $D \cdot E$  differs from 1 by a multiple of  $\phi$ .

Alice makes E and n public. All the other quantities here are kept secret.

The encryption is done as follows: Bob, who wants to send a plain text message P to Alice, that is an integer between 0 and n-1, computes the ciphertext integer  $C=P^E\pmod{n}$ . (In other words, raise P to the power E, divide the result by n, and C is the remainder). Then, Bob sends C to Alice.

For decrypting the message, Alice uses the secret decryption number D for finding the plain text P by computing:  $P = C^D \pmod{n}$ .

This method is currently secure, since in order to determine the secret decryption key D (for decrypting the message), the intruder should factor the 200 or so digits number n, which is a very hard task.

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# 1.6. First cryptosystems which are based on the braid groups

In this section, we describe first cryptosystems which are based on the braid groups. We start with the definition of some apparently hard problems which the cryptosystems are based on. After that, we describe first two key-exchange protocols which are based on the braid group. We finish the section with some more cryptosystems based on the braid group.

# 1.6.1. Underlying problems for cryptosystems in the braid group

We list here several apparently hard problems in the braid group, which are the base of many cryptosystems in the braid group:

• Conjugacy Decision Problem: Given  $u, w \in B_n$ , determine whether they are conjugate, i.e., there exists  $v \in B_n$  such that

$$w = v^{-1}uv$$

• Conjugacy Search Problem: Given conjugate elements  $u, w \in B_n$ , find  $v \in B_n$  such that

$$w = v^{-1}uv$$

• Multiple Simultaneous Conjugacy Search Problem: Given m pairs of conjugate elements  $(u_1, w_1), \ldots, (u_m, w_m) \in B_n$  which are all conjugated by the same element. Find  $v \in B_n$  such that

$$w_i = v^{-1}u_iv, \quad \forall i \in \{1, \dots, m\}$$

• Decomposition Problem:  $u \notin G \leq B_n$ . Find  $x, y \in G$  such that w = xuy.

## 1.6.2. Key-exchange protocols based on the braid group

In this section, we present two key-exchange protocols which are based on apparently hard problems in the braid group. After the transmitter and receiver agree on a shared secret key, they can use a symmetric cryptosystem for transmitting messages in the insecure channel.

### 1.6.2.1. Anshel-Anshel-Goldfeld key-exchange protocol

The following scheme was proposed theoretically by Anshel, Anshel and Goldfeld [5], and implemented in the braid group by Anshel, Anshel, Fisher and Goldfeld [4].

This scheme assumes that the Conjugacy Search Problem is difficult enough (so this scheme, as well as the other schemes described below, would keep its interest, even if it turned out that braid groups are not relevant, since it might be implemented in other groups).

Let G be a subgroup of  $B_n$ :

$$G = \langle g_1, \dots, g_m \rangle, \qquad g_i \in B_n$$

The secret keys of Alice and Bob are words  $a \in G$  and  $b \in G$  respectively.

The key-exchange protocol is as follows:

Protocol 1.6.

Public keys:  $\{g_1, \ldots, g_m\} \subset B_n$ . Private keys: Alice: a; Bob: b.

Alice: Sends Bob publicly the conjugates:  $ag_1a^{-1}, \ldots, ag_ma^{-1}$ . Bob: Sends Alice publicly the conjugates:  $bg_1b^{-1}, \ldots, bg_mb^{-1}$ .

Shared secret key:  $K = aba^{-1}b^{-1}$ 

K is indeed a shared key, since if  $a=x_1\cdots x_k$  where  $x_i=g_j^{\pm 1}$  for some j, then Alice can compute  $ba^{-1}b^{-1}=(bx_k^{-1}b^{-1})\cdots (bx_1^{-1}b^{-1})$  and hence Alice knows  $K=a(ba^{-1}b^{-1})$ . Similarly, Bob can compute  $aba^{-1}$ , and hence he knows  $K=(aba^{-1})b^{-1}$ .

The security is based on the difficulty of a variant to the Conjugacy Search Problem in  $B_n$ , namely the Multiple Conjugacy Search Problem, in which one tries to find a conjugating braid starting not from one single pair of conjugate braids  $(g, aga^{-1})$ , but from a finite family of such pairs  $(g_1, ag_1a^{-1}), \ldots, (g_m, ag_ma^{-1})$  obtained using the same conjugating braid. It should be noted that the Multiple Conjugacy Search Problem may be easier than the original Conjugacy Search Problem.

In [4], it is suggested to work in  $B_{80}$  with m = 20 and short initial braids  $g_i$  of length 5 or 10 Artin generators.

**Remark 1.2.** We simplified a bit the protocol given by Anshel-Anshel-Goldfeld, but the principle remains the same. Moreover, in their protocol,

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they used not the braids themselves, but their images under the colored Burau representation of the braid group defined by Morton [95] (see Section 1.8.4.1 below).

## 1.6.2.2. Diffie-Hellman-type key-exchange protocol

Following the commutative idea for achieving a shared secret key of Diffie-Hellman, Ko et al. [72] propose a key-exchange protocol based on the braid group and some commutative property of some of its elements. Although braid groups are not commutative, we can find large subgroups such that each element of the first subgroup commutes with each element of the second. Indeed, braids involving disjoint sets of strands commute. Similar approach appears also in the Algebraic Eraser Scheme (see [6] and Section 1.9.4 here).

Note that this scheme was proposed independently in [118] in the context of a general, unspecified noncommutative semigroup with difficult conjugacy problem, but the braid groups were not mentioned there explicitly.

Denote by  $LB_n$  (resp.  $UB_n$ ) the subgroup of  $B_n$  generated by  $\sigma_1, \ldots, \sigma_{m-1}$  (resp.  $\sigma_{m+1}, \ldots, \sigma_{m-1}$ ) with  $m = \lfloor \frac{n}{2} \rfloor$ . Then, every braid in  $LB_n$  commutes with every braid in  $UB_n$ .

Here is Ko et al. key-exchange protocol:

# Protocol 1.7.

Public key: one braid p in  $B_n$ .

Private keys: Alice:  $s \in LB_n$ ; Bob:  $r \in UB_n$ .

Alice: Sends Bob  $p' = sps^{-1}$ . Bob: Sends Alice  $p'' = rpr^{-1}$ 

Shared secret key:  $K = srpr^{-1}s^{-1}$ 

K is a shared key since Alice can compute  $K = sp''s^{-1}$  and Bob can compute K = rp'r-1, and both are equal to K since s and r commute.

The security is based on the difficulty of the Conjugacy Search Problem in  $B_n$ , or, more exactly, on the difficulty of the following variant, which can be called the Diffie-Hellman-like Conjugacy Problem:

**Problem 1.4.** Given a braid p in  $B_n$ , and the braids  $p' = sps^{-1}$  and  $p'' = rpr^{-1}$ , where  $s \in LB_n$  and  $r \in UB_n$ , find the braid  $rp'r^{-1}$ , which is also  $sp''s^{-1}$ .

The suggested parameters are n = 80, i.e. to work in  $B_{80}$ , with braids specified using (normal) sequences of length 12, i.e., sequences of 12 permutation braids (see [23]).

### 1.6.3. Encryption and decryption

The following scheme is proposed by Ko et al. [72]. We continue with the same notation of Ko et al. Assume that h is a public collision-free one-way hash function of  $B_n$  to  $\{0,1\}^{\mathbb{N}}$ , i.e., a computable function such that the probability of having  $h(b_2) = h(b_1)$  for  $b_2 \neq b_1$  is negligible (collision-free), and retrieving b from h(b) is infeasible (one-way) (for some examples see Dehornoy [29, Section 4.4] and Myasnikov [99]).

We start with  $p \in B_n$  and  $s \in LB_n$ . Alice's public key is the pair (p, p') with  $p' = sps^{-1}$ , where s is Alice's private key. For sending the message  $m_B$ , which we assume lies in  $\{0,1\}^{\mathbb{N}}$ , Bob chooses a random braid r in  $UB_n$  and he sends the encrypted text  $m''_B = m_B \oplus h(rp'r^{-1})$  (using  $\oplus$  for the Boolean operation "exclusive-or", i.e. the sum in  $\mathbb{Z}/2\mathbb{Z}$ ), together with the additional datum  $p'' = rpr^{-1}$ . Now, Alice computes  $m_A = m'' \oplus h(sp''s^{-1})$ , and we have  $m_A = m_B$ , which means that Alice retrieves Bob's original message.

Indeed, because the braids r and s commute, we have (as before):

$$sp''s^{-1} = srpr^{-1}s^{-1} = rsps^{-1}r^{-1} = rp'r^{-1}$$
.

and, therefore,  $m_A = m_B \oplus h(rp'r^{-1}) \oplus h(rp'r^{-1}) = m_B$ .

The security is based on the difficulty of the Diffie-Hellmann-like Conjugacy Problem in  $B_n$ . The recommended parameters are as in Ko et al's exchange-key protocol (see Section 1.6.2.2).

### 1.6.4. Authentication schemes

Three authentication schemes were introduced by Sibert, Dehornoy and Girault [117], which are based on the Conjugacy Search problem and Root Extraction Problem. Concerning the cryptanalysis of the Root Extraction Problem, see [63].

We present here their first scheme. This scheme is related to Diffie-Hellman based exchange-key in its idea of verifying that the secret key computed at the two ends is the same.

Note that any encryption scheme can be transformed into an authentication scheme, by sending to Alice both an encrypted version and a hashed

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image of the same message m, then requesting her to reply with the deciphered message m (she will do it only if the hashed image of the deciphered message is the same as the one sent by Bob).

Their first scheme is based on the difficulty of Diffie-Hellman-like Conjugacy Problem. It uses the fact that braids involving disjoint families of strands commute. The data consist of a public key, which is a pair of braids, and of Alice's private key, also a braid. We assume that n is even, and denote by  $LB_n$  (resp.  $UB_n$ ) the subgroup of  $B_n$  generated by  $\sigma_1, \ldots, \sigma_{\frac{n}{2}-1}$ , i.e., braids where the  $\frac{n}{2}$  lower strands only are braided (resp. in the subgroup generated by  $\sigma_{\frac{n}{2}+1}, \ldots, \sigma_{n-1}$ ). The point is that every element in  $LB_n$  commutes with every element in  $UB_n$ , and alternative subgroups with this property could be used instead. We assume that H is a fixed collision-free hash function from braids to sequences of 0's and 1's or, possibly, to braids.

## • Phase 1. Key generation:

- (1) Choose a public braid b in  $B_n$  such that the Diffie-Hellman-like Conjugacy Problem for b is hard enough;
- (2) Alice chooses a secret braid s in  $LB_n$ , her private key; she publishes  $b' = sbs^{-1}$ ; the pair (b, b') is her public key.

# • Phase 2. Authentication phase:

- (1) Bob chooses a braid r in  $UB_n$ , and sends the challenge  $x = rbr^{-1}$  to Alice;
- (2) Alice sends the response  $y = H(sxs^{-1})$  to Bob, and Bob checks  $y = H(rb'r^{-1})$ .

For active attacks, the security is ensured by the hash function H: if H is one-way, these attacks are ineffective.

Two more authentication schemes were suggested by Lal and Chaturvedi [76]. Their cryptanalysis are discussed in [63; 122].

# 1.7. Attacks on the conjugacy search problem using Summit Sets

In this section, we explain the algorithms for solving the Conjugacy Decision Problem and the Conjugacy Search Problem (CDP/CSP) in braid groups which are based on Summit sets. These algorithms are given in [51; 43; 44;

46; 53; 55]. We start with the basic idea, and then we continue with its implementations.

We follow here the excellent presentation of Birman, Gebhardt and González-Meneses [14]. For more details, see their paper.

### 1.7.1. The basic idea

Given an element  $x \in B_n$ , the algorithm computes a finite subset  $I_x$  of the conjugacy class of x which has the following properties:

- (1) For every  $x \in B_n$ , the set  $I_x$  is finite, non-empty and only depends on the conjugacy class of x. It means that two elements  $x, y \in B_n$  are conjugate if and only if  $I_x = I_y$ .
- (2) For each  $x \in B_n$ , one can compute efficiently a representative  $\tilde{x} \in I_x$  and an element  $a \in B_n$  such that  $a^{-1}xa = \tilde{x}$ .
- (3) There is a finite algorithm which can construct the whole set  $I_x$  from any representative  $\tilde{x} \in I_x$ .

Now, for solving the CDP/CSP for given  $x, y \in B_n$  we have to perform the following steps.

- (a) Find representatives  $\tilde{x} \in I_x$  and  $\tilde{y} \in I_y$ .
- (b) Using the algorithm from property (3), compute further elements of  $I_x$  (while keeping track of the conjugating elements), until either:
  - (i)  $\tilde{y}$  is found as an element of  $I_x$ , proving x and y to be conjugate and providing a conjugating element, or
  - (ii) the entire set  $I_x$  has been constructed without encountering  $\tilde{y}$ , proving that x and y are not conjugate.

We now survey the different algorithms based on this approach.

In Garside's original algorithm [51], the set  $I_x$  is the Summit Set of x, denoted SS(x), which is the set of conjugates of x having maximal infimum.

Remark 1.3. All the algorithms presented below for the different types of Summit Sets work also for Garside groups (defined by Dehornoy and Paris [36]), which are a generalization of the braid groups. In our survey, for simplification, we present them in the language of braid groups. For more details on the Garside groups and the generalized algorithms, see [14].

