GREEN CHICKEN EXAM - NOVEMBER 2012

GREEN CHICKEN AND STEVEN J. MILLER

Question 1: The Green Chicken is planning a surprise party for his grandfather and grandmother. The sum of the ages of the grandmother and grandfather is 154. The grandmother is twice as old as the grandfather was when the grandmother was as old as the grandfather is now. Who is older, the grandmother or the grandfather, and by how many years?

Question 2: A row in a classroom has n seats. Let s_n be the number of ways non-empty sets of students can sit in the row so that no student is sitting next to another student. (For example, a row of three seats could contain a single student in any of the seats, or a pair of students in the two outer seats. Thus $s_3 = 4$.) Find s_{15}.

Question 3: A theorem of Relue states that there are constants a_0, a_1, a_2, a_3 and a_4 such that

$$\cos(4x) \ = \ a_4 \cos^4(x) + a_3 \cos^3(x) + a_2 \cos^2(x) + a_1 \cos(x) + a_0.$$

What is $a_0 + a_1 + a_2 + a_3 + a_4$?

Question 4: Consider the soon to be popular new game of random M&Ms. The way the game works is some number, k, of M&Ms is chosen. Middlebury and Williams each start off with k M&Ms. Each team independently tosses a fair coin at the same time; any team that tosses a head eats one of their M&Ms, while any team that tosses a tail does not eat. The first team to eat all their M&Ms wins, unless both eat their last M&M at the same time, in which case the game is a tie. If $k = 3$ what is the probability of a tie? Express your answer as a fraction in lowest terms.

Question 5: Life is full of disappointments, of partial results to what we want. For example, Chebyshev proved that there exist constants A and B, $.8 < A < 1 < B < 1.2$, such that

$$\frac{Ax}{\log x} \ < \ \pi(x) \ < \ \frac{Bx}{\log x},$$

where $\pi(x)$ is the number of primes at most x. He wanted to prove that $\pi(x)$ was $x/\log x$ (this is true, and is the celebrated Prime Number Theorem), but had to settle for this. Speaking of settling, using Chebyshev's Theorem prove for any integer M there exists an even integer $2k$ such that there are at least M primes p with $p + 2k$ also prime. *Unfortunately my proof has $2k$ depending on M. If you can solve this with $2k$ independent of M, you'll have just proved the Twin Prime Conjecture, namely, there are infinitely many primes p such that $p + 2$ is also prime! You'll also automatically bring back the Green Chicken for your school.*

Question 6: A graph G is a collection of vertices V and edges E connecting pairs of vertices. Consider the following graph. The vertices are the integers $\{2, 3, 4, \dots, 2012\}$. Two vertices are connected by an edge if they share a divisor greater than 1; thus 30 and 1593 are connected by an edge as 3 divides each, but 30 and 49 are not.

The coloring number of a graph is the smallest number of colors needed so that each vertex is colored ***and*** if two vertices are connected by an edge, then those two vertices are not colored the same.

The Brown Chicken says the coloring number of this graph is at most 9. Prove she is wrong. Find the correct coloring number (if you can't find the exact answer, find upper and / or lower bounds). Prove your answers.

Hint: A complete graph on n vertices is a graph with n vertices such that any pair of two vertices are connected by an edge. It might be useful to note that the coloring number of the complete graph on n vertices is n.

Here are the problems and solutions.

Question 1: The Green Chicken is planning a surprise party for his grandfather and grandmother. The sum of the ages of the grandmother and grandfather is 154. The grandmother is twice as old as the grandfather was when the grandmother was as old as the grandfather is now. Who is older, the grandmother or the grandfather, and by how many years?

Solution 1: The difficulty in this problem is translating the text to equations and solving. There are several ways to solve it. Let x be the grandfather's current age, and y the grandmother's current age. Since the sum of their ages is 154, we get

$$x + y \ = \ 154.$$

The second condition is harder to translate, but becomes

$$y \ = \ 2\left[x - (y - x)\right].$$

Now it's easy. The second equation becomes $y = 2(2x - y)$. As $x = 154 - y$ (so $2x = 308 - 2y$) we find

$$y \ = \ 2(308 - 2y - y),$$

or

$$7y \ = \ 616$$

which implies $y = 88$ and thus $x = 66$, so the grandmother is 22 years older.

