GREEN CHICKEN PROBLEMS - NOVEMBER 15, 2014 – 36TH COMPETITION

NO CALCULATORS, CELL-PHONES, ...; WRITTEN BY S. J. MILLER

Question 1: An integer greater than 1 is prime if its only divisors are 1 and itself, otherwise it is called composite (the number 1 is neither prime nor composite, but instead is called a unit). While we have known for over 2000 years that there are infinitely many primes, it was only within the last 200 years that a proof was given that the arithmetic progression $\{an+b\}_{n=1}^{\infty}$ contains infinitely many primes if and only if the positive integers a and b share no prime divisors. Prove that for *every* positive integer b relatively prime to 2014 that the arithmetic progression $\{2014n+b\}_{n=1}^{\infty}$ contains infinitely many composite numbers.

Question 2: There are 10 horses, named horse $1, 2, \dots, 10$; they get their name from how many minutes it takes them to run around the circular race track (horse k runs it in k minutes). At time zero all the horses are together at the start of the race. The horses start running, and keep running around the circular track at their constant speeds. (a) Will all the horses align again at the starting point before minute 2014? (b) If T is the smallest time (after the starting time of zero) such that ***at least half*** of all the horses are again at the starting point, what is T?

Question 3: Conway's see-and-say (or the look-and-say) sequence has fascinated people for years, and is a fun, non-standard example of a sequence. Starting with $a_1 = 1$, we define a_{n+1} as the sequence obtained by saying the previous sequence aloud. The first few terms are 1, 11 (one one), 21 (two ones), 1211 (one two, one one), 111221 (one one, one two, two ones), 312211 (three ones, two twos, one one). Greenie noticed something very interesting. The largest number in the 2014$^{\text{th}}$ term is 36, and 2014 is the 36$^{\text{th}}$ Green Chicken contest! Is Greenie correct, or did she make a mistake? If she is wrong what is the largest number in the 2014$^{\text{th}}$ term? Prove all your claims.

Question 4: We say a positive integer n is k-ladderful if there are k consecutive primes $p_1, p_2, \dots, p_k$ such that $p_1 p_2^2 p_3^3 \cdots p_k^k$ divide n, and no higher power of these primes divide n (thus p_1^2 does not divide n and so on); for example, 2263800 is 3-ladderful as it equals $2^3 \cdot 3 \cdot 5^2 \cdot 7^3 \cdot 11$ (so the ladder comes from 3, 5 and 7). Let $\mathcal{L}_{2014}(x)$ be the set of all 2014-ladderful numbers at most x. Prove a positive percentage of all integers are 2014-ladderful; this means $\lim_{x\to\infty} \#\mathcal{L}_{2014}(x)/x > 0$.

Question 5: It wouldn't be a math competition if there weren't at least one problem involving the year, though this does seem to be getting a little excessive! Consider all sets of 2014 distinct positive integers, where each integer is at least 36. For each set, look at all the products of four distinct elements. (a) What is the largest number of distinct products? (b) What is the fewest number of distinct products? Prove your claims.

Question 6: The Green Chicken is standing at the origin of the real line. She flips a fair coin 2014 times; each toss is independent of the others and each toss has a 50% chance of being a head and a 50% chance of being a tail. If she tosses a head on the k^{th} toss she moves k units to the right, while if she tosses a tail on the k^{th} toss she moves k units to the left. What is the probability the Green Chicken returns to the origin after tossing the coin 2014 times?

Here are the problems and solutions to the 2014 Green Chicken Competition.

Question 1: An integer greater than 1 is prime if its only divisors are 1 and itself, otherwise it is called composite (the number 1 is neither prime nor composite, but instead is called a unit). While we have known for over 2000 years that there are infinitely many primes, it was only within the last 200 years that a proof was given that the arithmetic progression $\{an+b\}_{n=1}^{\infty}$ contains infinitely many primes if and only if the positive integers a and b share no prime divisors. Prove that for *every* positive integer b relatively prime to 2014 that the arithmetic progression $\{2014n+b\}_{n=1}^{\infty}$ contains infinitely many composite numbers.