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## 1.7.2. The Super Summit Sets

The Summit Set are improved by El-Rifai and Morton [43], who consider  $I_x = SSS(x)$ , the Super Summit Set of x, consisting of the conjugates of x having minimal canonical length  $\ell(x)$ . They also show that SSS(x) is the set of conjugates of x having maximal infimum and minimal supremum, at the same time. El-Rifai and Morton [43] show that SSS(x) is finite. In general, SSS(x) is much smaller than SS(x). For example, take the element  $x = \Delta_4 \sigma_1 \sigma_1 \in B_4$ ,  $SSS(x) = \{\Delta_4 \cdot \sigma_1 \sigma_3\}$  while

$$SS(x) = \{ \Delta_4 \cdot \sigma_1 \sigma_3, \Delta_4 \cdot \sigma_1 \cdot \sigma_1, \Delta_4 \cdot \sigma_3 \cdot \sigma_3 \}$$

(the factors in each left normal form are separated by a dot) [14, page 8]. Starting by a given element x, one can find an element  $\tilde{x} \in SSS(x)$  by a sequence of special conjugations, called *cyclings* and *decyclings*:

**Definition 1.8.** Let  $x = \Delta^p x_1 \cdots x_r \in B_n$  be given in Garside's normal form and assume r > 0.

The cycling of x, denoted by  $\mathbf{c}(x)$  is:

$$\mathbf{c}(x) = \Delta^p x_2 \cdots x_r \tau^{-p}(x_1),$$

where  $\tau$  is the involution which maps  $\sigma_i$  to  $\sigma_{n-i}$ , for all  $1 \leq i \leq n$ .

The decycling of x, denoted by  $\mathbf{d}(x)$  is:

$$\mathbf{d}(x) = x_r \Delta^p x_1 x_2 \cdots x_{r-1} = \Delta^p \tau^p(x_r) x_1 x_2 \cdots x_{r-1}.$$

If r = 0, we have  $\mathbf{c}(x) = \mathbf{d}(x) = x$ .

Note that  $\mathbf{c}(x) = (\tau^{-p}(x_1))^{-1}x(\tau^{-p}(x_1))$  and  $\mathbf{d}(x) = x_r^{-1}xx_r$ . This means that for an element of positive canonical length, the cycling of x is computed by moving the first permutation braid of x to the end, while the decycling of x is computed by moving the last permutation braid of x to the front. Moreover, for every  $x \in B_n$ ,  $\inf(x) \leq \inf(\mathbf{c}(x))$  and  $\sup(x) \geq \sup(\mathbf{d}(x))$ .

Note that the above decompositions of  $\mathbf{c}(x)$  and  $\mathbf{d}(x)$  are not, in general, Garside's normal forms. Hence, if one wants to perform iterated cyclings or decyclings, one needs to compute the left normal form of the resulting element at each iteration.

Given x, one can use cyclings and decyclings to find an element in SSS(x) in the following way: Suppose that we have an element  $x \in B_n$  such that  $\inf(x)$  is not equal to the maximal infimum in the conjugacy class of x. Then, we can increase the infimum by repeated cycling (due to

[18; 43]): there exists a positive integer  $k_1$  such that  $\inf(\mathbf{c}^{k_1}(x)) > \inf(x)$ . Therefore, by repeated cycling, we can conjugate x to another element  $\hat{x}$  of maximal infimum. Once  $\hat{x}$  is obtained, if the supremum is not minimal in the conjugacy class, we can decrease its supremum by repeated decycling. Again, due to [18; 43], there exists an integer  $k_2$  such that  $\sup(\mathbf{d}^{k_2}(\hat{x})) < \sup(\hat{x})$ . Hence, using repeated cycling and decycling a finite number of times, one obtains an element in SSS(x).

If we denote by m the length of  $\Delta$  in Artin generators and r is the canonical length of x, then we have (see [18; 43]):

**Prop 1.2.** A sequence of at most rm cyclings and decyclings applied to x produces a representative  $\tilde{x} \in SSS(x)$ .

Now, we have to explore all the set SSS(x). We have the following result (see [43]):

**Prop 1.3.** Let  $x \in B_n$  and  $V \subset SSS(x)$  be non-empty. If  $V \neq SSS(x)$ , then there exist  $y \in V$  and a permutation braid s such that  $s^{-1}ys \in SSS(x) \setminus V$ .

Since SSS(x) is a finite set, the above proposition allows us to compute the whole SSS(x). More precisely, if one knows a subset  $V \subset SSS(x)$  (we start with:  $V = \{\tilde{x}\}$ ), one conjugates each element in V by all permutation braids (n! elements). If one encounters a new element z with the same canonical length as  $\tilde{x}$  (which is a new element in SSS(x)), then add z to V and start again. If no new element is found, this means that V = SSS(x), and we are done.

One important remark is that this algorithm not only computes the set SSS(x), but it also provides conjugating elements joining the elements in SSS(x).

Now the checking if x and y are conjugate, is done as follows: Compute representatives  $\tilde{x} \in \mathrm{SSS}(x)$  and  $\tilde{y} \in \mathrm{SSS}(y)$ . If  $\inf(\tilde{x}) \neq \inf(\tilde{y})$  or  $\sup(\tilde{x}) \neq \sup(\tilde{y})$ , then x and y are not conjugate. Otherwise, start computing  $\mathrm{SSS}(x)$  as described above. The elements x and y are conjugate if and only if  $\tilde{y} \in \mathrm{SSS}(x)$ . Note that if x and y are conjugate, an element conjugating x to y can be found by keeping track of the conjugations during the computations of  $\tilde{x}$ ,  $\tilde{y}$  and  $\mathrm{SSS}(x)$ . Hence, it solves the Conjugacy Decision Problem and the Conjugacy Search Problem simultaneously.

From the algorithm, we see that the computational cost of computing SSS(x) depends mainly in two ingredients: the size of SSS(x) and the number of permutation braids. In  $B_n$ , all known upper bounds for the size of

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SSS(x) are exponential in n, although it is conjectured that for fixed n, a polynomial bound in the canonical length of x exists [44].

Franco and González-Meneses [46] reduce the size of the set we have to conjugate with, by the following observation:

**Prop 1.4.** Let  $x \in B_n$  and  $y \in SSS(x)$ . For every positive braid u there is a unique  $\leq$ -minimal element  $c_y(u)$  satisfying  $u \leq c_y(u)$  and  $(c_y(u))^{-1}y(c_y(u)) \in SSS(x)$ .

**Definition 1.9.** Given  $x \in B_n$  and  $y \in SSS(x)$ , we say that a permutation braid  $s \neq 1$  is *minimal* for y with respect to SSS(x) if  $s^{-1}ys \in SSS(x)$ , and no proper prefix of s satisfies this property.

It is easy to see that the number of minimal permutation braids for y is bounded by the number of Artin's generators.

Now, we have:

**Prop 1.5.** Let  $x \in B_n$  and  $V \subseteq SSS(x)$  be non-empty. If  $V \neq SSS(x)$ , then there exist  $y \in V$  and a generator  $\sigma_i$  such that  $c_y(\sigma_i)$  is a minimal permutation braid for y, and  $(c_y(\sigma_i))^{-1}y(c_y(\sigma_i)) \in SSS(X) \setminus V$ .

Using these proposition, the SSS(x) can be computed as in [43], but instead of conjugating each element  $y \in SSS(x)$  by all permutation braids (n! elements), it suffices to conjugate y by the minimal permutation braids  $c_y(\sigma_i)$   $(1 \le i \le n-1, n-1 \text{ elements})$ .

Figure 1.15 (taken from [29]) summarizes the solution of the conjugacy problem using the Super Summit Set for an element b.

Note that the algorithm computes a directed graph whose vertices are the elements in SSS(x), and whose arrows are defined as follows: for any two elements  $y, z \in SSS(x)$ , there is an arrow labeled by the minimal permutation braid  $p_i$  starting at y and ending at z if  $p_i^{-1}yp_i = z$ .

An example for such a graph can be seen in Figure 1.16, for the set  $SSS(\sigma_1)$  in  $B_4$  (taken from [14, pp. 10–11]). Note that there are exactly 3 arrows starting at every vertex (the number of Artin generators of  $B_4$ ). In general, the number of arrows starting at a given vertex can be smaller or equal, but never larger than the number of generators.

Hence, the size of the set of permutation braids is no longer a problem for the complexity of the algorithm (since we can use the minimal permutation braids instead), but there is still a big problem to handle: The size of SSS(x) is, in general, very big. The next improvement tries to deal with this.

### Braid Group Cryptography

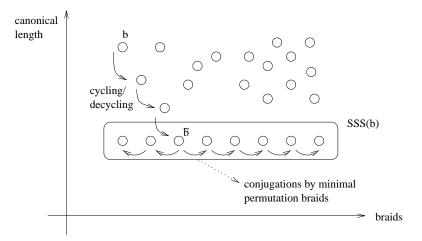


Fig. 1.15. Solving the conjugacy problem: going to the SSS and then exploring it (the points represent the conjugates of b)

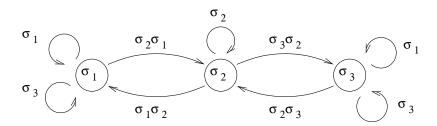


Fig. 1.16. The graph of  $SSS(\sigma_1)$  in  $B_4$ 

# 1.7.3. The Ultra Summit Sets

Gebhardt [53] defines a small subset of SSS(x) satisfying all the good properties described above, so that a similar algorithm can be used to compute it. The definition of this new subset appears after observing that the cycling function maps SSS(x) to itself. As SSS(x) is finite, iterated cycling of any representative of SSS(x) must eventually become periodic. Hence it is natural to define the following:

**Definition 1.10.** Given  $x \in B_n$ , the *Ultra Summit Set* of x, USS(x), is the set of elements  $y \in SSS(x)$  such that  $\mathbf{c}^m(y) = y$  for some m > 0.

Hence, the Ultra Summit Set USS(x) consists of a finite set of disjoint orbits, closed under cycling (see some schematic example in Figure 1.17).

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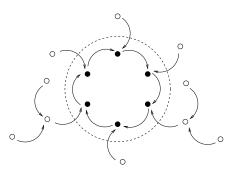


Fig. 1.17. Action of cycling inside the Super Summit Set; the elements of the Ultra Summit Set are in black and perform some orbits under cycling (taken from [29, Figure 4])

# **Example 1.3.** [14] One has

$$USS(\sigma_1) = SSS(\sigma_1) = SS(\sigma_1) = \{\sigma_1, \dots, \sigma_{n-1}\},\$$

and each element corresponds to an orbit under cycling, since  $\mathbf{c}(\sigma_i) = \sigma_i$  for i = 1, ..., n - 1.

A more interesting example is given by the element

$$x = \sigma_1 \sigma_3 \sigma_2 \sigma_1 \cdot \sigma_1 \sigma_2 \cdot \sigma_2 \sigma_1 \sigma_3 \in B_4.$$

In this example, USS(x) has 6 elements, while SSS(x) has 22 elements. More precisely, USS(x) consists of 2 closed orbits under cycling:  $USS(x) = O_1 \cup O_2$ , each one containing 3 elements:

$$O_1 = \left\{ \begin{array}{l} \sigma_1 \sigma_3 \sigma_2 \sigma_1 \cdot \sigma_1 \sigma_2 \cdot \sigma_2 \sigma_1 \sigma_3, \\ \sigma_1 \sigma_2 \cdot \sigma_2 \sigma_1 \sigma_3 \cdot \sigma_1 \sigma_3 \sigma_2 \sigma_1, \\ \sigma_2 \sigma_1 \sigma_3 \cdot \sigma_1 \sigma_3 \sigma_2 \sigma_1 \cdot \sigma_1 \sigma_2 \end{array} \right\},$$

$$O_2 = \left\{ \begin{array}{l} \sigma_3 \sigma_1 \sigma_2 \sigma_3 \cdot \sigma_3 \sigma_2 \cdot \sigma_2 \sigma_3 \sigma_1, \\ \sigma_3 \sigma_2 \cdot \sigma_2 \sigma_3 \sigma_1 \cdot \sigma_3 \sigma_1 \sigma_2 \sigma_3, \\ \sigma_2 \sigma_3 \sigma_1 \cdot \sigma_3 \sigma_1 \sigma_2 \sigma_3 \cdot \sigma_3 \sigma_2 \end{array} \right\}.$$

Notice that  $O_2 = \tau(O_1)$ .

Note also that the cycling of every element in USS(x) gives another element which is already in left normal form, hence iterated cyclings corresponds to cyclic permutations of the factors in the left normal form. Elements which satisfies this property are called rigid (see [14]).

**Remark 1.4.** The size of the Ultra Summit Set of a *generic* braid of canonical length  $\ell$  is either  $\ell$  or  $2\ell$  [53]. This means that, in the generic case,

Ultra Summit Sets consist of one or two orbits (depending on whether  $\tau(O_1) = O_1$  or not), containing rigid braids. But, there are exceptions: for example, the following braid in  $B_{12}$ :

$$E = (\sigma_2 \sigma_1 \sigma_7 \sigma_6 \sigma_5 \sigma_4 \sigma_3 \sigma_8 \sigma_7 \sigma_{11} \sigma_{10}) \cdot (\sigma_1 \sigma_2 \sigma_3 \sigma_2 \sigma_1 \sigma_4 \sigma_3 \sigma_{10}) \cdot (\sigma_1 \sigma_3 \sigma_4 \sigma_{10}) \cdot (\sigma_1 \sigma_{10}) \cdot (\sigma_1 \sigma_{10} \sigma_9 \sigma_8 \sigma_7 \sigma_{11}) \cdot (\sigma_1 \sigma_2 \sigma_7 \sigma_{11})$$

has an Ultra Summit Set of size 264, instead of the expected size 12 (see [15, Example 5.1]).

In the case of braid groups, the size and structure of the Ultra Summit Sets happen to depend very much on the geometrical properties of the braid, more precisely, on its Nielsen-Thurston type: periodic, reducible or Pseudo-Anosov (see [14; 15]).