Here is a longer solution.

Let's say the grandmother was born in year m and the grandfather in year f, and currently it is year y. The grandmother's age is $y - m$ and the grandfather's is $y - f$. We have three unknowns: m, f, y. We need to solve for these. Note, of course, that there is a certain indefiniteness that helps us. We can add any amount to *all* of the unknowns without changing anything, as all that really matters are the ages. Thus, we can take any of the variables to be zero. This will give us our third equation; we'll choose $f = 0$ (though we could have chosen $y = 2012$ if we wished).

The first condition becomes $(y - m) + (y - f) \ = \ 154$, which we rewrite as

$$2y - m - f \ = \ 154.$$

The second condition is $y - m \ = \ 2\left((y - f) + (m - f)\right)$, which is the same as

$$y + 3m - 4f \ = \ 0.$$

As promised, we now take

$$f \ = \ 0$$

to get our three equations. Collecting, we see we have to solve the system

$$\begin{aligned} 2y - m - f &= 154 \\ y + 3m - 4f &= 0 \\ f &= 0. \end{aligned}$$

Using $f = 0$ our system reduces to

$$\begin{aligned} 2y - m &= 154 \\ y + 3m &= 0. \end{aligned}$$

There are lots of ways to solve this. A simple one is to multiply the second equation by 2 and then subtract:

$$\begin{aligned} 2y - m &= 154 \\ 2y + 6m &= 0 \end{aligned}$$

implies

$$-7m \ = \ 154,$$

which implies $m = -22$.

Now that we know $m = -22$ we can use $y + 3m = 0$ to get $y = 66$.

Thus the grandmother's current age is $y - m = 66 - (-22) = 88$, while the grandfather is $y - f = 66 - 0 = 66$. Thus the grandmother is older by 22 years.

Question 2: A row in a classroom has n seats. Let s_n be the number of ways non-empty sets of students can sit in the row so that no student is sitting next to another student. (For example, a row of three seats could contain a single student in any of the seats, or a pair of students in the two outer seats. Thus $s_3 = 4$.) Find s_{15}.

Solution 2: We find a recurrence relation for the s_n's. We solve for s_{n+2} in terms of the earlier terms. There are two possibilities for the position $n + 2$: we don't have anyone there, or we do.

If we don't have anyone, then we have to seat people in the remaining $n + 1$ spots, with the provision that at least one spot is chosen and no two chosen spots are adjacent. This is the definition of s_{n+1}.

If however we have someone sitting in seat $n+2$ then no one can sit in seat $n+1$. We now choose the number of ways to seat people in the first n spots subject to the provision that no two people are seated together. This is $s_n + 1$. **The reason we must add 1 is that it is okay to have no seats chosen, as in this case there's already a non-empty subset as we have someone in seat $n + 2$.**

We thus find the recurrence relation

$$s_{n+2} = s_{n+1} + s_n + 1.$$

We need some initial conditions to solve it. A little inspection gives $s_1 = 1$ and $s_2 = 2$ (note this does give $s_3 = 4$ as claimed). We then find $s_4 = 3 + 1 + 1 = 5$. We march down, with each term one more than the sum of the previous two:

$$1, 2, 4, 7, 12, 20, 33, 54, 88, 143, 232, 376, 609, 986, 1596.$$

Thus our answer is 1596.

This is sequence A001595 from Sloane's On-line Encyclopedia of Integer Sequences; see `http://oeis.org/A000071`.

Here is some Mathematica code to quickly generate the sequence.

```
a = 1;
b = 1;
For[n = 3, n <= 15, n++,
  {
   x = a + b + 1;
   Print[x];
   a = b;
   b = x;
   }];
```

The sequence is quite interesting. Note each term is *exactly* one less than a Fibonacci number; if F_n is the n^{th} Fibonacci number (with the sequence going 0, 1, 1, 2, 3, 5, 8, 13, $\dots$, with $F_0 = 0$), then $s_n = F_{n+2} - 1$. We can prove this. Consider

$$s_{n+2} = s_{n+1} + s_n + 1.$$

If we add 1 to both sides we get

$$s_{n+2} + 1 = s_{n+1} + 1 + s_n + 1.$$

Letting $t_n = s_n + 1$ we find

$$t_{n+2} = t_{n+1} + t_n;$$

note this is the relation the Fibonaccis satisfy!. Taking into account the initial conditions, we see that the t_n's are shifted Fibonacci numbers, and thus the s_n's are the shifted Fibonaccis minus one.