Solution 1: As it requires no additional work, we prove this problem in greater generality and study all arithmetic progressions of the form $\{an+b\}$ with a, b positive integers. If a and b share a common factor $d > 1$, it's easy, as every term is a multiple of d. If $b > 1$ then consider the subsequence of n with $n = bm$; all those numbers are multiples of b. The hard case is when $b = 1$, but we can solve this directly. It's good to look at a special subsequence again; let's take n to be a nice function of a, say $n = f(m, a)$. This gives us terms of the form $f(m, a)a+1$. If $f(m, a) = 1$ then we get $a+1$, so this suggests trying $f(m, a) = g(m)(a+1)+1$, as this way we'll get $an+1 = g(m)(a+1)a+a+1 = (a+1)(ag(m)+1)$. We can take any increasing integer valued function g; let's make life easy and take $g(m) = m$. To recap: when $b = 1$ look at n of the form $n = (a+1)m+1$ as $an+1 = a(a+1)m+a+1 = (a+1)(am+1)$. For another choice, look at n of the form $n = am^2 + 2m$, as $an + 1 = a^2m^2 + 2am + 1 = (am + 1)^2$.

Question 2: There are 10 horses, named horse $1, 2, \dots, 10$; they get their name from how many minutes it takes them to run around the circular race track (horse k runs it in k minutes). At time zero all the horses are together at the start of the race. The horses start running, and keep running around the circular track at their constant speeds. (a) Will all the horses align again at the starting point before minute 2014? (b) If T is the smallest time (after the starting time of zero) such that ***at least half*** of all the horses are again at the starting point, what is T?

Solution 2: (a) We want all the horses at the starting point together. The answer to that is the least common multiple of $1, 2, \dots, 10$. All that matters is the highest power of primes dividing these numbers: we have $2^3, 3^2, 5$ and 7. Thus the answer is their product, or 2520. As this exceeds 2014, the horses do not all align at the starting point before minute 2014.

(b) Now we don't need all the horses, but only half. We can do the five smallest numbers: $1, 2, 3, 4, 5$; the least common multiple is the product of $3, 4$ and 5 or 60. If we looked at powers of 2, we would have $1, 2, 4, 8$ and this is not enough. We could throw in 3, which gives us a time of 24. Things are worse if we throw in anyone else. We can do a little better, though; once we put 3 in the mix we might as well put 6 in (as we have 2) and drop 8. This leads to horses $1, 2, 3, 4$ and 6; the answer is now the product of 3 and 4, or 12. A little work shows there is nothing better. Using a horse whose time is a multiple of 5 or 7 leads to something greater than 12 (and if 5 is bad, 10 can't be better!). We have similar problems if horses 8 or 9 are involved.

Question 3: Conway's see-and-say (or the look-and-say) sequence has fascinated people for years, and is a fun, non-standard example of a sequence. Starting with $a_1 = 1$, we define a_{n+1} as the sequence obtained by saying the previous sequence aloud. The first few terms are 1, 11 (one one), 21 (two ones), 1211 (one two, one one), 111221 (one one, one two, two ones), 312211 (three ones, two twos, one one). Greenie noticed something very interesting. The largest number in the 2014$^{\text{th}}$ term is 36, and 2014 is the 36$^{\text{th}}$ Green Chicken contest! Is Greenie correct, or did she make a mistake? If she is wrong what is the largest number in the 2014$^{\text{th}}$ term? Prove all your claims.

Solution 3: Greenie is sadly wrong; the largest number is 3. The proof follows from two facts: (1) once a number appears in a term, it appears in all subsequent terms; (2) no number 4 or more appears.

To prove the first claim, notice that once we have a number we have to say it every time afterwards. Thus once we have a 3, all subsequent terms have a 1, 2 and 3. We just need to prove we cannot have a 4 or larger number. We give the proof that there is never a 4; the proof that there are no 5s, 6s, 7s, ... follows similarly. How can we get a 4? We need to have a term of the form $xyyyyz$, where y is different from x and z (though x and z can be the same). Imagine we had $x1111z$. We got this by reading the previous line aloud. If the x goes with the first 1, it would be read, "we have x ones, one one, ..."; however, we would have then combined the x with at least the second 1. Thus we cannot have $x1111z$. If the x doesn't go with the 1 then we read, "one one, one one, ..."; however, we would have combined the first and third ones in this case. Thus we cannot ever have $x1111y$. A similar analysis holds for $xyyyyz$, and notice the proof proceeds identically if we had $xyyyyyz$, $xyyyyyyz$ and so on.