The algorithm given in [53] to solve the CDP/CSP in braid groups (using Ultra Summit Sets) is analogous to the previous ones, but this time one needs to compute USS(x) instead of SSS(x). In order to do this, we first have to obtain an element  $\hat{x} \in \text{USS}(x)$ . We do this as follows: take an element  $\tilde{x} \in \text{SSS}(x)$ . Now, start cycling it. Due to the facts that cycling an element in SSS(x) will result in another element in SSS(x) and that the Super Summit Set of x is finite, we will have two integers  $m_1, m_2$  ( $m_1 < m_2$ ), which satisfy:

$$\mathbf{c}^{m_1}(\tilde{x}) = \mathbf{c}^{m_2}(\tilde{x})$$

When having this, the element  $\hat{x} = \mathbf{c}^{m_1}(\tilde{x})$  is in USS(x), since:

$$\mathbf{c}^{m_2-m_1}(\hat{x}) = \hat{x}.$$

After finding a representative  $\hat{x} \in \text{USS}(x)$ , we have to explore all the set USS(x). This we do using the following results of Gebhardt [53] (which are similar to the case of the Super Summit Set):

**Prop 1.6.** Let  $x \in B_n$  and  $y \in \mathrm{USS}(x)$ . For every positive braid u there is a unique  $\leq$ -minimal element  $c_y(u)$  satisfying  $u \leq c_y(u)$  and  $(c_y(u))^{-1}y(c_y(u)) \in \mathrm{USS}(x)$ .

**Definition 1.11.** Given  $x \in B_n$  and  $y \in USS(x)$ , we say that a permutation braid  $s \neq 1$  is a *minimal* for y with respect to USS(x) if  $s^{-1}ys \in USS(x)$ , and no proper prefix of s satisfies this property.

It is easy to see that the number of minimal permutation braids for y is bounded by the number of Artin's generators.

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Now, we have:

**Prop 1.7.** Let  $x \in B_n$  and  $V \subseteq \text{USS}(x)$  be non-empty. If  $V \neq \text{USS}(x)$ , then there exist  $y \in V$  and a generator  $\sigma_i$  such that  $c_y(\sigma_i)$  is a minimal permutation braid for y, and  $(c_y(\sigma_i))^{-1}y(c_y(\sigma_i)) \in \text{USS}(X) \setminus V$ .

In [53], it is shown how to compute the minimal permutation braids (they are called there *minimal simple elements* in the Garside group's language) corresponding to a given  $y \in \text{USS}(x)$  (a further discussion on the minimal simple elements with some examples can be found in [15]). Hence, one can compute the whole USS(x) starting by a single element  $\hat{x} \in \text{USS}(x)$ , and then we are done.

For a better characterization of the minimal permutation braids, let us introduce some notions related to a braid given in a left normal form (see [15]):

**Definition 1.12.** Given  $x \in B_n$  whose left normal form is  $x = \Delta^p x_1 \cdots x_r$  (r > 0), we define the *initial factor of* x as  $\iota(x) = \tau^{-p}(x_1)$ , and the *final factor of* x as  $\varphi(x) = x_r$ . If r = 0 we define  $\iota(\Delta^p) = 1$  and  $\varphi(\Delta^p) = \Delta$ .

**Definition 1.13.** Let u, v be permutation braids such that  $uv = \Delta$ . The right complement of  $u, \partial(u)$ , is defined by  $\partial(u) = u^{-1}\Delta = v$ .

Note that a cycling of x is actually a conjugation of x by the initial factor  $\iota(x)$ :  $\mathbf{c}(x) = \iota(x)^{-1}x\iota(x)$ , and a decycling of x is actually a conjugation of x by the inverse of final factor  $\varphi(x)^{-1}$ :  $\mathbf{d}(x) = \varphi(x)x\varphi(x)^{-1}$ .

The notions of Definition 1.12 are closely related (see [14]):

**Lemma 1.1.** For every  $x \in B_n$ , one has  $\iota(x^{-1}) = \partial(\varphi(x))$  and  $\varphi(x^{-1}) = \partial^{-1}(\iota(x))$ .

The following proposition from [15] characterizes the minimal permutation braids for x as prefixes of x or of  $x^{-1}$ :

**Prop 1.8.** Let  $x \in \mathrm{USS}(x)$  with  $\ell(x) > 0$  and let  $c_x(\sigma_i)$  be a minimal permutation braid for x. Then  $c_x(\sigma_i)$  is a prefix of either  $\iota(x)$  or  $\iota(x^{-1})$ , or both.

As in the case of the Super Summit Set, the algorithm of Gebhardt [53] not only computes USS(x), but also a graph  $\Gamma_x$ , which determines the

conjugating elements. This graph is defined as follows.

**Definition 1.14.** Given  $x \in B_n$ , the directed graph  $\Gamma_x$  is defined by the following data:

- (1) The set of vertices is USS(x).
- (2) For every  $y \in \text{USS}(x)$  and every minimal permutation braid s for y with respect to USS(x), there is an arrow labeled by s going from y to  $s^{-1}ys$ .

**Example 1.4.** Let us give some example for the graph  $\Gamma_x$ . We follow [15, Example 2.10].

Let  $x = \sigma_1 \sigma_2 \sigma_3 \sigma_2 \cdot \sigma_2 \sigma_1 \sigma_3 \cdot \sigma_1 \sigma_3 \in B_4$ . This braid A is Pseudo-Anosov and rigid. A computation shows that USS(x) has exactly two cycling orbits, with 3 elements each, namely:

$$x_1 = \left\{ \begin{array}{l} x_{1,1} = \sigma_1 \sigma_2 \sigma_3 \sigma_2 \cdot \sigma_2 \sigma_1 \sigma_3 \cdot \sigma_1 \sigma_3, \\ x_{1,2} = \sigma_2 \sigma_1 \sigma_3 \cdot \sigma_1 \sigma_3 \cdot \sigma_1 \sigma_2 \sigma_3 \sigma_2, \\ x_{1,3} = \sigma_1 \sigma_3 \cdot \sigma_1 \sigma_2 \sigma_3 \sigma_2 \cdot \sigma_2 \sigma_1 \sigma_3 \end{array} \right\},$$

$$x_2 = \left\{ \begin{array}{l} x_{2,1} = \sigma_1 \sigma_3 \sigma_2 \sigma_1 \cdot \sigma_2 \sigma_1 \sigma_3 \cdot \sigma_1 \sigma_3, \\ x_{2,2} = \sigma_2 \sigma_1 \sigma_3 \cdot \sigma_1 \sigma_3 \cdot \sigma_1 \sigma_3 \sigma_2 \sigma_1, \\ x_{2,3} = \sigma_1 \sigma_3 \cdot \sigma_1 \sigma_3 \sigma_2 \sigma_1 \cdot \sigma_2 \sigma_1 \sigma_3 \end{array} \right\}.$$

The graph  $\Gamma_x$  of  $\mathrm{USS}(x)$  is illustrated in Figure 1.18. The solid arrows are conjugations by minimal permutation braids which are prefixes of the initial factors, while the dashed arrows are conjugations by minimal permutation braids which are prefixes of the final factors. Note that the definitions imply that the cycles  $x_1$  and  $x_2$  of  $\mathrm{USS}(x)$  are connected by solid arrows.

Concerning the complexity of this algorithm for solving the Conjugacy Search Problem, the number  $m_2$  of times one needs to apply cycling for finding an element in USS(x) is not known in general. Nevertheless, in practice, the algorithm based on the Ultra Summit Set is substantially better for braid groups (see [14]). For more information on the Ultra Summit Set and its structure, see [15].

**Remark 1.5.** One might think that for a given element  $x \in B_n$ , it is possible that its Ultra Summit Set with respect to the Garside normal form will be different from its Ultra Summit Set with respect to the *right* 

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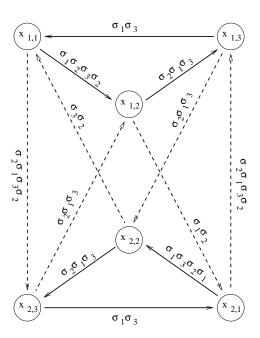


Fig. 1.18. The graph of  $USS(\sigma_1\sigma_2\sigma_3\sigma_2\sigma_2\sigma_1\sigma_3\sigma_1\sigma_3) \subset B_4$ 

normal form (see Section 1.3.1). If this happens, it is possible that even though one of the Ultra Summit Sets is large, the other will be small.

Gebhardt and González-Meneses [54] show that at least for *rigid braids*, the size of the above two Ultra Summit Sets is equal, and their associated graphs are isomorphic (a braid w is called *rigid*, if the cycling of w,  $\mathbf{c}(w)$ , is already given in Garside normal form, with no need for changing the permutation braids; see also [14, Section 3] and Example 1.3 here). They conjecture that this is the situation for any braid.

# 1.7.4. Some variants of the Ultra Summit Sets

In this section, we sketch some variants of the Super Summit Sets and the Ultra Summit Sets suggested by several authors.

# 1.7.4.1. Reduced Super Summit Sets

Lee, in his thesis [81] (2000), suggests a variant of the Super Summit Set, which is actually a subset of the Ultra Summit Set which was defined later

(2005) by Gebhardt:

**Definition 1.15.** The *Reduced Super Summit Set of x*, denote by RSSS(x), is:

$$RSSS(x) = \{ y \in C(x) | \mathbf{c}^m(y) = y \text{ and } \mathbf{d}^n(y) = y \text{ for some } m, n \ge 1 \}.$$

where C(x) is the conjugacy class of x

Lee's motivation to look on RSSS(x) comes from the facts that it is still easy to find algorithmically an element in RSSS(x) for a given x, this set is invariant under cyclings and decyclings, and this set is usually smaller than SSS(x).

Indeed, it is easy to see (by [43] and [55]) that:

$$RSSS(x) \subseteq USS(x) \subseteq SSS(x)$$

Lee indicates that there is no known algorithm to generate RSSS(x) without generating SSS(x) before. Despite this, he has succeeded to compute RSSS(x) in polynomial time for the case of rigid braids in  $B_4$ .

1.7.4.2. A general cycling operation and its induced set

Zheng [126] suggests to generalize the idea of cyclings. He defines:

**Definition 1.16.** The cycling operation of order q on x is the conjugation  $\mathbf{c}_q(x) = s^{-1}xs$ , where s is the maximal common prefix of x and  $\Delta^q$ . (this will be denoted in the next section as:  $s = x \wedge \Delta^q$ ).

The corresponding set is:

$$G_q = \{ x \in B_n \mid \mathbf{c}_q^N(x) = x \text{ for some } N > 0 \}.$$

The new cycling operations are indeed natural generalizations of the cycling and decycling operation:

$$\mathbf{c}(x) = \tau^{-\inf(x)} \left( \mathbf{c}_{\inf(x)+1}(x) \right), \qquad \mathbf{d}(x) = \mathbf{c}_{\sup(x)-1}(x).$$

Recall that C(x) is the conjugacy class of x. For getting the Super Summit Sets and the Ultra Summit Sets in the language of  $G_q$ , we define:

$$\inf_s(x) = \max\{\inf(y) \mid y \in C(x)\}, \qquad \sup_s(x) = \min\{\sup(y) \mid y \in C(x)\}.$$

Hence, we get that:

$$SSS(x) = C(x) \cap \left(\bigcap_{q \in \{\inf_s(x), \sup_s(x)\}} G_q\right),\,$$

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$$USS(x) = C(x) \cap \left( \bigcap_{q \in \{\inf_{s}(x), \inf_{s}(x) + 1, \sup_{s}(x)\}} G_q \right).$$

Zheng [126] defines a new summit set:

$$C^*(x) = C(x) \cap \left(\bigcap_{q \in \mathbb{Z}} G_q\right) = C(x) \cap \left(\bigcap_{\inf_s(x) \le q \le \sup_s(x)} G_q\right).$$

It is straight-forward that:

$$C^*(x) \subseteq \mathrm{USS}(x) \subseteq \mathrm{SSS}(x)$$
.

Given an element x, computing an element  $\hat{x} \in C^*(x)$  is done by applying iterated general cyclings  $\mathbf{c}_q$  until getting repetitions, for  $\inf(x) < q < \sup(x)$ . A more complicated algorithm is presented for finding the whole  $C^*(x)$  (see [126, Algorithm 3.8]). Having these ingredients for  $C^*(x)$ , we can solve the Conjugacy Search Problem based on  $C^*(x)$ .

Zheng [126, Section 6] presents some computational results, and he emphasizes that the new set  $C^*(x)$  is important especially for the case of reducible braids, where there are cases that USS(x) = SSS(x).

#### 1.7.4.3. Stable Super Summit Sets and Stable Ultra Summit Sets

The stable Super Summit Sets and stable Ultra Summit Sets were defined simultaneously by Birman, Gebhardt and González-Meneses [14] and Lee and Lee [78]:

**Definition 1.17.** Given  $x \in B_n$ , The stable Super Summit Set of x is defined as:

$$SSSS(x) = \{ y \in USS(x) \mid y^m \in USS(x^m), \forall m \in \mathbb{Z} \}.$$

The stable Ultra Summit Set of x is defined as:

$$SU(x) = \{ y \in USS(x) \mid y^m \in USS(x^m), \forall m \in \mathbb{Z} \}.$$

Birman, Gebhardt and González-Meneses [14, Proposition 2.23] and Lee and Lee [78, Theorem 6.1(i)] have proved that for every  $x \in B_n$  the stable sets SSSS(x) and SU(x) are non-empty.

We give here an example from [78], which shows that: (i) the stable Super Summit Set is different from both the Super Summit Set and the Ultra Summit Set; (ii) one cannot obtain an element of the stable Super Summit Set by applying only cyclings and decyclings.