Noticing that our series is one less than the Fibonaccis suggests another proof. Let $q_n = s_n + 1$; we may interpret this as counting the number of valid configurations *but now allowing the empty configuration*. As we allow the empty configuration, our new series is one more than the initial. Now $q_1 = 2$, $q_2 = 3$ and $q_3 = 5$. We find $q_{n+2} = q_{n+1} + q_n$. We get this by considering $n+2$ seats. If the last is taken then the second to last cannot, and there are q_n valid configurations from this point onward. If we do not take the last seat, then there are q_{n+1} valid configurations from this point onward. We see we have a shifted Fibonacci sequence, and the rest of the analysis is similar to before. We find $q_{15} = 1597$, so $s_{15} = 1596$.

Question 3: A theorem of Relue states that there are constants a_0, a_1, a_2, a_3 and a_4 such that

$$\cos(4x) \ = \ a_4 \cos^4(x) + a_3 \cos^3(x) + a_2 \cos^2(x) + a_1 \cos(x) + a_0.$$

What is $a_0 + a_1 + a_2 + a_3 + a_4$?

Solution 3: Since the two sides are equal, we may take any value of x. The simplest is $x = 0$. As the cosine of zero is 1, we get the sum of the unknown coefficients is 1.

For completeness, we prove the formula. We start with the double angle identity:

$$\cos(2x) \ = \ \cos(x)\cos(x) - \sin(x)\sin(x).$$

Using $\sin^2(x) = 1 - \cos^2(x)$ we see that $\cos(2x) = 2\cos^2(x) - 1$. Replacing x with $2x$ gives $\cos(4x) = 2\cos^2(2x) - 1$. We now substitute for $\cos(2x)$ and find $\cos(4x) = 2(2\cos^2(x) - 1)^2 - 1$, or $\cos(4x)$ equals

$$2(4\cos 4(x) - 4\cos^2(x) + 1) - 1 \ = \ 8\cos^4(x) - 8\cos^2(x) + 1.$$

Thus $a_4 = 8 = -a_2$, $a_3 = 0 = a_1$ and $a_0 = 1$.

Question 4: Consider the soon to be popular new game of random M&Ms. The way the game works is some number, k, of M&Ms is chosen. Middlebury and Williams each start off with k M&Ms. Each team independently tosses a fair coin at the same time; any team that tosses a head eats one of their M&Ms, while any team that tosses a tail does not eat. The first team to eat all their M&Ms wins, unless both eat their last M&M at the same time, in which case the game is a tie. If $k = 3$ what is the probability of a tie? Express your answer as a fraction in lowest terms.

Solution 4: Let's let $x_{m,w}$ denote the probability of a tie when Middlebury has m M&M's and Williams has w. On each turn, there are four possibilities, each occurring with probability 1/4: only Middlebury eats an M&M, both eat, just Williams eats, or neither eats. Note, and this is the important observation, that if neither eats it's as if the turn didn't happen. Thus, we may define a turn as repeatedly tossing pairs of coins until one of the first three possibilities happen, and we see that one-third of the time only Middlebury eats an M&M on a turn, one-third of the time both eat, and one-third of the time neither eats.

We are thus considering a related game where on each turn either both schools lose one M&M, or exactly one school loses an M&M. As we start with both having 3 M&Ms, the game ends either after 3, 4, 5, or 6 moves. Why? Fastest it can be is 3 moves, with at least one school losing an M&M each turn. It can't take 7 turns or more, because by the Pigeon-Hole Principle each turn at least one school loses an M&M, and thus after 7 turns one would lose at least 4 M&Ms.