Question 4: We say a positive integer n is k-ladderful if there are k consecutive primes $p_1, p_2, \dots, p_k$ such that $p_1 p_2^2 p_3^3 \cdots p_k^k$ divide n, and no higher power of these primes divide n (thus p_1^2 does not divide n and so on); for example, 2263800 is 3-ladderful as it equals $2^3 \cdot 3 \cdot 5^2 \cdot 7^3 \cdot 11$ (so the ladder comes from 3, 5 and 7). Let $\mathcal{L}_{2014}(x)$ be the set of all 2014-ladderful numbers at most x.

Prove a positive percentage of all integers are 2014-ladderful; this means $\lim_{x\to\infty} \#\mathcal{L}_{2014}(x)/x > 0$.

Solution 4: It suffices to consider 2014-ladderful numbers where the primes are the first 2014 primes. Let $P = 2 \cdot 3^2 \cdots p_{2014}^{2014}$. For each n, consider $P(nP+1)$. As P and $nP+1$ are relatively prime (if d divides both then d divides $nP+1$ minus $n \cdot P$ or 1), these numbers are divisible by just the right powers of the first 2014 primes. Notice this is an arithmetic progression – it's all numbers of the form $nP^2 + P$, and thus in the limit the fraction of numbers of this form is $1/P^2$. This proves a positive percentage of numbers up to x are 2014-ladderful.

Question 5: It wouldn't be a math competition if there weren't at least one problem involving the year, though this does seem to be getting a little excessive! Consider all sets of 2014 distinct positive integers, where each integer is at least 36. For each set, look at all the products of four distinct elements. (a) What is the largest number of distinct products? (b) What is the fewest number of distinct products? Prove your claims.

Solution 5: The number of ways to choose 4 numbers from 2014 is $\binom{2014}{4}$. If our 2014 numbers are all distinct primes then no two products can be equal (by the Fundamental Theorem of arithmetic), so this is the largest number of distinct products and a realization of it (fyi, $\binom{2014}{4} = 683,489,813,501$; the 4 is because this is the fourth year I've written problems for the Green Chicken Contest).

For (b), every time we have distinct factors we have an opportunity for products to differ. The fewest distinct products will happen when our numbers are of the form $p, p^2, \dots, p^{2014}$ for some prime p. We are thus left with counting how many distinct sums there are when we take four distinct elements of $\{1, 2, \dots, 2014\}$ (we could have started our exponents at 0, but it doesn't really matter). The smallest the sum can be is $1 + 2 + 3 + 4 = 10$, the largest it can be is $2011 + 2012 + 2013 + 2014 = 8050$. All that is left is to show we can obtain every number in between; if we do the answer is $8050 - 10 + 1 = 8041$. Consider the following chain: $1 + 2 + 3 + 4 = 10$, $1 + 2 + 3 + 5 = 11$, $1 + 2 + 3 + 6 = 12$, $\dots$, $1 + 2 + 3 + 2014 = 2020$, $1 + 2 + 4 + 2014 = 2021$, $\dots$, $1+2+2013+2014 = 4030$, $1+3+2013+2014 = 4031$, $\dots$, $1+2012+2013+2014 = 6040$, $2+2012+2013+2014 = 6041$, $\dots$, $2011 + 2012 + 2013 + 2014 = 8050$. Thus all the possible sums can be attained.

Question 6: The Green Chicken is standing at the origin of the real line. She flips a fair coin 2014 times; each toss is independent of the others and each toss has a 50% chance of being a head and a 50% chance of being a tail. If she tosses a head on the k^{th} toss she moves k units to the right, while if she tosses a tail on the k^{th} toss she moves k units to the left. What is the probability the Green Chicken returns to the origin after tossing the coin 2014 times?

Solution 6: The probability is zero. One way to see this is by a parity argument. Let ϵ_k be 1 if the k^{th} toss was a head and -1 if it was a tail. The Green Chicken's location after 2014 tosses is $\epsilon_1 \cdot 1 + \epsilon_2 \cdot 2 + \cdots + \epsilon_{2014} \cdot 2014$. This is either an even number or an odd number, and the *only* way to return to the origin is to have moved an even distance. Notice, however, that our sum is *always* odd. To see this, we observe that the parity is unchanged if we replace each ϵ_k with 1 (if it was already 1 there is no change, while if it is -1 we have just added an even number). Thus the parity of the amount Greenie has moved is the same as the parity of $1+2+\cdots+2014$. As the sum of the first n integers is $n(n+1)/2$ we see this is $2014 \cdot 2015/2 = 1007 \cdot 2015$, which is odd (the actual value is 2,029,105). Thus there is *no* chance of returning after 2014 tosses.