**Example 1.5.** [78, page 11] Consider the positive 4-braid monoid  $B_4^+$ . Let

$$g_1 = \sigma_1 \sigma_2 \sigma_3$$
,  $g_2 = \sigma_3 \sigma_2 \sigma_1$ ,  $g_3 = \sigma_1 \sigma_3 \sigma_2$ ,  $g_4 = \sigma_2 \sigma_1 \sigma_3$ .

Note that  $g_i$ 's are permutation braids and conjugate to each other.

It is easy to see that

$$SSS(g_1) = USS(g_1) = \{g_1, g_2, g_3, g_4\}.$$

Now, we show that the stable Super Summit Set of  $g_1$  is different from the Super/Ultra Summit Set of  $g_1$ . The normal forms of  $g_i^2$  are as follows:

$$g_1^2 = (\sigma_1 \sigma_2 \sigma_3 \sigma_1 \sigma_2) \sigma_3; \quad g_2^2 = (\sigma_3 \sigma_2 \sigma_1 \sigma_3 \sigma_2) \sigma_1; \quad g_3^2 = \Delta; \quad g_4^2 = \Delta.$$

Therefore,  $\inf(g_1^2) = \inf(g_2^2) = 0$  and  $\inf(g_3^2) = \inf(g_4^2) = 1$ . Hence,

$$SSSS(g_1) = \{g_3, g_4\}.$$

Note that  $\mathbf{c}^k(g_i) = \mathbf{d}^k(g_i) = g_i$  for  $i = 1, \dots, 4$  and all k > 1. In particular, we cannot obtain an element of the stable Super Summit Set by applying only cyclings and decyclings to  $g_1$  or  $g_2$ .

A finite-time algorithm for computing the stable Super Summit Sets (i.e. when given  $x \in B_n$ , first compute an element  $\hat{x} \in SSS(x)$  and then compute the whole set SSSS(x)) is given by Lee and Lee in [80, Section 6].

Birman, Gebhardt and González-Meneses [14, page 27] remark that their proof for the non-emptiness of the stable Ultra Summit Set (Proposition 2.23 there) actually yields an algorithm for computing this set.

Zheng [126], as a continuation of his idea of general cyclings, suggests to generalize also the stable sets. He defines:

**Definition 1.18.**  $\mathbf{c}_{p,q}(x) = s^{-1}xs$ , where s is the maximal common prefix of  $x^p$  and  $\Delta^q$  (i.e.,  $s = x^p \wedge \Delta^q$ ).

The corresponding set is:

$$G_{p,q} = \{ x \in B_n \mid \mathbf{c}_{p,q}^N(x) = x \text{ for some } N > 0 \}.$$

Note that  $\mathbf{c}_q(x^p) = (\mathbf{c}_{p,q}(x))^p$ , so applying a  $\mathbf{c}_q$  operation on  $x^p$  is equivalent to applying a  $\mathbf{c}_{p,q}$  operation on x. In particular,  $x^p \in G_q$  if and only if  $x \in G_{p,q}$ .

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Similarly, one can define:

$$C^{[m,n],*}(x) = C(x) \cap \left(\bigcap_{m \le p \le n, q \in \mathbb{Z}} G_{p,q}\right).$$

Zheng claims, that with a suitable modification, the algorithms for computing  $C^*(x)$  can be used to compute the set  $C^{[m,n],*}(x)$ .

An even more generalized set is:

$$C^{*,*}(x) = C(x) \cap \left(\bigcap_{p,q \in \mathbb{Z}} G_{p,q}\right),$$

but currently there is no algorithm for computing it, because he does not know how to bound the order p. Nevertheless, Zheng [126, Theorem 7.3] have proved that the set  $C^{*,*}(x)$  is nonempty.

The set  $C^{*,*}(x)$  is indeed a generalization of the stable sets, since:

$$SSSS(x) = C(x) \cap \left( \bigcap_{p \ge 1, q \in \{\inf_{s}(x^{p}), \sup_{s}(x^{p})\}} G_{p,q} \right),$$

$$SU(x) = C(x) \cap \left( \bigcap_{p \ge 1, q \in \{\inf_s(x^p), \inf_s(x^p) + 1, \sup_s(x^p)\}} G_{p,q} \right).$$

By the non-emptiness result of Zheng, we have an alternative proof that the stable sets are nonempty.

### 1.7.5. Cyclic sliding

The last step up-to-date for seeking a polynomial-time solution to the conjugacy search problem has been done by Gebhardt and González-Meneses [55; 56].

Their idea is introducing a new operation, called *cyclic sliding*, and they suggest to replace the usual cycling and decycling operations by this new one, as it is more natural from both the theoretical and computational points of view. Then, the Ultra Summit Set USS(x) of x, will be replaced by its analogue for cyclic sliding: the set of *sliding circuits*, SC(x). The sets of sliding circuits and their elements naturally satisfy all the good properties that were already shown for Ultra Summit Sets, and sometimes even better properties: For example, for elements of canonical length 1, cycling and decycling are trivial operations, but cyclic sliding is not.

One more advantage of considering the set SC(x) is that it yields a simpler algorithm to solve the Conjugacy Decision Problem and the Conjugacy Search Problem in the braid group. The worst case complexity of the algorithm is not better than the previously known ones [53], but it is conceptually simpler and easier to implement. The details of the implementation and the study of complexity are presented in [56].

For any two braids u, v, let us denote  $u \wedge v$  to be the largest common prefix of u and v (the notation comes from the corresponding operation on the lattice generated by the partial order  $\leq$  on the elements of  $B_n$ , see Section 1.3.1).

The following is an interesting observation:

**Observation 1.19.** Given two permutation braids u and v, the decomposition  $u \cdot v$  is *left-weighted* if  $\partial(u) \wedge v = \varepsilon$  or, equivalently, if  $uv \wedge \Delta = u$ . The condition  $\partial(u) \wedge v = \varepsilon$  actually means that if we move any crossing from v to u, then u will not be anymore a permutation braid.

By this observation, it is easy to give a procedure to find the left-weighted factorization of the product of two permutation braids u and v as follows. If the decomposition uv is not left-weighted, this means that there is a nontrivial prefix  $s \leq v$  such that us is still a permutation braid (i.e.  $s \leq \partial(u)$ ). The maximal element which satisfies this property is  $s = \partial(u) \wedge v$ . Therefore, for transforming the decomposition uv into a left-weighted one, we have to slide the prefix  $s = \partial(u) \wedge v$  from the second factor to the first one. That is, write v = st and then consider the decomposition uv = (us)t, with us as the first factor and t as the second one. The decomposition  $us \cdot t$  is left-weighted by the maximality of s. This action will be called local sliding (see Figure 1.19).

Motivated by the idea of local sliding, one wants now to do a cycling in the same manner. Given a braid in a left normal form  $x = \Delta^p x_1 \cdots x_r$ , we want now to slide a part of  $x_1$  to  $x_r$ . This will be done by conjugating a prefix of  $\tau^{-p}(x_1)$ . The appropriate prefix is:  $\partial(x_r) \wedge \tau^{-p}(x_1)$ , which is equal to:  $\iota(x^{-1}) \wedge \iota(x)$ . Hence, Gebhardt and González-Meneses [55] define:

**Definition 1.20.** Given  $x \in B_n$ , define the cyclic sliding  $\mathfrak{s}(x)$  of x as the conjugate of x by  $\mathfrak{p}(x) = \iota(x^{-1}) \wedge \iota(x)$ , that is:

$$\mathfrak{s}(x) = \mathfrak{p}(x)^{-1} x \mathfrak{p}(x).$$

By a series of results, Gebhardt and González-Meneses [55, Section 3, Results 3.4-3.10] show that the cyclic sliding is indeed a generalization

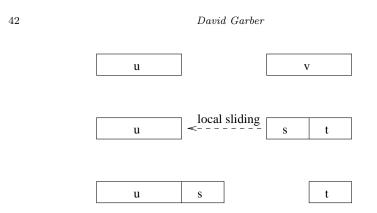


Fig. 1.19. An illustration of a local sliding

of cycling and decycling, and the fact that for every  $x \in B_n$ , iterated application of cyclic sliding eventually reaches a period, that is, there are integers  $N \geq 0$  and M > 0 such that  $\mathfrak{s}^{M+N}(x) = \mathfrak{s}^N(x)$ .

Now, one can define the set of sliding circuits of x:

**Definition 1.21.** An element  $y \in B_n$  belongs to a sliding circuit if  $\mathfrak{s}^m(y) = y$  for some  $m \ge 1$ .

Given  $x \in B_n$ , the set of sliding circuits of x, denoted by SC(x), is the set of all conjugates of x which belong to a sliding circuit.

Note that SC(x) does not depend on x but only on its conjugacy class. Hence, two elements  $x, y \in B_n$  are conjugate if and only if SC(x) = SC(y). Therefore, the computation of SC(x) and of one element of SC(y) will solve the Conjugacy Decision Problem in  $B_n$ .

The set SC(x) is usually much smaller than USS(x). For example, for

$$B_{12} \ni x = \sigma_7 \sigma_8 \sigma_7 \sigma_6 \sigma_5 \sigma_4 \sigma_9 \sigma_8 \sigma_7 \sigma_6 \sigma_5 \sigma_4 \sigma_3 \sigma_2 \sigma_{10} \sigma_9 \sigma_8 \sigma_7 \sigma_6 \sigma_5 \sigma_4 \sigma_3 \cdot \\ \cdot \sigma_2 \sigma_1 \sigma_{11} \sigma_{10} \sigma_9 \sigma_8 \sigma_7 \sigma_6 \sigma_5 \sigma_4 \sigma_3 \sigma_2 \sigma_1$$

we have that |SC(x)| = 6, but |SSS(x)| = |USS(x)| = 126498 (see [55, Section 5], based on an example from [57]). On the other hand, the size of the set SC(x) still might be exponential in the length of x (for example, if  $\delta = \sigma_{n-1} \cdots \sigma_1 \in B_n$ , one has  $|SC(\delta)| = 2^{n-2} - 2$  [55, Proposition 5.1]).

Gebhardt and González-Meneses have proved [55, Proposition 3.13] that:

$$SC(x) = RSSS(x)$$

for x satisfying  $\ell_s(x) > 1$  (where  $\ell_s(x) = \sup_s(x) - \inf_s(x)$ , i.e. the canonical

length of elements in the Super Summit Set of x), and

$$SC(x) \subseteq RSSS(x)$$

for x satisfying  $\ell_s(x) = 1$ , and in general SC(x) is a *proper* subset of RSSS(x) in this case.

They remark that the case  $\ell_s(x) = 1$  in which the sets differ is not irrelevant, since, for example, a periodic braid x which is not conjugate to a power of  $\Delta$  has  $\ell_s(x) = 1$ , but the conjugacy problem for such braids is far from being easy [16].

As in the previous Summit Sets, the algorithm to solve the CDP/CSP in braid groups (using sliding circuits) starts by obtaining an element  $\hat{x} \in SC(x)$ . We do this as follows: take an element x. Now, apply iterated cyclic sliding on it. Due to the periodic property of the sliding operation, we will have two integers  $m_1, m_2$  ( $m_1 < m_2$ ), which satisfy:

$$\mathfrak{s}^{m_1}(x) = \mathfrak{s}^{m_2}(x).$$

When having this, the element  $\hat{x} = \mathfrak{s}^{m_1}(x)$  is in SC(x), since:

$$\mathfrak{s}^{m_2-m_1}(\hat{x}) = \hat{x}.$$

After finding a representative  $\hat{x} \in SC(x)$ , we have to explore all the set SC(x). This we do in a similar way to the Ultra Summit Set case: There are  $\leq$ -minimal elements which conjugate an element in SC(x) to another element there. The number of such possible minimal conjugators for a given element in SC(x) is bounded by the number of Artin generators). Hence, one can compute the whole SC(x) starting by a single element  $\hat{x} \in SC(x)$ , and then we are done (for more information, see [55, Section 4.1] and [56])

Again, as in the previous Summit Sets, the algorithm of Gebhardt and González-Meneses [55] not only computes SC(x), but also a graph SCG(x), which determines the conjugating elements. This graph is defined as follows.

**Definition 1.22.** Given  $x \in B_n$ , the directed graph SCG(x) is defined by the following data:

- (1) The set of vertices is SC(x).
- (2) For every  $y \in SC(x)$  and every minimal permutation braid s for y with respect to SC(x), there is an arrow labeled by s going from y to  $s^{-1}ys$ .

More information about these sorts of Summit sets can be found in the series of papers [14; 15; 16] and [77; 78; 80].

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### 1.7.6. An updated summary of the theoretical solution for the conjugacy search problem

In this section, we give an updated summary for the current status of the complexity of the theoretical solution for the Conjugacy Search Problem. We follow here the nice presentation of González-Meneses in his talk at Singapore (2007) [59].

As already mentioned, according to Nielsen-Thurston geometric classification (based on [102] and [120]), there are three types of braids: periodic braids, reducible braids and pseudo-Anosov braids.

A braid  $\alpha$  is called *periodic* if there exist integers k, m such that  $\alpha^k = \Delta^{2m}$ . A braid  $\alpha$  is called *reducible* if it preserves a family of curves, called a *reduction system*. A braid is called *pseudo-Anosov* if it is neither periodic nor reducible.

For the case of **periodic braids**, Birman, Gebhardt and González-Meneses [16] present a polynomial-time algorithm for solving the conjugacy search problem. Almost at the same time, Lee and Lee [79] suggest another entirely different solution for this case.