Let M denote a turn where only Middlebury loses an M&M, W a turn where only Williams does, and B a turn when both do. The probability of a tie is the sum of the probabilities of a tie with exactly 3, 4, 5 or 6 turns. It's convenient to break things up into the number of B's we have (0, 1, 2 or 3); note it can't be zero as the last term must be a B. The possible sequences (and their permutations) are:

- 3 B's: BBB: this is the only permutation, happens with probability $(1/3)^3$.
- 2 B's: BWMB: there are 6 permutations: we have to end with a B, but there are $3! = 6$ ways to permute B, W and M: the sum of the six probabilities here is $6 \cdot (1/3)^4$.
- 1 B: WWMMB: There are $\frac{4!}{2!2!} = 6$ ways to permute WWMM (if the four letters were distinct there would be 4!, but as the two W's are the same we divide by 2!, and similarly divide by 2! for the two M's): the sum of the six probabilities here is $6 \cdot (1/3)^5$.

Adding the three cases gives

$$(1/3)^3 + 6(1/3)^4 + 6(1/3)^5 \;=\; \frac{3+6+2}{81} \;=\; \frac{11}{81}.$$

It turns out this is the same as the following formula:

$$\sum_{k=0}^{3} \binom{6-2k+k}{k}\binom{6-2k}{3-k}(1/3)^{6-k+1},$$

which suggests a generalization to both starting with m M&M's (6 is $2m$, 3 is m except in 1/3, which is always 1/3).

Here is a more 'visual' solution, for $k = 4$ (to really highlight the method).

Consider the following chart (see Figure 1).

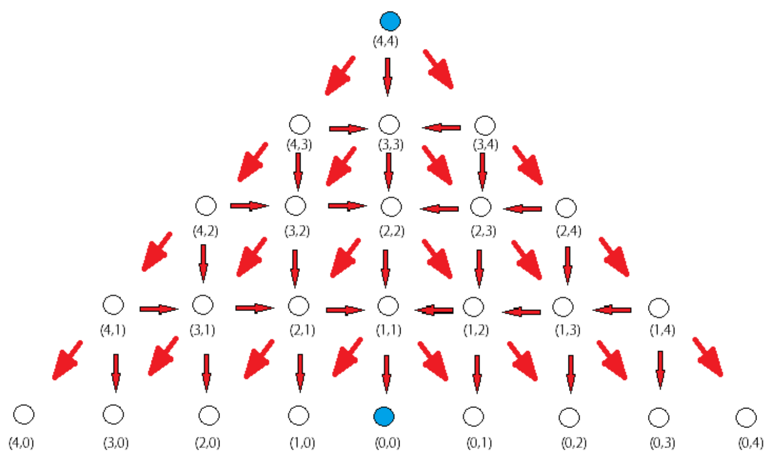

FIGURE 1. The M&M game when $k = 4$.

We start at (4,4) and have a tie if we end at (0,0). The game ends as soon as we reach any vertex in the last row. Each vertex has three arrows denoting the three possibilities; we have a 1/3 probability of taking any arrow. Thus, to find the probability of ending in a tie, we need to count how many paths there are from (4,4) to (0,0) and weight each path by $1/3^r$, where r is the number of arrows.

There is one path of length 4, which thus contributes $1/3^4$. There are 10 paths of length 5, contributing $5/3^5$. It starts getting harder to count all the paths of length six; there's a real danger of missing a few. Also, with long paths (such as those of length 7) we can start going down and to the left, then cross over into the right part, then cross back to the middle!

We can simplify things a bit. To get a tie, we have to reach (1,1), and from that point onward there's a 1/3 chance of a tie. Thus, we just have to find the probability of reaching (1,1) starting from (4,4).

We start with probability 1 of being at (4,4). We then see how this probability 'falls' down the tree. After one move, we're in each vertex of the second level with probability 1/3. See Figure 2.

We now continue. We take the two outer vertices with probability, and have the probability 'fall' down. Each sends one-third of its probability to three vertices. We have Figure 3.

We now take the vertex with probability 5/9 and send its probability to its three children, getting Figure 4.

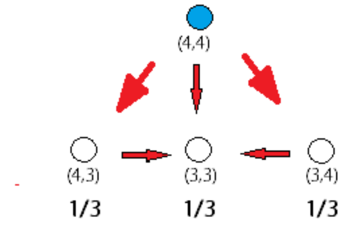

FIGURE 2. The M&M game when $k = 4$, going down one level.