For the case of **reducible braids**, there is a result of Gebhardt and González-Meneses [59] that these braids fall into exactly two cases:

- (1) The braid  $\alpha$  is conjugate to a braid with a *standard* reducing curve, which means that the reducing curves are round circles, and hence the Conjugacy Search Problem can be decomposed into smaller problems (inside the tubes).
  - There is only one problem here: the conjugate braid (with a standard reducing curve) is in  $USS(\alpha)$ , and for reaching it, one has to make an unknown number of cycling/decycling (or sliding) steps.
- (2) The braid  $\alpha$  is rigid (i.e. a cycling of the Garside normal form of  $\alpha$  is left-weighted as written, or alternatively, it is a fixed point with respect to cyclic slidings).

For the case of **pseudo-Anosov braids**: Due to a result of Birman, Gebhardt and González-Meneses [14, Corollary 3.24], there exists a small power of a pseudo-Anosov braid which is conjugate to a rigid braid. Another result [58] claims that in the case of pseudo-Anosov braids, the conjugating elements of the pair (x, y) and the pair  $(x^m, y^m)$  coincide, and hence instead of solving the Conjugacy Search Problem in the pair (x, y), one can solve it in the pair  $(x^m, y^m)$ . Therefore, one can restrict himself to the case of

rigid braids.

If we summarize all cases, we get that the main challenges in this direction are:

- (1) Solve the Conjugacy Search Problem for rigid braids in polynomial
- (2) Given a braid x, find a polynomial bound for the number of cycling/decycling steps one has to perform for reaching an element in USS(x).

#### 1.8. More attacks on the conjugacy search problem

There are some more ways to attack the Conjugacy Search Problem, apart of solving it completely. In this section, we present some techniques to attack the Conjugacy Search Problem without actually solving it theoretically.

#### 1.8.1. A heuristic algorithm using the Super Summit Sets

Hofheinz and Steinwandt [65] use a heuristic algorithm for attacking the Conjugacy Search Problem which is the basis of the cryptosystems of Anshel-Anshel-Goldfeld [4] and Ko et al. [72].

Their algorithm is based on the idea that it is probable that if we start with two elements in the same conjugacy class, their representatives in the Super Summit Set will not be too far away, i.e. one representative is a conjugation of the other by a permutation braid.

So, given a pair (x, x') of braids, where  $x' = s^{-1}xs$ , we do the following steps:

- (1) By a variant of cycling (adding a multiplication by  $\Delta$  to the first permutation braid, based on [82, Proposition 1]) and decycling, we find  $\tilde{x} \in SSS(x)$  and  $\tilde{x}' \in SSS(x')$ .
- (2) Try to find a permutation braid P, such that  $\tilde{x}' = P^{-1}\tilde{x}P$ .

In case we find such a permutation braid P, since we can follow after the conjugators in the cycling/decycling process, at the end of the algorithm we will have at hand the needed conjugator for breaking the cryptosystem. Note that we do not really need to find exactly s, since each  $\tilde{s}$  which satisfies  $x' = \tilde{s}^{-1}x\tilde{s}$  will do the job as well and reveal the shared secret key.

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Their experiments show that they succeed to reveal the shared secret key in almost 100% of the cases in the Anshel-Anshel-Goldfeld protocol (where the cryptosystem is based on the Multiple Simultaneous Conjugacy Problem) and in about 80% of the cases in the Diffie-Hellman-type protocol.

Note that their attack is special to cryptosystems which are based on the conjugacy problem, since it depends very much on the fact that x and x' are conjugate.

### 1.8.2. Reduction of the Conjugacy Search Problem

Maffre [87; 88] presents a deterministic, polynomial algorithm that reduces the Conjugacy Search Problem in braid group.

The algorithm is based on the decomposition of braids into products of canonical factors and gives a partial factorization of the secret: a divisor and a multiple. The tests which were performed on different keys of existing protocols showed that many protocols in their current form are broken and that the efficiency of the attack depends on the random generator used to create the key.

#### 1.8.3. Length-based attacks

A different probabilistic attack on the braid group cryptosystems is the *length-based attack*. In this section, we will sketch its basic idea, and different variants of this attack on the braid group cryptosystems. We finish this section with a short discussion about the applicability of the length-based attack to other groups.

### 1.8.3.1. The basic idea

The basic idea was introduced by Hughes and Tannenbaum [67].

Let  $\ell$  be a length function on the braid group  $B_n$ . In the Conjugacy Search Problem, we have an instance of (p, p') where  $p' = s^{-1}ps$ , and we look for s. The idea of a probabilistic length-based attack to this problem is: if we can write  $s = s'\sigma_i$  for a given i, then the length  $\ell(\sigma_i s^{-1}ps\sigma_i^{-1})$  should be strictly smaller than the length  $\ell(\sigma_j s^{-1}ps\sigma_i^{-1})$  for  $j \neq i$ .

Thus, for using such an attack, one should choose a good length function on  $B_n$  and run it iteratively till we get the correct conjugator.

### 1.8.3.2. Choosing a length function

In [49], we suggest some length functions for this purposes. The first option is the *Garside length*, which is the length of the Garside normal form by means of Artin generators (i.e. if  $w = \Delta_n^r P_1 P_2 \cdots P_k$ , then  $\ell_{Gar}(w) = r|\Delta| + |P_1| + |P_2| + \cdots + |P_k|$ ).

A better length function is the *Reduced Garside length* (which is called *Mixed Garside length* in [44]). The motivation for this length function is that a part of the negative powers of  $\Delta_n$  can be canceled with the positive permutation braids. Hence, it is defined as follows: if  $w = \Delta_n^{-r} P_1 P_2 \cdots P_k$ , then:

$$\ell_{\text{RedGar}}(w) = \ell_{\text{Gar}}(w) - 2 \sum_{i=1}^{\min\{r,k\}} |P_i|.$$

This length function is much more well-behaved, and hence it gives better performances. But even this length function did not give a break of the cryptosystems (by the basic length-based attack).

In [64], Hock and Tsaban checked the corresponding length functions for the Birman-Ko-Lee presentation, and they found out that the reduced length function with respect to the Birman-Ko-Lee presentation behaves even better than the reduced Garside length function.

#### 1.8.3.3. The memory approach

The main contribution of [48] is new improvements to the length-based attack.

First, it introduces a new approach which uses memory: In the basic length-based attack, we hold each time only the best conjugator so far. The problem with this is that sometimes a prefix of the correct conjugator is not the best conjugator at some iteration and hence it is thrown out. In such a situation, we just miss the correct conjugator in the way, and hence the length-based algorithm fails. Moreover, even if we use a 'look ahead' approach, which means that instead of adding one generator in each iteration we add several generators in each iteration, we still get total failure for the suggested parameters, and some success for small parameters [49].

In the memory approach, we hold each time a given number (which is the size of the memory) of possible conjugators which are the best among all the other conjugators of this length. In the next step, we add one more generator to all the conjugators in the memory, and we choose again only the best ones among all the possibilities. In this approach, in a successful

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search, we will often have the correct conjugator in the first place of the memory.

The results of [48] show that the length-based attack with memory is applicable to the cryptosystems of Anshel-Anshel-Goldfeld and Ko et al, and hence their cryptosystems are not secure. Moreover, the experiments show that if we increase the size of the memory, the success rate of the length-based attack with memory becomes higher.

## 1.8.3.4. A different variant of Length-based attack by Myasnikov and Ushakov

Recently, Myasnikov and Ushakov [100] suggested a different variant of the length-based approach.

They start by mentioning the fact that the geodesic length, i.e. the length of the shortest path in the corresponding Cayley graph, seems to be the best candidate for a length function in the braid group, but there is no known efficient algorithm for computing it. Moreover, it was shown by Paterson and Razborov [104] that the set of geodesic braids in  $B_n$  is co-NP complete. On the other hand, many other length functions are bad for the length-based attacks (like the canonical length, which is the number of permutation braids in the Garside normal form).

As a length function, they choose some approximation function for the geodesic length: they use Dehornoy's handles reduction and conjugations by  $\Delta$  (this length function appears in [96; 97]). This length function satisfies  $|a^{-1}ba| > |b|$  for almost all a and b.

Next, they identify a type of braid word, which they call *peaks*, which causes problems to the Length-based attacks:

**Definition 1.23.** Let G be a group, and let  $\ell_G$  be a length function on G, and  $H = \langle w_1, \ldots, w_k \rangle$ . A word  $w = w_{i_1} \cdots w_{i_n}$  is called an n-peak in H relative to  $\ell_G$  if there is no  $1 \leq j \leq n-1$  such that

$$\ell_G(w_{i_1}\cdots w_{i_n}) \ge \ell_G(w_{i_1}\cdots w_{i_j}).$$

An example of a *commutator-type peak* is given in [100, Example 1]: if  $a_1 = \sigma_{39}^{-1} \sigma_{12} \sigma_7 \sigma_3^{-1} \sigma_1^{-1} \sigma_{70} \sigma_{25} \sigma_{24}^{-1}$  and  $a_2 = \sigma_{42} \sigma_{56}^{-1} \sigma_8 \sigma_{18}^{-1} \sigma_{19} \sigma_{73} \sigma_{33}^{-1} \sigma_{22}^{-1}$  then their commutator is a peak:  $a_1^{-1} a_2^{-1} a_1 a_2 = \sigma_7 \sigma_8^{-1}$ .

The main idea behind their new variant of the Length-based attack is to add elements from the corresponding subgroup to cut the peaks. By an investigation of the types of peaks, one can see that this is done by adding to the vector of elements all the conjugators and commutators of its elements. By this way, the Length-based attack will be more powerful. For more information and for an exact implementation, see [100].

#### 1.8.3.5. Applicability of the length-based approach

One interesting point about the length-based approach is that it is applicable not only for the Conjugacy Search Problem, but also for solving equations in groups. Hence, it is a threat also to the Decomposition Problem and for the Shifted Conjugacy Problem which was introduced by Dehornoy (see [30] and Section 1.9.3 below).

Moreover, the length-based approach is applicable in any group which has a reasonable length function, e.g. the Thompson group, as indeed has been done by Ruinskiy, Shamir and Tsaban (see [108] and Section 1.11.1.2 below).

#### 1.8.4. Attacks based on linear representations

A different way to attack these cryptographic schemes is by using linear representations of the braid groups. The basic idea is to map the braid groups into groups of matrices, in which the Conjugacy Search Problem is easy. In this way, we might solve the Conjugacy Search Problem of  $B_n$  by lifting the element from the group of matrices back to the braid group  $B_n$ .

For more information on the linear representations of the braid group, we refer the reader to the surveys of Birman and Brendle [13] and Paris [103].

#### 1.8.4.1. The Burau representations

The best known linear representation of the braid group  $B_n$  is the Burau representation [21]. We present it here (we partially follow [82]).

The Burau representation is defined as follows. Let  $\mathbb{Z}[t^{\pm 1}]$  be the ring of Laurent polynomials  $f(t) = a_k t^k + a_{k+1} t^{k+1} + \cdots + a_m t^m$  with integer coefficients (and possibly with negative degree terms). Let  $\mathrm{GL}_n(\mathbb{Z}[t^{\pm 1}])$  be the group of  $n \times n$  invertible matrices over  $\mathbb{Z}[t^{\pm 1}]$ . The Burau representation is a homomorphism  $B_n \to \mathrm{GL}_n(\mathbb{Z}[t^{\pm 1}])$  which sends a generator  $\sigma_i \in B_n$ 

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to the matrix:

$$\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & 1 - t & t & & \\ & 1 & 0 & & \\ & & & \ddots & \\ & & & 1 \end{pmatrix} \in GL_n(\mathbb{Z}[t^{\pm 1}]),$$

where 1 - t occurs in row and column i of the matrix.

This representation is reducible, since it can be decomposed into the trivial representation of dimension 1 and an irreducible representation  $B_n \to \mathrm{GL}_{n-1}(\mathbb{Z}[t^{\pm 1}])$  of dimension n-1, called the reduced Burau representation, which sends a generator  $\sigma_i \in B_n$  to the matrix:

$$C_{i}(t) = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & t - t & 1 & \\ & & & 1 & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \in GL_{n-1}(\mathbb{Z}[t^{\pm 1}]),$$

where t occurs in row i of the matrix. If i = 1 or i = n - 1, the matrix is truncated accordingly (see [82]).

Note that these matrices satisfy the braid group's relations:

$$C_i(t)C_i(t) = C_i(t)C_i(t)$$
 for  $|i - j| > 2$ 

$$C_i(t)C_{i+1}(t)C_i(t) = C_{i+1}(t)C_i(t)C_{i+1}(t)$$
 for  $i = 1, ..., n-1$ 

The Burau representation of  $B_n$  is faithful for n=3 and it is known to be unfaithful for  $n \geq 5$  (i.e. the map from  $B_n$  to the matrices is not injective) [93; 94; 83; 10]. The case of n=4 remains unknown. In the case of  $n \geq 5$ , the kernel is very small [123], and the probability that different braids admit the same Burau image is negligible.

Here is a variant of the Burau representation introduced by Morton [95]. The colored Burau matrix is a refinement of the Burau matrix by assigning  $\sigma_i$  to  $C_i(t_{i+1})$ , so that the entries of the resulting matrix have several variables. This naive construction does not give a group homomorphism.

Thus the induced permutations are considered simultaneously. We label the strands of an n-braid by  $t_1, \ldots, t_n$ , putting the label  $t_j$  on the strand which starts from the jth point on the right.