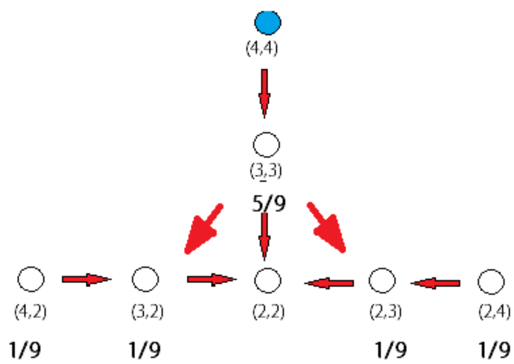

FIGURE 3. The M&M game when $k = 4$, removing probability from two outer.

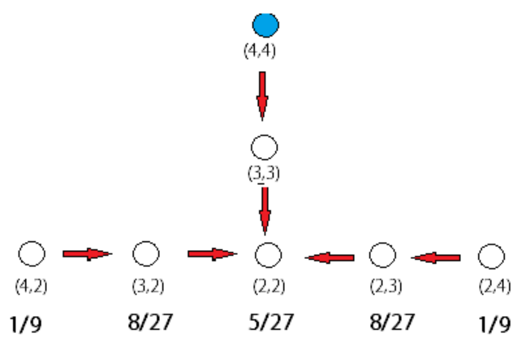

FIGURE 4. The M&M game when $k = 4$, removing probability from central vertex.

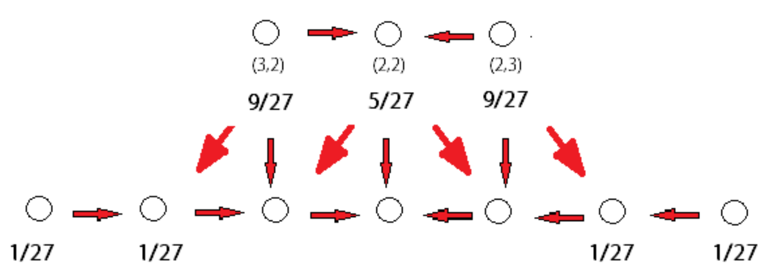

FIGURE 5. The M&M game when $k = 4$, removing probability from the (4,2) and (2,4) vertices.

We continue, and move the probability from the (4,2) and (2,4) vertices in Figure 5.

We now move the probability down from the (3,2) and (2,3) vertices; see Figure 6.

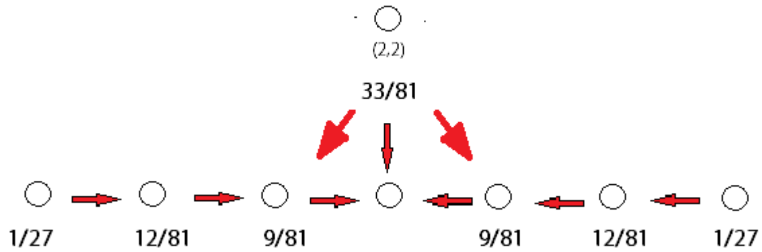

FIGURE 6. The M&M game when $k = 4$, removing probability from the (3,2) and (2,3) vertices.

We now move the probability down from the (2,2) vertex; see Figure 7.

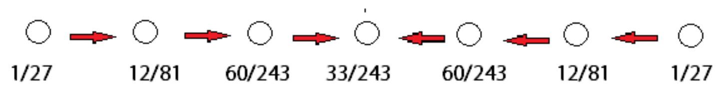

FIGURE 7. The M&M game when $k = 4$, removing probability from the (2,2) vertex.

All we have to do is deal with the final row. All that matters is the probability that makes it to the center; anything that falls lower doesn't contribute. Thus, the two probabilities of 1/27 at the end each send probabilities of 1/81 to three vertices, but the only one that matters is the one to the left (or right). Figure 8 shows what we now have.

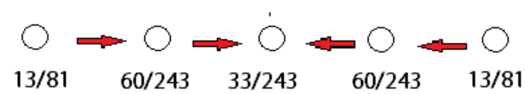

FIGURE 8. The M&M game when $k = 4$, removing probability from (4,1) and (1,4) vertices.

Continuing, we move the probability in again; Figure 9 shows the result.

FIGURE 9. The M&M game when $k = 4$, removing probability from (3,1) and (1,3) vertices.

We then continue and move the probabilities from the end vertices on the end, giving a probability of $\frac{1}{3}\frac{73}{24} + \frac{33}{243} + \frac{1}{3}\frac{73}{243}$ or $\frac{245}{729}$ for the remaining vertex, the (1,1) vertex. We had already said our answer is just 1/3 of this, as that's the probability we get to (0,0). Thus the answer is 245/2187, or about 11%.

There are other ways to solve the problem. One way is to take the solution below and proceed via differentiating identities, but the algebra is not pleasant.

Here's another nice solution to the problem, though it uses far more math than you are expected to know! We have

$$x_{k,k} = \sum_{i=k}^{\infty} \binom{i-1}{k-1}^2 \frac{1}{4^i}.$$

Why is this the case? Imagine we have a tie. We have a string of coin flips. For each team, they either eat an M&M on their turn, or don't, with each event happening with probability 1/2. If there is a tie, we must have some number of coin tosses, say i. Clearly $i \geq k$. We know the last toss must be heads for both, which happens with probability $1/2 \cdot 1/2 = 1/4$. For each team, what must happen in the other $i-1$ tosses? We must have exactly $k-1$ heads for each team. For Middlebury, that probability is $\binom{i-1}{k-1} 1/2^{i-1}$, and similarly for Williams. By independence we multiply and then sum over i. Amazingly, there turns out to be a simple closed form expression for this! Typing

```
Sum[Binomial[i - 1, k - 1]^2 / 4^i, {i, k, Infinity}]
```

into Mathematica yields

```
4^-k Hypergeometric2F1[k, k, 1, 1/4]
```

(so, depending on your point of view, there *is* a closed form solution). The hypergeometric functions arise in all sorts of areas in mathematical physics; courses are devoted to them as so many of our 'special' functions are hypergeometric functions at particular arguments. For more, see

http://en.wikipedia.org/wiki/Hypergeometric_function.

There are advanced techniques to prove such facts. Fortunately, this case isn't too bad and can be done without too much trouble. Again, this is *not* something to be able to get on an exam; a competition is yet one more opportunity to teach. I'm spending the time writing the solution to this up in great detail so that, at your leisure over the next N years, you can read a bit about the hypergeometric series, and at least be aware that there *are* methods to solve problems such as these!

We first need some notation. We start with the Pochhammer symbol: for $n \geq 0$, let

$$(x)_n \;=\; \frac{\Gamma(x+n)}{\Gamma(x)} \;=\; x(x+1)\cdots(x+n-1).$$

Next, we define the hypergeometric function ${}_2F_1$:

$${}_2F_1(a,b,c;z) \;=\; \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}.$$

This can be shown to converge for all $|z| < 1$ so long as c is not a negative integer. We now turn to our sum; we'll need $\Gamma(m+1) = m!$ and $\Gamma(1) = 1$. We have

$$\begin{aligned}
\sum_{i=k}^{\infty} \binom{i-1}{k-1}^2 \frac{1}{4^i} &= \sum_{n=0}^{\infty} \binom{k-1+n}{k-1}^2 \frac{1}{4^{k+n}} \\
&= \frac{1}{4^k} \sum_{n=0}^{\infty} \frac{(k-1+n)!}{(k-1)!n!} \frac{(k-1+n)!}{(k-1)!n!} \frac{1}{4^n} \\
&= \frac{1}{4^k} \sum_{n=0}^{\infty} \frac{\Gamma(k+n)}{\Gamma(k)} \frac{\Gamma(k+n)}{\Gamma(k)} \frac{\Gamma(1)}{\Gamma(n+1)} \frac{(1/4)^n}{n!} \\
&= \frac{1}{4^k} \sum_{n=0}^{\infty} \frac{(k)_n (k)_n}{(1)_n} \frac{(1/4)^n}{n!} \;=\; \frac{{}_2F_1(k,k,1;1/4)}{4^k}.
\end{aligned}$$

It may seem silly, but if Middlebury and Williams each start with $k = 1/2$ of an M&M, the probability of a tie is about 53.6591%. If we have 2012 M&Ms, we get a tie approximately 0.444788746532% of the time. Approximately.