Now we define:

**Definition 1.24.** Let  $a \in B_n$  be given by a word  $\sigma_{i_1}^{e_1} \cdots \sigma_{i_k}^{e_k}$ ,  $e_j = \pm 1$ . Let  $t_{j_r}$  be the label of the under-crossing strand at the rth crossing. Then the colored Burau matrix  $M_a(t_1, \ldots, t_n)$  of a is defined by

$$M_a(t_1,\ldots,t_n) = \prod_{r=1}^k (C_{i_r}(t_{j_r}))^{e_r}.$$

The permutation group  $S_n$  acts on  $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_{n-1}^{\pm 1}]$  from left by changing variables: for  $\alpha \in S_n$ ,  $\alpha(f(t_1, \ldots, t_n)) = f(t_{\alpha(1)}, \ldots, t_{\alpha(n)})$ . Then  $S_n$  also acts on the matrix group  $\mathrm{GL}_{n-1}(\mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}])$  entry-wise: for  $\alpha \in S_n$  and  $M = (f_{ij})$ , then  $\alpha(M) = (\alpha(f_{ij}))$ . Then we have

**Definition 1.25.** The colored Burau group  $CB_n$  is:

$$S_n \times GL_{n-1}(\mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}])$$

with multiplication  $(\alpha_1, M_1) \cdot (\alpha_2, M_2) = (\alpha_1 \alpha_2, (\alpha_2^{-1} M_1) M_2)$ . The colored Burau representation  $C: B_n \to CB_n$  is defined by  $C(\sigma_i) = ((i, i + 1), C_i(t_{i+1}))$ .

It is easy to see the following:

- (1)  $CB_n$  is a group, with identity element  $(e, I_{n-1})$  and  $(\alpha, M)^{-1} = (\alpha^{-1}, \alpha M^{-1})$ ,
- (2)  $C(\sigma_i)$ 's satisfy the braid relations and so  $C: B_n \to CB_n$  is a group homomorphism.
- (3) for  $a \in B_n$ ,  $C(a) = (\pi_a, M_a)$ , where  $\pi_a$  is the induced permutation and  $M_a$  is the colored Burau matrix.

Using the Burau representation, the idea of Hughes [66] to attack the Anshel-Anshel-Goldfeld scheme [4; 5] is as follows: take one or several pairs of conjugate braids (p, p') associated with the same conjugating braids. Now, it is easy to compute their classical Burau image and to solve the Conjugate Search Problem in the linear group.

In general, this is not enough for solving the Conjugate Search Problem in  $B_n$ , because there is no reason for the conjugating matrix that has been found to belong to the image of the Burau representation, or that one can

find a possible preimage. Since the kernel of the classical Burau representation is small [123], there is a non-negligible probability that we will find the correct conjugator and hence we break the cryptosystem.

In a different direction, Lee and Lee [82] indicate a weakness in the Anshel-Anshel-Goldfeld protocol in a different point. Their shared key is the colored Burau representation of a commutator element.

The motivation for this attack is that despite the change of variables in the colored Burau matrix by permutations, the matrix in the final output, which is the shared key, is more manageable than braids. They show that the security of the key-exchange protocol is based on the problems of listing all solutions to some Multiple Simultaneous Conjugacy Problems in a permutation group and in a matrix group over a finite field. So if both of the two listing problems are feasible, then we can guess correctly the shared key, without solving the Multiple Simultaneous Conjugacy Problem in braid groups.

Note that Lee-Lee attack is special to this protocol, since it uses the colored Burau representation of a commutator element, instead of using the element itself. In case we change the representation in the protocol, this attack is useless.

#### 1.8.4.2. The Lawrence-Krammer representation

The Lawrence-Krammer representation is another linear representation of  $B_n$ , which is faithful [11; 74]. It associates with every braid in  $B_n$  a matrix of size  $\binom{n}{2}$  with entries in a 2-variable Laurent polynomial ring  $\mathbb{Z}[t^{\pm 1}, q^{\pm 1}]$ .

Cheon and Jun [24] develop an attack against the scheme of Diffie-Hellman-type protocol based on the Lawrence-Krammer representation: as in the case of the Burau representation, it is easy to compute the images of the involved braids in the linear group and to solve the Conjugacy Problem there, but in general, there is no way to lift the solution back to the braid groups.

But, since we only have to find a solution to the derived Diffie-Hellman-like Conjugacy Problem:

**Problem 1.5.** Given  $p, sps^{-1}$  and  $rpr^{-1}$ , with  $r \in LB_n$  and  $s \in UB_n$ , find  $(rs)p(rs)^{-1}$ .

Taking advantage of the particular form of the Lawrence-Krammer matrices, which contain many 0's, Cheon and Jun obtain a solution with a polynomial complexity and they show that, for the parameters suggested

by Ko et al. [72], the procedure is doable, and so the cryptosystem is not secure.

# 1.9. Newly suggested braid group cryptosystems, their cryptanalysis and their future applications

In this section, we present recent updates on some problems in the braid group, on which one can construct a cryptosystem. We also discuss some newly suggested braid group cryptosystems.

### 1.9.1. Cycling problem as a potential hard problem

In their fundamental paper, Ko et al. [72] suggested some problems which can be considered as hard problems, on which one can construct a cryptosystem. One of the problems is the *Cycling Problem*:

**Problem 1.6.** Given a braid y and a positive integer t such that y is in the image of the operator  $\mathbf{c}^t$ . Find a braid x such that  $\mathbf{c}^t(x) = y$ .

Maffre, in his thesis [86], shows that the Cycling Problem for t = 1 has a very efficient solution. That is, if y is the cycling of some braid, then one can find x such that  $\mathbf{c}(x) = y$  very fast.

Following this result, Gebhardt and Gonzáles-Meneses [54] have shown that the general Cycling Problem has a polynomial solution. The reason for that is the following result: The cycling operation is surjective on the braid group [54]. Hence, one can easily find the tth preimage of y under this operation.

Note that the decycling operation and cyclic sliding operation are surjective too (the decycling operation is a composition of surjective maps:  $\mathbf{d}(x) = (\tau(\mathbf{c}(x^{-1})))^{-1}$ , and the cyclic sliding operation can be written as a composition of a cycling and a decycling [55, Lemma 3.8]). Hence, these problems cannot be considered as hard problems, on which one can construct a cryptosystem [60].

It will be interesting to find new operations on the braid group which their solution can be consider as an hard problem, on which one can construct a cryptosystem.

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#### 1.9.2. A cryptosystem based on the shortest braid problem

A different type of problem consists in finding the shortest words representing a given braid (see Dehornoy [29, Section 4.5.2]). This problem depends on a given choice of a distinguished family of generators for  $B_n$ , e.g., the  $\sigma_i$ 's or the band generators of Birman-Ko-Lee.

We consider this problem in  $B_{\infty}$  which is the group generated by an infinite sequences of generators  $\{\sigma_1, \sigma_2, \dots\}$  subject to the usual braid relations.

The Minimal Length Problem (or Shortest Word Problem) is:

**Problem 1.7.** Starting with a word w in the  $\sigma_i^{\pm 1}$ 's, find the shortest word w' which is equivalent to w, i.e., that satisfies  $w' \equiv w$ .

This problem is considered to be hard due to the following result of Paterson and Razborov [104]:

**Prop 1.9.** The Minimal Length Problem (in Artin's presentation) is co-NP-complete.

This suggests introducing new schemes in which the secret key is a short braid word, and the public key is another longer equivalent braid word. It must be noted that the NP-hardness result holds in  $B_{\infty}$  only, but it is not known in  $B_n$  for fixed n.

The advantage of using an NP-complete problem lies in the possibility of proving that some instances are difficult; however, from the point of view of cryptography, the problem is not to prove that some specific instances are difficult (worst-case complexity), but rather to construct relatively large families of provably difficult instances in which the keys may be randomly chosen.

Based on some experiments, Dehornoy [29] suggests that braids of the form  $w(\sigma_1^{e_1}, \sigma_2^{e_2}, \dots, \sigma_n^{e_n})$  with  $e_i = \pm 1$ , i.e., braids in which, for each i, at least one of  $\sigma_i$  or  $\sigma_i^{-1}$  does not occur, could be relevant.

The possible problem of this approach is that the shortest word problem in  $B_n$  for a fixed n is not so hard. In  $B_3$ , there is polynomial-time algorithms for the shortest word problem (see [8] and [124] for the presentation by the Artin generators and [125] for the presentation by band generators). Also, this problem was solved in polynomial time in  $B_4$  for the presentation by the band generators ([70] and [81, Chapter 5]). For small fixed n, Wiest [124] conjectures for an efficient algorithm for finding shortest representatives in  $B_n$ . Also, an unpublished work [50] indicates that a heuristic algorithm

based on a random walk on the Cayley graph of the braid group might give good results in solving the Shortest Word Problem.

In any case, a further research is needed here in several directions:

- (1) Cryptosystem direction: Can one suggest a cryptosystem based on the shortest word problem in  $B_{\infty}$ , for using its hardness due to Paterson-Razborov?
- (2) **Cryptanalysis direction**: What is the final status of the shortest word problem in  $B_n$  for a fixed n?
- (3) **Cryptanalysis direction**: What is the hardness of the Shortest Word Problem in the Birman-Ko-Lee's presentation?

## 1.9.3. A cryptosystem based on the Shifted Conjugacy Search Problem

Dehornoy [30] has suggested an authentication scheme which is based on the Shifted Conjugacy Search Problem.

Before we describe the scheme, let us define the Shifted Conjugacy Search Problem. Let  $x, y \in B_{\infty}$ . We define:

$$x * y = x \cdot dy \cdot \sigma_1 \cdot dx^{-1}$$

where dx is the *shift* of x in  $B_{\infty}$ , i.e. d is the injective function on  $B_{\infty}$  which sends the generator  $\sigma_i$  to the generator  $\sigma_{i+1}$  for each  $i \geq 1$ . In this context, the Shifted Conjugacy Search Problem is:

**Problem 1.8.** Let  $s, p \in B_{\infty}$  and p' = s \* p. Find a braid  $\tilde{s}$  satisfying  $p' = \tilde{s} * p$ .

Now, the suggested scheme is based on the Fiat-Shamir authentication scheme: We assume that S is a set and  $(F_s)_{s\in S}$  is a family of functions of S to itself that satisfies the following condition:

$$F_r(F_s(p)) = F_{F_r(s)}(F_r(p)), \quad r, s, p \in S$$

Alice is the prover who wants to convince Bob that she knows the secret key s. Then the scheme works as follows:

Protocol 1.26.

Public key: Two elements  $p, p' \in S$  such that  $p' = F_s(p)$ .

Private keys: Alice:  $s \in S$ .

Alice: Chooses a random  $r \in S$  and sends Bob  $x = F_r(p)$  and  $x' = F_r(p')$ .

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Bob: Chooses a random bit c and sends it to Alice.

Alice: If c = 0, sends y = r (then Bob checks:  $x = F_y(p)$  and  $x' = F_y(p')$ );

If c = 1, sends  $y = F_r(s)$  (then Bob checks:  $x' = F_y(x)$ ).

Dehornoy [30] suggests to implement this scheme on Left-Distributive(LD)-systems. A LD-system is a set S with a binary operation which satisfies:

$$r * (s * p) = (r * s) * (r * p).$$

The Fiat-Shamir-type scheme on LD-systems works as follows:

Protocol 1.27.

Public key: Two elements  $p, p' \in S$  such that p' = s \* p.

Private keys: Alice:  $s \in S$ .

Alice: Chooses a random  $r \in S$  and sends Bob x = r \* p and x' = r \* p'.

Bob: Chooses a random bit c and sends it to Alice.

Alice: If c = 0, sends y = r (then Bob checks: x = y \* p and x' = y \* p');

If c = 1, sends y = r \* s (then Bob checks: x' = y \* x).

Now, one can use the shifted conjugacy operation as the \* operation on  $B_{\infty}$  in order to get a LD-system. So, in this way, one can achieve an authentication scheme on the braid group with a non-trivial operation [30].

**Remark 1.6.** For attacking the Shifted Conjugacy Search Problem, one cannot use the Summit Sets theory, since it is not a conjugation problem anymore. Nevertheless, one still can apply on it the length-based attack, since it is still an equation with x.

Longrigg and Ushakov [84] cryptanalyze the suggestion of Dehornoy, and they show that they can break the scheme (e.g. 24% of success rate for keys of length 100 in  $B_{40}$ ). Their idea is that in general cases they can reduce the Shifted Conjugacy Search Problem into the well-studied Conjugacy Search Problem. Based on some simple results, they construct an algorithm for solving the Shifted Conjugacy Search Problem in two steps:

- (1) Find a solution  $s' \in B_{n+1}$  for the equation  $p'\delta_{n+1}^{-1} = s'd(p)\sigma_1\delta_{n+1}^{-1}$  in  $B_{n+1}$ . This part can be done using the relevant Ultra Summit Set.
- (2) Correct the element  $s' \in B_{n+1}$  to obtain a solution  $s \in B_n$ . This can be done by finding a suitable element  $c \in C_{B_{n+1}}(d(p)\sigma_1\delta_{n+1}^{-1})$  (the centralizer of  $d(p)\sigma_1\delta_{n+1}^{-1}$  in  $B_{n+1}$ ).

The algorithm for computing centralizers presented in [45] is based on computing the Super Summit Set, which is hard in general (note that actually the Super Summit Set can be replaced by the Ultra Summit Set and the Sliding Circuits set in Franco and González-Meneses' algorithm [60]). Hence, Longrigg and Ushakov use some subgroup of the centralizer which is much easier to work with.

In the last part of their paper, they discuss possibilities for hard instances for Dehornoy's scheme, which will resist their attack. Their attack is based on two ingredients:

(1) The Conjugacy Search Problem is easy for the pair

$$(p'\delta_{n+1}^{-1}, d(p)\sigma_1\delta_{n+1}^{-1})$$

in  $B_{n+1}$ .

(2) The centralizer  $C_{B_{n+1}}(d(p)\sigma_1\delta_{n+1}^{-1})$  is "small" (i.e. isomorphic to an Abelian group of small rank).

Hence, if one can find keys for which one of the properties above is not satisfied, then the attack probably fails.

With respect to this scheme, it is interesting to check (see also [30]):

- (1) **Cryptanalysis direction:** What is the success rate of a length-based attack on this scheme?
- (2) **Cryptanalysis direction:** Can one develop a theory for the Shifted Conjugacy Search Problem which will be parallel to the Summit Sets theory?
- (3) **Cryptosystem direction:** Can one suggest a LD-system on the braid group, which will be secure for the length-based attack?
- (4) **Cryptosystem direction:** Can one find keys for which the properties above are not satisfied, and for which Longrigg-Ushakov's attack fails?
- (5) **Cryptosystem direction:** Can one suggest a LD-system on a different group, which will be secure?

#### 1.9.4. Algebraic Eraser

Recently, Anshel, Anshel, Goldfeld and Lemieux [6] introduce a new scheme for a cryptosystem which is based on combinatorial group theory. We will present here the main ideas of the scheme and the potential attacks on it.

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#### 1.9.4.1. The scheme and the implementation

We follow the presentation of [69]. Let G be a group acting on a monoid M on the left, that is, to each  $g \in G$  and each  $a \in M$ , we associate a unique element denoted  ${}^g a \in M$ , such that:

$${}^{1}a = a;$$
  ${}^{gh}a = {}^{g}({}^{h}a);$   ${}^{g}(ab) = {}^{g}a \cdot {}^{g}b$ 

for all  $a, b \in M$  and  $g, h \in G$ . The set  $M \times G$ , with the operation  $(a, g) \circ (b, h) = (a \cdot {}^{g}b, gh)$  is a monoid, which is denoted by  $M \rtimes G$ .

Let N be a monoid, and  $\varphi: M \to N$  a homomorphism. The algebraic eraser operation is the function  $\star: (N \times G) \times (M \rtimes G) \to (N \times G)$  defined by:

$$(a,g) \star (b,h) = (a\varphi(gb), gh)$$

The function  $\star$  satisfies the following identity:

$$((a,g) \star (b,h)) \star (c,r) = (a,g) \star ((b,h) \circ (c,r))$$

for all  $(a, g) \in N \times G$  and  $(b, h), (c, r) \in M \rtimes G$ .

We say that two submonoids A, B of  $M \rtimes G$  are  $\star$ -commuting if

$$(\varphi(a), g) \star (b, h) = (\varphi(b), h) \star (a, g)$$

for all  $(a, g) \in A$  and  $(b, h) \in B$ . In particular, if A, B \*-commute, then:  $\varphi(a)\varphi(g) = \varphi(b)\varphi(h)$  for all  $(a, g) \in A$  and  $(b, h) \in B$ .

Based on these settings, Anshel, Anshel, Goldfeld and Lemieux suggest the *Algebraic Eraser Key Agreement Scheme*. It consists on the following public information:

- (1) A positive integer m.
- (2)  $\star$ -commuting submonoids A, B of  $M \rtimes G$ , each given in terms of a generating set of size k.
- (3) Elementwise commuting submonoids C, D of N.

Here is the protocol:

Protocol 1.28.

Alice: Chooses  $c \in C$  and  $(a_1, g_1), \ldots, (a_m, g_m) \in A$ , and sends  $(p, g) = (c, 1) \star (a_1, g_1) \star \cdots \star (a_m, g_m) \in N \times G$  (where the  $\star$ -multiplication is carried out from left to right) to Bob.

Bob: Chooses  $d \in D$  and  $(b_1, h_1), \ldots, (b_m, h_m) \in B$ , and sends  $(q, h) = (d, 1) \star (b_1, h_1) \star \cdots \star (b_m, h_m) \in N \times G$  to Alice.

Alice and Bob can compute the shared key:

$$(cq,h)\star(a_1,g_1)\star\cdots\star(a_m,g_m)=(dp,g)\star(b_1,h_1)\star\cdots\star(b_m,h_m)$$

For the reason why it is indeed a shared key, see [6] and [69].

Anshel, Anshel, Goldfeld and Lemieux apply their general scheme to a particular case, which they call *Colored Burau Key Agreement Protocol* (CBKAP):

Fix a positive integers n and r, and a prime number p. Let  $G = S_n$ , the symmetric group on the n symbols  $\{1, \ldots, n\}$ . The group  $G = S_n$  acts on  $GL_n(\mathbb{F}_p(t_1, \ldots, t_n))$  by permuting the variables  $\{t_1, \ldots, t_n\}$  (note that in this case the monoid M is in fact a group, and hence, the semi-direct product  $M \rtimes G$  also forms a group, with inversion  $(a, g)^{-1} = (g^{-1}a^{-1}, g^{-1})$  for all  $(a, g) \in M \rtimes G$ ).

Let  $N = GL_n(\mathbb{F}_p)$ . The group  $M \rtimes S_n$  is the subgroup of  $GL_n(\mathbb{F}_p(t_1,\ldots,t_n)) \rtimes S_n$ , generated by  $(x_1,s_1),\ldots,(x_{n-1},s_{n-1})$ , where  $s_i=(i,i+1)$ , and  $x_i=C_i(t_i)$  (see page 50 above), for  $i=2,\ldots,n-1$ . Recall that the colored Burau group  $M\rtimes G$  is a representation of Artin's braid group  $B_n$ , determined by mapping each Artin generator  $\sigma_i$  to  $(x_i,s_i)$ ,  $i=1,\ldots,n-1$ .

 $\varphi: M \to GL_n(\mathbb{F}_p)$  is the evaluation map sending each variable  $t_i$  to a fixed element  $\tau \in \mathbb{F}_p$ . Let  $C = D = \mathbb{F}_p(\kappa)$  is the group of matrices of the form:

$$\ell_1 \kappa^{j_1} + \cdots + \ell_r \kappa^{j_r}$$

with  $\kappa \in GL_n(\mathbb{F}_p)$  of order  $p^n - 1, \ell_1, \ldots, \ell_r \in \mathbb{F}_p$ , and  $j_1, \ldots, j_r \in \mathbb{Z}$ .

Commuting subgroups of  $M \rtimes G$  are chosen in a similar way to  $LB_n$  and  $UB_n$  in Section 1.6.2.2. This part is done by a Trusted Third Party (TTP), before the key-exchange protocol starts.

Fix  $I_1, I_2 \subseteq \{1, \ldots, n-1\}$  such that for all  $i \in I_1$  and  $j \in I_2$ , |i-j| > 2, and  $|I_1|$  and  $|I_2|$  are both  $\leq n/2$ . Then, define  $L = \langle \sigma_i : i \in I_1 \rangle$  and  $U = \langle \sigma_j : j \in I_2 \rangle$ , subgroups of  $B_n$  generated by Artin generators. From the construction of  $I_1$  and  $I_2$ , L and U commute elementwise. Add to both groups the central element  $\Delta^2$  of  $B_n$ .

Now, they choose a secret random  $z \in B_n$ . Next, they choose  $w_1 = zw_1'z^{-1}, \ldots, w_k = zw_k'z^{-1} \in zLz^{-1}$  and  $v_1 = zv_1'z^{-1}, \ldots, v_k = zv_k'z^{-1} \in zUz^{-1}$ , each a product of t-many generators. Transform them into Garside's normal form, and remove all even powers of  $\Delta$ . Reuse the names  $w_1, \ldots, w_k; v_1, \ldots, v_k$  for the resulting braids. These braids are made public.

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Anshel, Anshel, Goldfeld and Lemieux have cryptanalyzed their scheme and the TTP protocol, and conclude that if the conjugating element z is known, there is a successful linear algebraic attack on CBKAP (see [6, Section 6]). On the other hand, if z is not known, this attack cannot be implemented. Moreover, they claim that the length-based attack is ineffective against CBKAP because  $w_i$  and  $v_i$  are not known and for some more reasons.

#### 1.9.4.2. The attacks

There are several attacks on this cryptosystem. Kalka, Teicher and Tsaban [69] attack the general scheme and then show that the attack can be applied to CBKAP, the specific implementation of the scheme.

For the general scheme, they show that the secret part of the shared key can be computed (under some assumptions, which also include the assumption that the keys are chosen with standard distributions). They do it in two steps: First they compute d and  $\varphi(b)$  up to a scalar, and using that they can compute the secret part of the shared key. They remark that if the keys are chosen by a distribution different from the standard, it is possible that this attack is useless (see [69, Section 8] for a discussion on this point).

In the next part, they show that the assumptions are indeed satisfied for the specific implementation of the scheme. The first two assumptions (that it is possible to generate an element  $(\alpha, 1) \in A$  with  $\alpha \neq 1$ , and that N is a subgroup of  $GL_n(\mathbb{F})$  for some field  $\mathbb{F}$  and some n) can be easily checked. The third assumption (that given an element  $g \in \langle s_1, \ldots, s_k \rangle$ , where  $(a_1, s_1), \ldots, (a_k, s_k) \in M \rtimes G$  are the given generators of A, then g can be explicitly expressed as a product of elements of  $\{s_1^{\pm}, \ldots, s_k^{\pm}\}$ , can be reformulated as the Membership Search Problem in generic permutation groups:

**Problem 1.9.** Given random  $s_1, \ldots, s_k \in S_n$  and  $s \in \langle s_1, \ldots, s_k \rangle$ , express s as a short (i.e. of polynomial length) product of elements from  $\{s_1^{\pm}, \ldots, s_k^{\pm}\}$ .

They provide a simple and very efficient heuristic algorithm for solving this problem in generic permutation groups. The algorithm gives expressions of length  $O(n^2 \log(n))$ , in time  $O(n^4 \log(n))$  and space  $O(n^2 \log(n))$ , and is the first practical one for  $n \geq 256$ . Hence, the third assumption is satisfied too. So the attack can be applied to the CBKAP implementation.

Myasnikov and Ushakov [101] attack the scheme of Anshel, Anshel, Goldfeld and Lemieux from a different direction. Anshel, Anshel, Goldfeld and Lemieux [6] discuss the security of their scheme and indicate that if the conjugator z generated randomly by the TTP algorithm is known, then one can attack their scheme by an efficient linear attack, which can reveal the shared key of the parties. The problem of recovering the exact z seems like a very difficult mathematical problem since it reduces to solving the system of equations:

$$\begin{cases} w_1 = \Delta^{2p_1} z w_1' z^{-1} \\ \vdots \\ w_k = \Delta^{2p_k} z w_k' z^{-1} \\ v_1 = \Delta^{2r_1} z v_1' z^{-1} \\ \vdots \\ v_k = \Delta^{2r_k} z v_k' z^{-1} \end{cases}$$

which has too many unknowns, since only the left hand sides are known. Hence, it might be difficult to find the original z.

The attack of Myasnikov and Ushakov is a variant of the length-based attack. It is based on the observation that actually any solution z' for the system of equations above can be used in a linear attack on the scheme. Hence, they start by recovering the powers of  $\Delta$  which were added, so one can peel the  $\Delta^{2p}$  part. In the next step, they succeed in revealing the conjugator z (or any equivalent solution z').

Experimental results with instances of the TTP protocol generated using |z|=50 (which is almost three times greater than the suggested value) showed 100% success rate. They indicate that the attack may fail when the length of z is large relative to the length of  $\Delta^2$  (for more details, see [101, Section 3.4]).

Chowdhury [27] shows that the suggested implementation of the Algebraic Eraser scheme to the braid group (the TTP protocol) is actually based on the Multiple Simultaneous Conjugacy Search Problem, and then it can be cracked. He gives some algorithms for attacking the implementation.

It will be interesting to continue the research on the Algebraic Eraser key-agreement scheme in several directions:

(1) **Cryptosystem direction**: Can one suggest a different distribution for the choice of keys, so the cryptosystem can resist the attack of

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Kalka-Teicher-Tsaban?

- (2) **Cryptosystem direction**: Can one suggest a different implementation (different groups, etc.) for the Algebraic Eraser scheme which can resist the attack of Kalka-Teicher-Tsaban?
- (3) **Cryptanalysis direction**: Can the *usual* length-based approach [48] be applied to attack the TTP protocol?
- (4) **General**: One should perform a rigorous analysis of the algorithm of Kalka-Teicher-Tsaban for the Membership Search Problem in generic permutation groups (see [69, Section 8]).

## 1.9.5. Cryptosystems based on the decomposition problem and the triple decomposition problem

This section deals with two cryptosystems which are based on different variants of the decomposition problem: Given  $a, b = xay \in G$ , find x, y.

Shpilrain and Ushakov [113] suggest the following protocol, which is based on the decomposition problem:

Protocol 1.29.

Public key:  $w \in G$ .

Alice: chooses an element  $a_1 \in G$  of length  $\ell$ , chooses a subgroup of the centralizer  $C_G(a_1)$ , and publishes its generators  $A = \{\alpha_1, \ldots, \alpha_k\}$ .

*Bob*: chooses an element  $b_2 \in G$  of length  $\ell$ , chooses a subgroup of  $C_G(b_2)$ , and publishes its generators  $B = \{\beta_1, \dots, \beta_m\}$ .

Alice: chooses a random element  $a_2 \in \langle B \rangle$  and sends publicly the normal form  $P_A = N(a_1wa_2)$  to Bob.

*Bob*: chooses a random element  $b_1 \in \langle A \rangle$  and sends publicly the normal form  $P_B = N(b_1 w b_2)$  to Alice.

Shared secret key:  $K_A = a_1 P_B a_2 = b_1 P_A b_2 = K_B$ .

Since  $a_1b_1 = b_1a_1$  and  $a_2b_2 = b_2a_2$ , we indeed have  $K = K_A = K_B$ , the shared secret key. Alice can compute  $K_A$  and Bob can compute  $K_B$ .