Question 5: Life is full of disappointments, of partial results to what we want. For example, Chebyshev proved that there exist constants A and B, $.8 < A < 1 < B < 1.2$, such that

$$\frac{Ax}{\log x} \;<\; \pi(x) \;<\; \frac{Bx}{\log x},$$

where $\pi(x)$ is the number of primes at most x. He wanted to prove that $\pi(x)$ was $x/\log x$ (this is true, and is the celebrated Prime Number Theorem), but had to settle for this. Speaking of settling, using Chebyshev's

Theorem prove for any integer M there exists an even integer $2k$ such that there are at least M primes p with $p+2k$ also prime. *Unfortunately my proof has $2k$ depending on M. If you can solve this with $2k$ independent of M, you'll have just proved the Twin Prime Conjecture, namely, there are infinitely many primes p such that $p+2$ is also prime! You'll also automatically win the Green Chicken for your school.*

Solution 5: This is a pigeon-hole problem. Consider the primes in $\{3,\dots,x\}$ with x even for convenience. The number of distinct pairs of primes (p,q) with $2<p<q$ is

$$\binom{\pi(x)-1}{2} = \frac{(\pi(x)-1)(\pi(x)-2)}{2} > \frac{\pi(x)^2}{3} > \frac{A^2x^2}{3\log^2 x}$$

for x large; note each pair gives rise to an even difference $q-p$ (with $0<q-p\le x/2$). The number of possible even differences between primes at most x is at most $x/2$, which means the average number of occurrences of each difference is

$$\frac{\binom{\pi(x)-1}{2}}{x/2} \ge \frac{A^2x^2/3\log^2 x}{x/2} = \frac{2A^2x}{3\log^2 x},$$

which tends to infinity with x. By the Pigeon-hole Principle (aka, Dirichlet's Box Principle), at least one of these differences must have at least the average number. Thus, given M we simply take an even x with $2A^2x/3\log^2 x > M$ and we are ensured of having at least one difference occurring this many times. So close and yet so far from the Twin Prime Conjecture!

Question 6: A graph G is a collection of vertices V and edges E connecting pairs of vertices. Consider the following graph. The vertices are the integers $\{2,3,4,\dots,2012\}$. Two vertices are connected by an edge if they share a divisor greater than 1; thus 30 and 1593 are connected by an edge as 3 divides each, but 30 and 49 are not.

The coloring number of a graph is the smallest number of colors needed so that each vertex is colored ***and*** if two vertices are connected by an edge, then those two vertices are not colored the same.

The Brown Chicken says the coloring number of this graph is at most 9. Prove she is wrong. Find the correct coloring number (if you can't find the exact answer, find upper and / or lower bounds). Prove your answers.

Hint: A complete graph on n vertices is a graph with n vertices such that any pair of two vertices are connected by an edge. It might be useful to note that the coloring number of the complete graph on n vertices is n.

Solution 6: The coloring number is at least 10, as the vertices 2, 4, 8, 16, 32, ..., $1024 = 2^{10}$ are all connected to each other, and thus we need at least 10 colors. Why? This is a complete graph with 10 vertices, and its coloring number is 10. As this subgraph of our graph has coloring number 10, the entire graph has coloring number at least 10.

We can get a very good lower bound easily. Instead of looking at powers of 2, we can look at the even numbers. There are 1006 even numbers, and each even number is connected to every other. Thus we have a complete graph with 1006 vertices, implying the coloring number is at least 1006.

It's easy to see the coloring number is at most $2012-\pi(2012)+1$, where $\pi(2012)$ is the number of primes at most 2012. Why? We can color all the primes the same color, as none are connected to any other. That's our plus 1; the $2012-\pi(2012)$ comes from a trivial bounding, using a different color for each remaining vertex.

Interestingly, our lower bound is the answer: the coloring number is 1006. To see this, choose 1006 colors and color each even number with one of these colors, never using the same color twice. Note we ***have*** to do this, as no two even numbers can share a color. We are left with coloring the odd numbers 3, 5, 7, 9, ..., 2011. We color the vertex $2k+1$ with the color of vertex $2k$. Note $2k+1$ and $2k$ can't share a factor d greater than 1 and are thus not connected. (If they shared a factor, it would have to divide their difference, which is 1). Since vertex $2k$ is the only vertex that has the color we want to use for vertex $2k+1$, we see that we have a valid coloring. We showed the coloring number must be at least 1006; since we've found a coloring that works with 1006 colors, we know this must be the answer.