They suggest the following values of parameters for the protocol:  $G = B_{64}$ ,  $\ell = 1024$ . For computing the centralizers, Alice and Bob should use the algorithm from [45], but actually they have to compute only some elements from them and not the whole sets.

Two key-exchange protocols which are based on a variant of the decomposition problem have been suggested by Kurt [75]. We describe here the second protocol which is an extension of the protocol of Shpilrain and Ushakov to the *triple decomposition problem*:

**Problem 1.10.** Given  $v = x_1^{-1}a_2x_2$ , find  $x_1 \in H$ ,  $a_2 \in A$  and  $x_2 \in H'$  where  $H = C_G(g_1, \ldots, g_{k_1}), H' = C_G(g'_1, \ldots, g'_{k_2})$ , and A is a subgroup of G given by its generators.

Here is Kurt's second protocol (his first protocol is similar): Let G be a non-commutative monoid with a large number of invertible elements.

Protocol 1.30.

Alice: picks two invertible elements  $x_1, x_2 \in G$ , chooses subsets  $S_{x_1} \subseteq C_G(x_1)$  and  $S_{x_2} \subseteq C_G(x_2)$ , and publishes  $S_{x_1}$  and  $S_{x_2}$ .

*Bob:* picks two invertible elements  $y_1, y_2 \in G$ , chooses subsets  $S_{y_1} \subseteq C_G(y_1)$  and  $S_{y_2} \subseteq C_G(y_2)$ , and publishes  $S_{y_1}$  and  $S_{y_2}$ .

Alice: chooses random elements  $a_1 \in G$ ,  $a_2 \in S_{y_1}$  and  $a_3 \in S_{y_2}$  as her private keys. She sends Bob publicly (u, v, w) where  $u = a_1x_1$ ,  $v = x_1^{-1}a_2x_2$ ,  $w = x_2^{-1}a_3$ .

*Bob*: chooses random elements  $b_1 \in S_{x_1}$ ,  $b_2 \in S_{x_2}$  and  $b_3 \in G$  as his private keys. He sends Alice publicly (p,q,r) where  $p=b_1y_1,\ q=y_1^{-1}b_2y_2,\ r=y_2^{-1}b_3$ .

Shared secret key:  $K = a_1b_1a_2b_2a_3b_3$ .

Indeed, K is a shared key, since Alice can compute  $a_1pa_2qa_3r = a_1b_1a_2b_2a_3b_3$  and Bob can compute  $ub_1vb_2wb_3 = a_1b_1a_2b_2a_3b_3$ .

As parameters, Kurt suggests to use  $G = B_{100}$  and each secret key should be of length 300 Artin generators.

Chowdhury [26] attacks the two protocols of Kurt, by observing that by some manipulations one can gather the secret information by solving only the Multiple Simultaneous Conjugacy Search Problem. Hence, the security of Kurt's protocols is based on the solution of the Multiple Simultaneous Conjugacy Search Problem. Since the Multiple Simultaneous Conjugacy Search Problem can be attacked by several methods, Chowdhury has actually shown that Kurt's protocols are not secure.

Although Shpilrain and Ushakov indicate that their key-exchange scheme resists length-based attack, it will be interesting to check if this indeed is the situation. Also, it is interesting to check if one can change the secrets of Kurt's protocols in such a way that it cannot be revealed by just solving the Simultaneous Conjugacy Search Problem. If such a change exists, one should check if the new scheme resists length-based attacks.

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#### 1.10. Future directions I: Alternate distributions

In this section and in the next section, we discuss some more future directions of research in this area and related areas. This section deals the interesting option of changing the distribution of the generators. In this way, one can increase the security of cryptosystems which are vulnerable when assuming a standard distribution. In the next section, we deal with some suggestions of cryptosystems which are based on different non-commutative groups, apart from the braid group.

For overcoming some of the attacks, one can try to change the distribution of the generators. For example, one can require that if the generator  $\sigma_i$  appears, then in the next place we give more probability for the appearance of  $\sigma_{i\pm 1}$ . In general, such a situation is called a *Markov walk*, i.e. the distribution of the choice of the next generator depends on the choice of the current chosen generator.

A work in this direction is the paper of Maffre [88]. After suggesting a deterministic polynomial algorithm that reduces the Conjugacy Search Problem in braid group (by a partial factorization of the secret), he proposes a new random generator of keys which is secure against his attack and the one of Hofheinz and Steinwandt [65].

This situation appears also in the Algebraic Eraser scheme (Section 1.9.4). The attack of Kalka, Teicher and Tsaban [69] assumes that the distribution of the generators is standard. They indicate that if the distribution is not standard, it is possible that the attack fails.

# 1.11. Future directions II: Cryptosystems based on different non-commutative groups

The protocols presented here for the braid groups can be applied to other non-commutative groups, so the natural question here is:

**Problem 1.11.** Can one suggest a different non-commutative group where the existing protocols on the braid group can be applied, and the cryptosystem will be secure?

We survey here some suggestions.

#### 1.11.1. Thompson group

When some of the cryptosystems on the braid groups were attacked, it was natural to look for different groups, with a hope that a similar cryptosystem on a different group will be more secure and more successful. The Thompson group is a natural candidate for such a group: there is a normal form which can computed efficiently, but the decomposition problem seems difficult. On this base, Shpilrain and Ushakov [112] suggest a cryptosystem.

In this section, we will define the Thompson group, the Shpilrain-Ushakov cryptosystem, and we discuss its cryptanalysis.

#### 1.11.1.1. Definitions and the Shpilrain-Ushakov cryptosystem

Thompson's group F is the infinite non-commutative group defined by the following generators and relations:

$$F = \langle x_0, x_1, x_2, \dots \mid x_i^{-1} x_k x_i = x_{k+1} \quad (k > i) \rangle$$

Each  $w \in F$  admits a unique normal form [22]:

$$w = x_{i_1} \cdots x_{i_r} x_{i_t}^{-1} \cdots x_{i_1}^{-1},$$

where  $i_1 \leq \cdots \leq i_r$ ,  $j_1 \leq \cdots \leq j_t$ , and if  $x_i$  and  $x_i^{-1}$  both occur in this form, then either  $x_{i+1}$  or  $x_{i+1}^{-1}$  occurs as well. The transformation of an element of F into its normal form is very efficient [112].

We define here a natural length function on the Thompson group:

**Definition 1.31.** The normal form length of an element  $w \in F$ , LNF(w), is the number of generators in its normal form: If  $w = x_{i_1} \cdots x_{i_r} x_{j_t}^{-1} \cdots x_{j_1}^{-1}$  is in normal form, then LNF(w) = r + t.

Shpilrain and Ushakov [112] suggest the following key-exchange protocol based on the Thompson group:

Protocol 1.32.

Public subgroups: A, B, W of F, where ab = ba for all  $a \in A, b \in B$ 

Public key: a braid  $w \in W$ .

Private keys: Alice:  $a_1 \in A, b_1 \in B$ ; Bob:  $a_2 \in A, b_2 \in B$ .

Alice: Sends Bob  $u_1 = a_1 w b_1$ .

Bob: Sends Alice  $u_2 = b_2 w a_2$ 

Shared secret key:  $K = a_1b_2wa_2b_1$ 

K is a shared key since Alice can compute  $K = a_1u_2b_1$  and Bob can compute  $K = b_2u_1a_2$ , and both are equal to K since  $a_1, a_2$  commute with  $b_1, b_2$ .

Here is a suggestion for implementing the cryptosystem [112]: Fix a natural number  $s \geq 2$ . Let  $S_A = \{x_0x_1^{-1}, \ldots, x_0x_s^{-1}\}$ ,  $S_B = \{x_{s+1}, \ldots, x_{2s}\}$  and  $S_W = \{x_0, \ldots, x_{s+2}\}$ . Denote by A, B and W the subgroups of F generated by  $S_A$ ,  $S_B$ , and  $S_W$ , respectively. A and B commute elementwise, as required.

The keys  $a_1, a_2 \in A$ ,  $b_1, b_2 \in B$  and  $w \in W$  are all chosen of normal form length L, where L is a fixed integer, as follows: Let X be A, B or W. Start with the unit word, and multiply it on the right by a (uniformly) randomly selected generator, inverted with probability  $\frac{1}{2}$ , from the set  $S_X$ . Continue this procedure until the normal form of the word has length L.

For practical implementation of the protocol, it is suggested in [112] to use  $s \in \{3, 4, ..., 8\}$  and  $L \in \{256, 258, ..., 320\}$ .

### 1.11.1.2. Length-based attack

We present some attacks on the Ushakov-Shpilrain cryptosystem.

As mentioned before, the length-based attack is applicable for any group with a reasonable length function. Ruinskiy, Shamir and Tsaban [108] applied this attack to the Thompson group.

As before, the basic length-based attack without memory always fails for the suggested parameters. If we add the memory approach, there is some improvement: for a memory of size 1024, there is 11% success. But if the memory is small (up to 64), even the memory approach always fails. They suggest that the reason for this phenomenon (in contrast to a significant success for the length-based attack with memory on the braid group) is that the braid group is much closer to the free group than the Thompson group, which is relatively close to an abelian group.

Their improvement is trying to avoid repetitions. The problem is that many elements return over and over again, and hence the algorithm goes into loops which make its way to the solution much difficult. The solution of this is holding a list of the already-checked conjugators, and when we generate a new conjugator, we check in the list if it has already appeared (this part is implemented by a hash table). In case of appearance, we just ignore it. This improvement increases significantly the success rate of the algorithm: instead of 11% for a memory of size 1024, we now have 49.8%, and instead of 0% for a memory of size 64, we now have 24%.

In the same paper [108], they suggest some more improvements for the length-based algorithm. One of their reasons for continuing with the improvements is the following interesting fact which was pointed out by Shpilrain [111]: there is a very simple fix for key-agreement protocols that are broken in probability less than p: Agree on k independent keys in parallel, and XOR them all to obtain the shared key. The probability of breaking the shared key is at most  $p^k$ , which is much smaller.

In a different paper, Ruinskiy, Shamir and Tsaban [116] attack the key agreement protocols based on non-commutative groups from a different direction: by using functions that estimate the distance of a group element to a given subgroup. It is known that in general the Membership Problem is hard, but one can use some heuristic approaches for determining the distance of an element to a given subgroup, e.g., to count the number of generators which are not in the subgroup.

They test it against the Shpilrain-Ushakov protocol, which is based on Thompson's group F, and show that it can break about half the keys within a few seconds.

#### 1.11.1.3. Special attack by Matucci

Some interesting special attack for the Ushakov-Shpilrain cryptosystem can be found in Kassabov and Matucci [91] and Matucci [90].

#### 1.11.2. Polycyclic groups

Eick and Kahrobaei [41] suggest to use polycyclic groups as the basis of a cryptosystem. These groups are a natural generalization of cyclic groups, but they are much more complex in their structure than cyclic groups. Hence, their algorithmic theory is more difficult and thus it seems promising to investigate classes of polycyclic groups as candidates to have a more substantial platform perhaps more secure.

Here is one presentation for polycyclic groups:

$$\langle a_1, \dots, a_n \mid a_i^{-1} a_j a_i = w_{ij}, a_i a_j a_i^{-1} = v_{ij}, a_k^{r_k} = u_{kk}, \text{ for } 1 \leq i < j \leq n, \quad k \in I \rangle$$

where  $I \subseteq \{1, \ldots, n\}$  and  $r_i \in \mathbb{N}$  if  $i \in I$  and the right hand sides  $w_{ij}, v_{ij}, u_{jj}$  of the relations are words in the generators  $a_{j+1}, \ldots, a_n$ . Using induction, it is straightforward to show that every element in the group defined by this presentation can be written in the form  $a_1^{e_1} \cdots a_n^{e_n}$  with  $e_i \in \mathbb{Z}$  and  $0 \le e_i < r_i$  if  $i \in I$  (see [119] for more information).

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Eick and Kahrobaei introduce a Diffie-Hellman-type key-exchange which is based on the polycyclic group. As in the braid groups' case, the cryptosystem is based on the fact that the word problem can be solved effectively in polycyclic groups, while the known solutions to the conjugacy problem are far less efficient. For more information, see [41].

In a different direction, Kahrobaei and Khan [68] introduce a non-commutative key-exchange scheme which generalizes the classical El-Gamal Cipher [42] to polycyclic groups.

### 1.11.3. Miller groups

Mahalanobis [89] suggested some Diffie-Hellman-type exchange key on Miller Groups [92], which are groups with an abelian automorphism group.

#### 1.11.4. Grigorchuk group

Garzon and Zalcstein [52] suggest a cryptosystem which is based on the word problem of the Grigorchuk group [62]. Both Petrides [105] and González-Vasco, Hofheinz, Martinez and Steinwandt [61] cryptanalyze this cryptosystem.

The Conjugacy Decision Problem in this group is also polynomial [85], so this problem cannot be served as a base for a cryptosystem.

# 1.11.5. Twisted conjugacy problem in the semigroup of $2 \times 2$ matrices over polynomials

Shpilrain and Ushakov [114] suggest an authentication scheme which is based on the *twisted conjugacy search problem*:

**Problem 1.12.** Given a pair of endomorphisms (i.e., homomorphisms into itself)  $\varphi, \psi$  of a group G and a pair of elements  $w, t \in G$ , find an element  $s \in G$  such that  $t = \psi(s^{-1})w\varphi(s)$  provided at least one such s exists.

Their suggested platform semigroup G is the semigroup of all  $2 \times 2$  matrices over truncated one-variable polynomials over  $\mathbb{F}_2$ , the field of two elements. For more details, see their paper.

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